Negatively Oriented Ideal Triangulations and a Proof of Thurston's Hyperbolic Dehn Filling Theorem

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October 27, 1999

ABSTRACT. We give a complete proof of Thurston's celebrated hyperbolic Dehn filling theorem, following the ideal triangulation approach of Thurston and Neumann-Zagier. We avoid assuming that a genuine ideal triangulation always exists, using only a partially flat one, obtained by subdividing an Epstein-Penner decomposition. This forces us to deal with negatively oriented tetrahedra. Our analysis of the set of hyperbolic Dehn filling coefficients is elementary and self-contained. In particular, it does not assume smoothness of the complete point in the variety of deformations.

MATHEMATICS SUBJECT CLASSIFICATION (1991): 57M50 (primary), 57Q15 (secondary).

Thurston's hyperbolic Dehn filling theorem is one of the greatest achievements in the geometric theory of 3-dimensional manifolds, and the basis of innumerable results proved over the last twenty years. The aim of this note is to give a complete and self-contained proof of this theorem, following the ideal triangulation approach due to Neumann and Zagier [20] (and inspired by Thurston [27]), rather than Thurston's original approach through representations ([27] again). The motivation for being interested in a proof via triangulations is threefold. First, it allows a more elementary and transparent description of the geometry after deformation. Second, it mirrors the way the theorem is implemented in practice by the software SnapPea [31].

Third, it permits to establish the asymptotic volume estimates which were the original remarkable achievement of [20].

In Section 1 we will provide some general background on 3-dimensional topology and hyperbolic geometry, and mention some of the many applications and consequences of the hyperbolic Dehn filling theorem. The expert may skip this section.

In our exposition of the proof of the hyperbolic Dehn filling theorem we will need to modify the argument in [20] in two relevant aspects, which we will explain in detail in this introduction, after giving the statement of the result itself. We include both the ordinary and the cone manifold case.

Theorem 0.1. *Let M be an orientable, non-compact, complete, finite-volume hyperbolic 3-manifold. Denote by $\overline{M}$ the compact manifold of which M is the interior, and by $T_1, \ldots, T_k$ the tori which constitute $\partial\overline{M}$. For all i, choose a basis λ_i, μ_i of $H_1(T_i)$. Denote by C the set of coprime pairs of integers, together with a symbol ∞. For $c_1, \ldots, c_k \in C$ denote by $M_{c_1 \cdots c_k}$ the manifold obtained from $\overline{M}$ as follows: if $c_i = \infty$, remove T_i; if $c_i = (p_i, q_i)$, glue to $\overline{M}$ along T_i the solid torus $D^2 \times S^1$, with the meridian $S^1 \times \{*\}$ being glued to a curve homologous to $p_i\lambda_i + q_i\mu_i$. Then:*

1. *There exists a neighbourhood $\mathcal{F}$ of $(\infty, \ldots, \infty)$ in C^k, where C is topologized as a subset of $S^2 = \mathbb{R}^2 \sqcup \{\infty\}$, such that for $(c_1, \ldots, c_k) \in \mathcal{F}$ the manifold $M_{c_1 \ldots c_k}$ admits a complete finite-volume hyperbolic structure.*

2. *Given any $c_1, \ldots, c_k \in C$, for sufficiently small positive real numbers $\vartheta_1, \ldots, \vartheta_k$, the manifold $M_{c_1 \ldots c_k}$ admits the structure of a complete finite-volume hyperbolic cone manifold, with cone locus given by the cores $\{0\} \times S^1$ of the solid tori glued to the T_i's such that $c_i \neq \infty$, where the cone angle is ϑ_i.*

The first difference in our proof with respect to [20] is that we start from a *partially flat* ideal triangulation of M, namely one in which some of the tetrahedra degenerate into flat quadrilaterals with distinct vertices. The existence of such a triangulation easily follows from a result of Epstein and Penner [8]. The argument in [20] was based on the assertion that M is itself obtained by Dehn filling from a hyperbolic manifold which admits a *genuine* ideal triangulation. The reader was addressed to a pre-print of Thurston, later published as [29], for the proof of the assertion, but the result appears to be missing in the printed form of Thurston's paper.

Some historical explanation about ideal triangulations is in order here. It was believed for quite some time by several people that the existence of *genuine* ideal triangulations could be proved as an easy consequence of the result of Epstein and Penner [8]. Eventually, this was recognized to be false, and general existence presently appears to be an open problem (see for instance [30] for sufficient conditions based on the Epstein-Penner decomposition, and [31] for experimental evidence). The first named author is responsible, among others, for the spreading of the erroneous belief that [8] implies existence of triangulations. In particular, the proof presented in [1] of Thurston's hyperbolic Dehn filling theorem is incomplete, because it assumes from the beginning that a genuine ideal triangulation exists.

Starting from an ideal triangulation which is partially flat, it becomes inevitable, when deforming the structure, to deal with negatively oriented tetrahedra, *i.e.* to consider positive-measure overlapping of the geometric tetrahedra. The original part of this paper consists of a careful analysis of this overlapping phenomenon. In particular, we explicitly show how to associate to a deformed triangulation a hyperbolic structure on the manifold, and we describe a developing map for this structure. The general problem of associating a geometric structure to a triangulation with some inverted tetrahedra is known to be a difficult one (see Section 2 for some hints). However, the special case which arises from subdivision of an Epstein-Penner decomposition is simple enough to allow us to construct and analyse the structure explicitly.

The second difference between [20] and our approach is that we do not attempt to prove smoothness of the complete point in the deformation space of the hyperbolic structure. Smoothness can actually be proved in the context of the *representation* rather than *triangulation* approach to deformations, see [14]. As mentioned in [14] and sketched in [27] and [7], the Dehn filling theorem can also be established using the representation approach only. However in this approach the proof of smoothness near the complete point relies on technical cohomology computations, so we have preferred to stick to the more elementary and geometric approach through triangulations. Thurston actually claims that smoothness can be established also in the context of triangulations, looking carefully at the equations which define the space of deformations (personal communication to the first named author, Berkeley, June 1998). Being unable to provide the details for this argument, we have decided not to establish smoothness, but to modify the proof in [20] to a possibly singular context. Our proof that the set of "good" filling parameters indeed covers a neighbourhood of $(\infty,\ldots,\infty)$ becomes somewhat more involved without assuming smoothness. It uses classical tools from the theory of stratifications and analytic spaces, which appear to be more suited to a local argument than tools coming from algebraic geometry, used for instance in [7].

Acknowledgements. We would like to thank the Universities of Pisa and Toulouse for travel and financial support during the preparation of this paper. We gratefully thank Riccardo Benedetti and Michel Boileau for many helpful conversations. In particular, it is a pleasure to acknowledge that the proof of Proposition 4.6 emerged from discussions with Benedetti. The first named author also thanks the Department of Mathematics of the University of Parma for its friendly hospitality.

1 Motivations and background

One of the most natural issues in topology is that of understanding (say, classifying) manifolds (say, closed oriented ones) of a given dimension n. The classification is well-known for $n = 2$ and known to be impossible for $n \geq 4$ (even if spectacular results have been and are being proved, in particular for $n = 4$). In dimension 3 the so-called uniformization (or geometrization) conjecture of Thurston [29] states that every manifold splits along a family of spheres and tori into canonical pieces which admit a geometric structure. Here a geometric structure is a complete Riemannian metric on the interior, locally isometric to a certain homogeneous model. With some natural restrictions, it turns out that only 8 geometries exist [23], and all of them but one are well-understood and "sporadic" among 3-manifolds. The most interesting geometry is the hyperbolic one, *i.e.* the geometry of constant sectional curvature -1, and according to Thurston's conjecture this geometry plays a central role in 3-dimensional topology. In the rest of this section we will state the most basic results of hyperbolic geometry in dimension 3, and we will mention some of the applications of Thurston's hyperbolic Dehn filling theorem, trying to emphasize its crucial role in the understanding of 3-manifolds. The next two results may be found in [1].

Theorem 1.1. *If a* 3*-manifold admits a complete finite-volume hyperbolic structure then it admits only one such structure up to isometry.*

This result is known as Mostow's rigidity theorem, and it holds in all dimensions $n \geq 3$. For $n = 2$ a hyperbolic metric corresponds to a Riemann surface structure, so there are continuous moduli.

Theorem 1.2. *Let M be an orientable* 3*-manifold which admits a complete finite-volume hyperbolic structure. Then:*

1. *M is the interior of a compact* 3*-manifold with boundary $\overline{M}$, where $\partial\overline{M}$ consists of a finite family of tori;*

2. *M is irreducible (every embedded S^2 bounds an embedded D^3) and atoroidal (every subgroup of $\pi_1(M)$ isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ is conjugate to the fundamental group of a component of $\partial\overline{M}$).*

This result is a consequence of Margulis' lemma, and it has analogues in all dimensions. Concerning the notion of "atoroidal" manifold, we mention that there are several variations on this definition, and their mutual relationships have been investigated until quite recently [12], [24], [4]. We do not want to get into this matter here.

It is a great achievement of Thurston, and an important step towards the geometrization conjecture, that a 3-manifold M satisfying the necessary conditions of Theorem 1.2, with certain obvious exceptions, actually is hyperbolic, *provided* either $\partial M \neq \emptyset$ or M is Haken (*i.e.* it contains a two-sided π_1-injective surface). In particular all link exteriors in S^3, with some well-understood families of exceptions, are hyperbolic [28]. Roughly speaking, the last fact stated means that most links are hyperbolic. On the other hand, the hyperbolic Dehn filling theorem implies that most surgeries on a hyperbolic link are hyperbolic. Putting these two facts together one deduces that a wealth of closed manifolds indeed are hyperbolic.

The hyperbolic Dehn filling theorem, in the case of cone-manifolds, is a key step in the proof of the orbifold theorem, another astonishing result of Thurston which has numerous consequences in 3-dimensional topology, detailed proofs of which are currently becoming available [2], [5], [11], [26], [36].

Thurston's hyperbolic Dehn filling theorem is a tool to provide infinite families of examples of closed hyperbolic 3-manifolds. Before the theorem was established, the only known ways to obtain closed hyperbolic 3-manifolds were to glue hyperbolic polyhedra by isometries, or to find the holonomy representation of the fundamental group of a Haken 3-manifold. Examples of manifolds obtained with these methods are the Seifert-Weber dodecahedral space [25] and some fibered manifolds due to Jørgensen [13], but the explicit computations involved are usually quite complicated. Thurston himself used his hyperbolic Dehn filling theorem to provide an infinite family of irreducible non-Seifert and non-Haken 3-manifolds. To our knowledge, the unique previous example of non-Seifert and non-Haken 3-manifold was the Seifert-Weber dodecahedral space, but it was shown only later to be non-Haken [10].

Another reason of importance of the hyperbolic Dehn filling theorem is that it describes precisely the accumulation points of the image of the volume function. The estimates established by Neumann and Zagier [20], which our proof shows to be valid in general, imply for instance the following:

Theorem 1.3. *Let M be an orientable, non-compact, complete, finite-volume hyperbolic* 3*-manifold. Then the set of volumes of hyperbolic manifolds obtained by Dehn filling on M accumulates from the left to the volume of M. In addition, if M has only one end then all but a finite number of volumes are different from each other.*

As a corollary of this theorem, it follows that almost all Dehn surgeries on a hyperbolic knot give different manifolds. This result can also be proven by looking at the length of the shortest non-trivial loop, but we do not know any other proof without using hyperbolic Dehn filling.

Referring to Theorem 1.3, it should also be noted that a result of Gromov and Thurston implies that *all* Dehn-filled manifolds obtained from M have a volume strictly smaller than the volume of M. Concerning convergence, a deeper fact than convergence of volumes actually holds. It turns out that the set of hyperbolic 3-manifolds has a natural topology in which the only accumulation points are cusped manifolds, and convergence corresponds exactly to Dehn filling along divergent sequences of parameters.

Many authors have investigated other properties of the family of hyperbolic 3-manifolds obtained by Dehn filling on a given manifold, and we mention here some of the results which have been obtained. The η-invariant and the Chern-Simons invariant, which are related to each other, have been studied on families Dehn-filled manifolds. It was proven by Yoshida [34] that the Chern-Simons invariant can be naturally viewed as the imaginary part of an invariant which has the volume as the real part. Meyerhoff studied the behaviour of the set of possible values for the Chern-Simons invariant on a family of manifolds obtained by Dehn surgery. In particular he proved that the range of the Chern-Simons invariant is dense [15] and he constructed infinitely many examples of pairs of manifolds with equal volume and different Chern-Simons invariants [16]. Refined computations of the asymptotic value of the Chern-Simons invariant and the η-invariant have been made by Neumann [19] and Neumann and Meyerhoff [17], using ideal triangulations in both cases. Ideal triangulations and their deformations have also been used by Yoshida [35] to understand the relationship between ideal points of the representation variety and incompressible surfaces, according to the work of Culler and Shalen [6].

As said in the introduction, the hyperbolic Dehn filling theorem was first stated in Thurston's notes [27]. It was proved in the particular case of the figure-eight knot complement, using ideal triangulations, and it was sketched in the general case using the representation variety. However, to have a rigorous proof, some more details than those mentioned in [27] are required about the variety of representations. One can complete the proof using either the cohomology results developed in Kapovich's book [14], or some trick as explained in [36]. The proof via triangulations has already been commented in the introduction.

2 Deformation of partially flat triangulations

We describe in this section how to subdivide an Epstein-Penner decomposition into a partially flat ideal triangulation, and how to associate to a modified choice of the moduli of the tetrahedra a deformed hyperbolic structure.

Convex ideal cellularization. Let us fix for the rest of the paper a manifold M as in the statement of Theorem 0.1. See for instance [1] or [22] for the appropriate definitions, and for the proof that indeed $M = \mathrm{int}(\overline{M})$ with $\partial\overline{M} = T_1 \sqcup \ldots \sqcup T_k$. It was proved in [8] that there exist convex ideal polyhedra P_α, $\alpha = 1, \ldots, \nu$, in $\mathbb{H}^3$ such that M is obtained from their disjoint union via face-pairings. Each face-pairing will be an isometry $\varphi_i : F_i \to F_i'$ between a codimension-1 face F_i of some P_α and one such face F_i' of some other P_α (possibly the same P_α, but $F_i \neq F_i'$). Here i ranges between 1 and half the total number of faces of the P_α's. Orientability implies that φ_i reverses the induced orientation, where the P_α's are oriented as subsets of $\mathbb{H}^3$. One way to express the fact that $M = \bigsqcup P_\alpha/\{\varphi_i\}$ is to say that the quotient of $\bigsqcup P_\alpha$ under the equivalence relation generated by the φ_i is homeomorphic to M, and, modulo this homeomorphism, the projection into M of the interior of each P_α is an orientation-preserving isometry. The reason for spelling out this definition is that later on we will need to deal with less obvious identification spaces. See [9] for the most general conditions under which a set of face-pairings on a set of polyhedra defines a manifold or an orbifold.

Partially flat triangulation. We choose now a vertex v_α in each P_α. Moreover, for each of the faces of P_α not containing v_α, we choose a vertex, and take cones from this vertex over the edges not containing it, to subdivide the face into triangles. Now we take cones from v_α over the triangles thus obtained. The result is that P_α has been subdivided into ideal hyperbolic tetrahedra. It will be convenient to call *facets* the triangles into which the original *faces* of P_α have been subdivided. If we now consider a face-pairing $\varphi_i : F_i \to F_i'$, it may or not be the case that φ_i respects the subdivisions of F_i and F_i' into facets. If subdivisions are not respected, we can insert geometrically flat ideal tetrahedra between F_i and F_i', to reconcile these subdivisions, as sketched in Fig. 1.

To be precise, assume F_i and F_i' have been triangulated by taking cones over vertices w_i and w_i' respectively. We identify ∂F_i to $\partial F_i'$ via φ_i, and refer to some abstract version γ_i of this loop, disjoint from the original polyhedra. If $w_i = w_i'$ in γ_i then the triangulations of F_i and F_i' match, and there is nothing to do. If w_i and w_i' are the endpoints of an edge e of γ_i, as in Fig. 1, then for every edge e' of γ_i disjoint from e we add the tetrahedron that is the join of e and e'. In the remaining cases the edge between w_i and w_i' is an interior edge of both the triangulations of the faces F_i and F_i'. Then we divide the faces along this edge and apply twice the previous construction.

From the topological point of view, we are led to consider the ideal triangulation $\mathcal{T}$ of M which consists of all the "fat" tetrahedra obtained by subdividing the P_α, together with the "flat" tetrahedra just inserted. Recall that a *topological ideal triangulation* of M is just a collection of orientation-

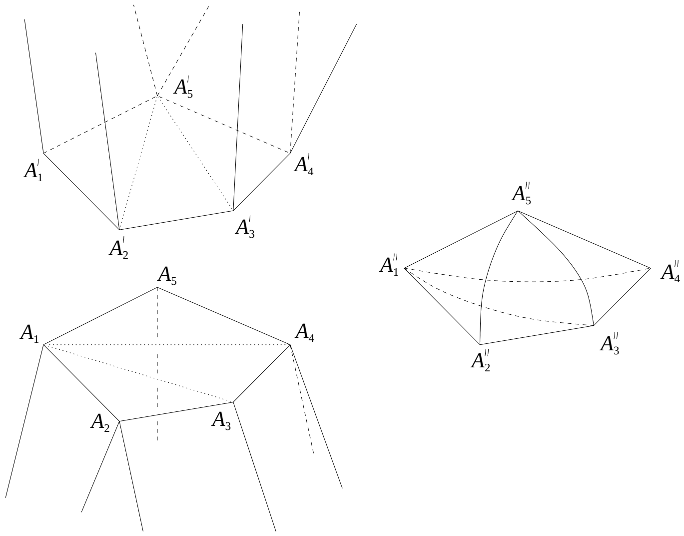

Figure 1: If two paired pentagonal faces as in the figure are subdivided by the dotted lines shown, we add the "flat" tetrahedra $(A''_1, A''_2, A''_3, A''_5)$ and $(A''_1, A''_3, A''_4, A''_5)$

reversing simplicial pairings between the faces of a finite number of copies of the standard tetrahedron, with the property that the identification space defined by the pairings is homeomorphic to the space $\widehat{M}$ obtained from $\overline{M}$ by collapsing each boundary component to a point. In particular, the name "fat" or "flat", used for a tetrahedron of $\mathcal{T}$, only refers to the way the tetrahedron arose from the original geometric subdivision of M. The tetrahedron in its own right, as a member of $\mathcal{T}$, is always "fat".

Even if the topological triangulation $\mathcal{T}$ does depend on the initial choice of vertices on the P_α, we will fix one such choice and refer to a definite $\mathcal{T}$.

Consistency and completeness equations. Recall now that if we fix a pair of opposite edges on the standard ideal tetrahedron Δ, the realizations of Δ as an oriented ideal tetrahedron in $\mathbb{H}^3$ are parametrized (up to oriented isometry) by the upper half-plane $\pi_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$, as described in Fig. 2. This correspondence easily extends to $\mathbb{R}\setminus\{0,1\}$ to cover the case where Δ flattens out to a quadrilateral with distinct vertices. We will interpret parameters in $-\pi_+$ as describing tetrahedra with negative orientation (in

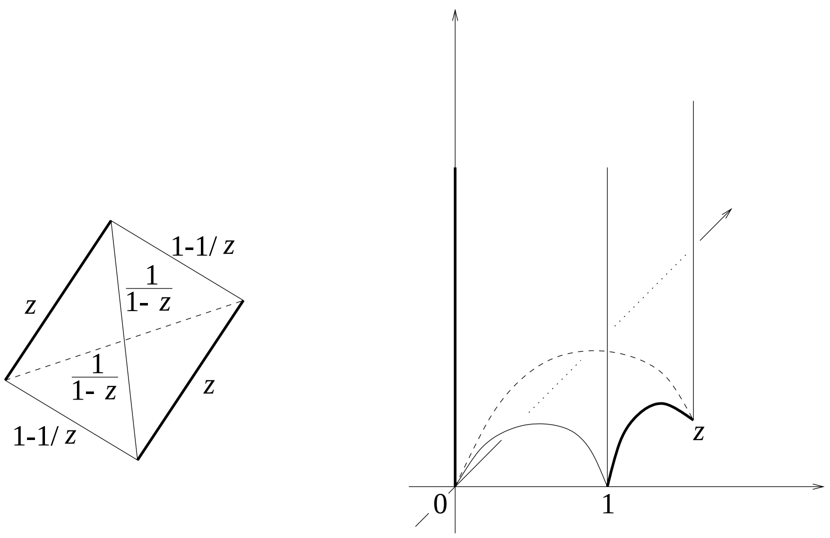

Figure 2: Moduli of an ideal tetrahedron, using the $\mathbb{C} \times (0, \infty)$ model of $\mathbb{H}^3$

particular, negative volume).

Given an ideal triangulation $\mathcal{T}$ of M consisting of tetrahedra $\Delta_1, \ldots, \Delta_n$, we can fix a pair of opposite edges on each Δ_j, choose a modulus $z_j \in \pi_+$ and ask ourselves if M admits a (complete) hyperbolic structure inducing on each Δ_j the structure described by z_j. The answer, which goes back to Thurston [27] (see also [1]), is given by two systems of equations in $z = (z_1, \ldots, z_n)$. We first have the *consistency* (also called *compatibility* equations $\mathcal{C}^*_{\mathcal{T}}(z)$, which prescribe that the product of the moduli around each edge should be 1 and the sum of the corresponding arguments should be 2π. The system $\mathcal{C}^*_{\mathcal{T}}(z)$ is satisfied if and only if there exists on M a (possibly incomplete) hyperbolic structure as mentioned. In practice one often needs to consider only the system $\mathcal{C}_{\mathcal{T}}(z)$ obtained by neglecting the condition on arguments, because close enough to a solution $z^{(0)}$ of $\mathcal{C}^*_{\mathcal{T}}$, the systems $\mathcal{C}_{\mathcal{T}}$ and $\mathcal{C}^*_{\mathcal{T}}$ are equivalent. The other system of equations $\mathcal{M}_{\mathcal{T}}(z)$ one needs to consider, called *completeness* equations, are rational equations in z determined by the combinatorics of $\mathcal{T}$, just as it happens for $\mathcal{C}_{\mathcal{T}}$. They have a geometrical meaning only when $\mathcal{C}^*_{\mathcal{T}}(z)$ holds. In this case a representation ρ_z of $H_1(\partial\overline{M})$ into the group of affine automorphisms of $\mathbb{C}$ is well-defined up to conjugation, and $\mathcal{M}_{\mathcal{T}}(z)$ means that the image of ρ_z consists of translations.

An exact combinatorial description of $\mathcal{M}_{\mathcal{T}}(z)$ is provided after the statement of Theorem 3.1.

Partially flat and negatively oriented solutions. The geometric meaning of $\mathcal{C}^*_{\mathcal{T}}(z)$ and $\mathcal{M}_{\mathcal{T}}(z)$ for $z_1, \ldots, z_n \in \pi_+$ is as follows. First one realizes the abstract face-pairings as isometries between the faces of the ideal tetrahedra in $\mathbb{H}^3$ corresponding to $z_1, \ldots, z_n$. The resulting identification space is homeomorphic to M, and a hyperbolic structure is defined away from the edges. Consistency equations $\mathcal{C}^*_{\mathcal{T}}(z)$ translate the fact that this structure extends to edges, and $\mathcal{M}_{\mathcal{T}}(z)$ translates completeness. As mentioned, the resulting systems $\mathcal{C}_{\mathcal{T}}$ and $\mathcal{M}_{\mathcal{T}}$ are rational and depend only on the combinatorics of $\mathcal{T}$. Moreover only denominators z_j and $1 - z_j$ appear, so it makes sense to consider solutions $z \in (\mathbb{C} \setminus \{0, 1\})^n$. This is not quite the case for $\mathcal{C}^*_{\mathcal{T}}(z)$, because for $z \in -\pi_+$ there is no obvious way to choose arguments for $z, 1/(1-z), 1-1/z$ so that their sum gives π. We will deal with this small subtlety below.

Even if one disregards the problem about arguments, the geometric interpretation of a solution z of $\mathcal{C}_{\mathcal{T}}$ is not so clear when some z_j is not in π_+. The idea is that if $z_j \in -\pi_+$ then Δ_j should overlap with some other $\Delta_{j'}$ with $z_{j'} \in \pi_+$ (actually, at least two of them, so that the algebraic number of tetrahedra covering each point is always 1), but it is not easy to turn this idea into a general formal definition. Actually, a general definition cannot work, as the following discussion shows. Consider the tame case where some z_j are in π_+ and some (but not all) are in $\mathbb{R} \setminus \{0, 1\}$. If we take the corresponding "fat" and "flat" tetrahedra in $\mathbb{H}^3$, we can still glue their faces together, but it was shown in [21] that the resulting identification space is in general not homeomorphic to M. If moduli in $-\pi_+$ are involved, the situation can of course get even worse.

The complete solution. We note first that for $z \in \mathbb{R} \setminus \{0, 1\}$ there is an obvious good choice for the arguments of $z, 1/(1-z), 1-1/z$, namely $\arg(t) = \pi$ for $t < 0$ and $\arg(t) = 0$ otherwise. So, it makes sense to consider partially flat solutions z of $\mathcal{C}^*_{\mathcal{T}}$. As mentioned, such a z does not have in general a geometric meaning. However, it was shown in [21] that if z is a solution also of $\mathcal{M}_{\mathcal{T}}$ then the identification space obtained from the fat and flat tetrahedra is indeed M, and a complete hyperbolic structure is naturally

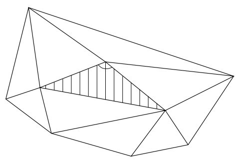

Figure 3: Foliation representing the flattening of a triangle

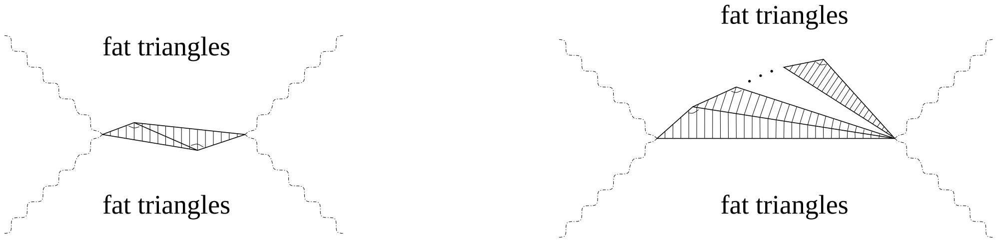

Figure 4: Foliated components arising from subdivision of an Epstein-Penner decomposition

defined. This result itself is not used in this paper, but we will employ the following technical tool introduced in [21] for the proof. To signify the flattening of a genuine triangle into a segment we will foliate the triangle, as sketched in Fig. 3. One of the main points in [21] is the proof that the simultaneous collapse of all the foliated components does not alter the topology.

Going back to the specific situation arising from the subdivision of an Epstein-Penner decomposition of M, we see that we can assign a modulus $z_j^{(0)}$ to each Δ_j in $\mathcal{T}$, where $z_j^{(0)} \in \pi_+$ if Δ_j lies in some P_α, and $z_j^{(0)} \in \mathbb{R}\setminus\{0,1\}$ if Δ_j is one of the tetrahedra we have inserted.

Lemma 2.1. $z^{(0)}$ *is a solution of* $\mathcal{C}^*_{\mathcal{T}}$ *and* $\mathcal{M}_{\mathcal{T}}$. *Moreover the foliated components arising on* $\partial\overline{M}$ *have one of the shapes described in Fig. 4.*

Proof of 2.1. The first assertion is obvious: we already know that M is complete hyperbolic, and $z^{(0)}$ corresponds to a geometric partially flat triangulation, so the geometric interpretation of $\mathcal{C}^*_{\mathcal{T}}$ and $\mathcal{M}_{\mathcal{T}}$ is the same as for genuine triangulations. The second assertion is easily proved by taking transversal sections in Fig. 1 near the ideal vertices. $\boxed{2.1}$

Foliated components as in Fig. 4 are called *bigons*. Lemma 2.1 implies that the foliated components on $\partial\overline{M}$ corresponding to $z^{(0)}$ are bigons intersecting each other only at their ends. This fact will be used in the sequel.

Remark 2.2. If one considers a general partially flat solution of $\mathcal{C}^*_{\mathcal{T}}$ and $\mathcal{M}_{\mathcal{T}}$, annular foliated components and more complicated intersections between components can appear on $\partial\overline{M}$, see [21]. This makes the analysis of the deformed structures considerably harder, and explains why we have decided to concentrate on solutions arising from Epstein-Penner decompositions.

Solutions near the complete solution. From now on we will only be concerned with solutions z of $\mathcal{C}_{\mathcal{T}}$ lying in an arbitrarily small neighbourhood $\mathcal{U}$ of $z^{(0)}$. Formally, all our statements should contain the phrase "$\mathcal{U}$ can be taken so small that...", but we will omit it systematically. We define $\mathcal{D} = \{z \in \mathcal{U} : \ \mathcal{C}_{\mathcal{T}}(z)\}$. We note first that on $\mathcal{U}$ the arguments can be defined by continuity also for the moduli in $-\pi_+$, and of course the resulting system $\mathcal{C}^*_{\mathcal{T}}$ is equivalent to $\mathcal{C}_{\mathcal{T}}$. For this reason we will henceforth leave the discussion of arguments in the background. Moreover we will assume that for $z \in \mathcal{U}$, if $z_j^{(0)} \in \pi_+$, then also $z \in \pi_+$. In other words, flat tetrahedra can become fat, flat, or negative, but fat tetrahedra stay fat.

It will be convenient to denote the generic abstract element of $\mathcal{T}$ by Δ_j, and by $\Delta_j(z)$ the geometric version of Δ_j corresponding to $z \in \mathcal{D}$. As mentioned, for $z_j \in -\pi_+$ one imagines $\Delta_j(z)$ to be negatively oriented, but we will only use directly those $\Delta_j(z)$ for which $z_j^{(0)}$, and hence z_j, lies in π_+. We will also use $P_\alpha(z^{(0)})$ to emphasize that we are considering the geometric polyhedron rather than the abstract one P_α. For all α, let J_α be the set of indices j such that Δ_j appears in the original subdivision of P_α. Consider also the set of face-pairings p_α corresponding to the triangles lying in the interior of P_α. In this context a face-pairing is just a combinatorial rule, but when the abstract tetrahedra are turned into geometric ones, an isometry is uniquely determined.

Lemma 2.3. *For $z \in \mathcal{D}$ and for all α, the tetrahedra $\Delta_j(z)$, $j \in J_\alpha$ can be assembled along p_α to give a (probably non-convex) ideal polyhedron $P_\alpha(z)$ in $\mathbb{H}^3$ with triangular faces, combinatorially equivalent (in particular, homeomorphic) to P_α (endowed with the* facets *structure).*

Proof of 2.3. We first note that, using the projective model of $\mathbb{H}^3$, ideal polyhedra can be viewed as compact Euclidean polyhedra with vertices on the unit sphere. Choosing a maximal tree in the graph corresponding to the pairing p_α, we can realize in $\mathbb{H}^3$ the $\Delta_j(z)$, $j \in J_\alpha$, so that the pairings in the tree are given by actual overlapping. Moreover we can define a map $f_j(z) : \Delta_j(z^{(0)}) \to \Delta_j(z)$, for instance using Euclidean coordinates and taking convex combinations of vertices. Since z satisfies $\mathcal{C}_\mathcal{T}(z)$, these $f_j(z)$'s match to give a map $F_\alpha(z) : P_\alpha(z^{(0)}) \to P_\alpha(z)$. Moreover $F_\alpha(z)$ is locally injective. To conclude we note that $P_\alpha(z)$ converges to the identity of $P_\alpha(z^{(0)})$ as z goes to $z^{(0)}$, and we use Euclidean compactness of $P_\alpha(z^{(0)})$ to deduce that $F_\alpha(z)$ is eventually injective. All conclusions easily follow. $\boxed{2.3}$

Using the combinatorial equivalence between $P_\alpha(z)$ and P_α, we can define the faces $F_i(z)$ also for the $P_\alpha(z)$. Each $F_i(z)$ will be a (probably non-planar) union of facets.

We define now an *abstract* polyhedron $\tilde{P}_\alpha$ by adding to P_α all the flat tetrahedra Δ_j arising from faces F_i contained in ∂P_α. Recall that we have artificially broken the symmetry of face-pairings using the notation F_i, F_i' for a pair of faces to be glued, so each flat tetrahedron is used once. The $\tilde{P}_\alpha$ have a natural facet structure on their boundary. Moreover, using the pairing of triangles in $\mathcal{T}$, we deduce a pairing of the facets of the $\tilde{P}_\alpha$, and the result of all these facet-pairings is M.

The idea is now to replace each $P_\alpha(z)$ by some $\tilde{P}_\alpha(z)$ having the same combinatorial structure as $\tilde{P}_\alpha$, so to obtain M from geometric polyhedra. As obvious, $\tilde{P}_\alpha(z)$ will result from elementary modifications on $P_\alpha(z)$, each modification coming from one of the faces F_i contained in ∂P_α. The elementary modification is itself obvious: $\tilde{P}_\alpha(z)$ will have the same vertices on $\partial\mathbb{H}^3$ as $P_\alpha(z)$, but facets (convex envelopes of triples of these vertices) will be taken according to the combinatorial structure of $\tilde{P}_\alpha$ rather than P_α. For example, consider the situation of Fig. 1. Let P_α be the polyhedron shown below in the figure, and let $F_i = (A_1, \ldots, A_5)$. The collection of facets of $P_\alpha(z)$ contains the triangles $(A_1(z), A_2(z), A_3(z))$, $(A_1(z), A_3(z), A_4(z))$, $(A_1(z), A_4(z), A_5(z))$. Now we replace these triangles by $(A_5(z), A_1(z), A_2(z))$, $(A_5(z), A_2(z), A_3(z))$, $(A_5(z), A_3(z), A_4(z))$, leaving all other facets of $P_\alpha(z)$ unchanged. The resulting collection of triangles still bounds an ideal polyhedron in $\mathbb{H}^3$, which we take as $\tilde{P}_\alpha(z)$. We will also denote by $\tilde{F}_i(z)$ the union of the modified facets.

Remark 2.4. Assume that under a face-pairing $\varphi_i : F_i \to F_i'$ no edge of the subdivisions of F_i and F_i' is matched (as in Fig. 1). Then the flat tetrahedra inserted come in a natural order starting from F_i and proceeding towards F_i' (in Fig. 1, first $(A_1'', A_5'', A_4'', A_3'')$ and then $(A_1'', A_5'', A_3'', A_2'')$). The transformation of $P_\alpha(z)$ into $\tilde{P}_\alpha(z)$ can be viewed as the result of successive transformations corresponding to the individual flat tetrahedra. Each transformation consists in replacing a quadrilateral, bent along one diagonal, with the quadrilateral having the same perimeter and bent along the other diagonal. If the dihedral angle at the first diagonal is more than π then the modulus of the corresponding tetrahedron is in π_+, and the tetrahedron is being added to P_α. If the angle is less than π, then the modulus is in $-\pi_+$, and the tetrahedron is being deleted. If the angle is π, the modulus is in $\mathbb{R} \setminus \{0, 1\}$ and we are only changing the combinatorial structure of the facets of P_α. When the pairing φ_i matches an edge of the subdivisions, this description must be repeated for both of the polygons into which F_i is divided by the matching edge.

Theorem 2.5. *1. The above-described modification of $P_\alpha(z)$ can be carried out simultaneously for all faces F_i.*

2. The resulting collection $\tilde{P}_\alpha(z)$, with the face structure given by the $\tilde{F}_i(z)$ and the $F_i'(z)$, is combinatorially equivalent to the original collection P_α.

3. Each pairing $\tilde{F}_i(z) \to F_i'(z)$ can be realized by an isometry.

4. The identification space resulting from the pairings is homeomorphic to M, and it can be endowed with a hyperbolic structure compatible with the structure defined on the interior of each $\tilde{P}_\alpha(z)$.

Proof of 2.5. It is again useful to identify hyperbolic ideal polyhedra with compact Euclidean polyhedra with vertices on the sphere. Using this point of view, let us consider the 1-skeleton $\Gamma_\alpha(z^{(0)})$ of a certain $P_\alpha(z^{(0)})$. On $\Gamma_\alpha(z^{(0)})$ we have certain simple circuits which correspond to the faces of P_α. Note that each circuit is contained in a plane, and the various planes form dihedral angles strictly less than π at the edges of $\Gamma_\alpha(z^{(0)})$. Now we consider the same circuits in the modified 1-skeleton $\Gamma_\alpha(z)$. By compactness, we easily see that for z close enough to $z^{(0)}$, the convex envelopes of any two distinct circuits meet at most in a common edge or vertex of $\Gamma_\alpha(z)$. This shows points 1, 2 and the first assertion in 4.

We show point 3 in the special case of Fig. 1, leaving to the reader the general case. The idea is to somehow realize in $\mathbb{H}^3$ the flat tetrahedra. Let x and y be the moduli along the edge (A_1'', A_5'') of the tetrahedra $(A_1'', A_2'', A_3'', A_5'')$ and $(A_1'', A_3'', A_4'', A_5'')$ respectively. Note that $x(z^{(0)}), y(z^{(0)}) \in (1, \infty)$. Now in the half-plane model of $\mathbb{H}^3$ we choose $A_1''(z) = \infty$, $A_5''(z) = 0$, $A_4''(z) = 1$, $A_3''(z) = x(z)$ and $A_2''(z) = y(z) \cdot x(z)$. Consistency of z along (A_1, A_4) and (A_1, A_3) implies that the unique $f \in \mathrm{Isom}^+(\mathbb{H}^3)$ such that $f(A_1(z)) = A_1''(z)$, $f(A_5(z)) = A_5''(z)$, and $f(A_4(z)) = A_4''(z)$, also enjoys $f(A_3(z)) = A_3''(z)$ and $f(A_2(z)) = A_2''(z)$. Similarly consistency along (A_5, A_2) and (A_5, A_3) implies that $g(A_l'(z)) = A_l''(z)$, $l = 1, \ldots, 5$, for some $g \in \mathrm{Isom}^+(\mathbb{H}^3)$. Now, the description of $\tilde{P}_\alpha$ given in Remark 2.4 implies that

$$\begin{aligned}\tilde{F}_i(z) = f^{-1}\Big((A_5''(z), A_1''(z), A_2''(z)) \ &\cup\ (A_5''(z), A_2''(z), A_3''(z)) \\ &\cup\ (A_5''(z), A_3''(z), A_4''(z))\Big)\end{aligned}$$

whence the conclusion.

The second assertion in point 4 follows from point 3 and consistency along the original edges of the P_α. $\boxed{2.5}$

3 Developing map and completion of deformed structures

We will denote in the sequel by $\mathfrak{h}(z)$ the hyperbolic structure on M constructed in Theorem 2.5 for $z \in \mathcal{D}$. In this section we will analyze the completion of $\mathfrak{h}(z)$, the key ingredient being the understanding of the developing map of cusps. We will first give the statement needed in Section 4 to conclude the proof of Theorem 0.1, then we will switch to a 2-dimensional setting, and later we will use the 2-dimensional construction to understand $\mathfrak{h}(z)$.

Statements of results. Let us return to the notation of Theorem 0.1 and slightly modify it in order to unify the two assertions (ordinary case and cone manifold case). Consider the set

$$G = \{\infty\} \cup \{g \in \mathbb{R}^2 \;:\; g = r \cdot (p,q) \text{ for some } r > 0 \text{ and relatively prime } p, q \in \mathbb{Z}\}.$$

(The motivation for the notation is that G consists of *G*eneralized filling coefficients, as opposed to the genuine *C*oefficients of the set C defined in Theorem 0.1.) For $g \in G \setminus \{\infty\}$ note that its expression as $r \cdot (p,q)$ is unique, and define $c(g) = (p,q)$, $\vartheta(g) = 2\pi/r$. Set $c(\infty) = \infty$. Topologize G as a subset of $\mathbb{R}^2 \cup \{\infty\} = S^2$. We can now restate Theorem 0.1 as follows:

Theorem 3.1. *Under the assumptions of Theorem 0.1 there exists a neighbourhood $\mathcal{F}$ of $(\infty,\ldots,\infty)$ in G^k such that for $(g_1,\ldots,g_k) \in \mathcal{F}$ the manifold $M_{c(g_1)\ldots c(g_k)}$ admits the structure of a complete finite-volume hyperbolic cone manifold, with cone locus given by the cores $\{0\} \times S^1$ of the solid tori glued to the T_i's such that $g_i \neq \infty$, where the cone angle is $\vartheta(g_i)$.*

It is perhaps worth noticing here that this statement is actually independent of the choice of the basis λ_i, μ_i of $H_1(T_i)$. In fact, a different choice is related through a matrix in $\mathrm{GL}(2,\mathbb{Z})$, which induces a homeomorphism of S^2 and preserves coprimality of integer pairs, and hence the function $\vartheta : G \to \mathbb{R}_+$ introduced above.

Theorem 3.1 is the result which we will establish in the rest of the paper. To summarize the content of the present section, we now go back to the notation of Section 2. Note first that for $z \in \mathcal{D}$ a homomorphism $h_i(z) : H_1(T_i) \to \mathbb{C}^*$ is defined by $h_i(z)([\gamma]) = (-1)^{\#\gamma_0} L_z(\gamma)$, where γ is a simplicial loop with respect to the triangulation of T_i induced by $\mathcal{T}$, $\#\gamma_0$ is the number of vertices of γ and $L_z(\gamma)$ is the product of all moduli along angles which γ leaves on its left on T_i. Recall that $\mathcal{M}_{\mathcal{T}}(z)$ is the system $\{h_i(z)(\lambda_i) = h_i(z)(\mu_i) = 1,\ i = 1,\ldots,k\}$. Note that $h_i(z^{(0)})(\lambda_i) = h_i(z^{(0)})(\mu_i) = 1$, so we can use the holomorphic branch log of the logarithm function enjoying $\log(1) = 0$ to define maps $u_i, v_i : \mathcal{D} \to \mathbb{C}$ as $u_i(z) = \log(h_i(z)(\lambda_i))$ and $v_i(z) = \log(h_i(z)(\mu_i))$. We will establish the following:

Theorem 3.2.

1. *For $z \in \mathcal{D}$, we have $u_i(z) = 0$ if and only if $v_i(z) = 0$.*
2. *If $z \in \mathcal{D}$ and $u_1(z) = \cdots = u_k(z) = 0$ then $z = z^{(0)}$.*

3. *The following limit exists and is not real:*

$$\tau_i = \lim_{z\in\mathcal{D},u_i(z)\neq 0,z\to z^{(0)}} \frac{v_i(z)}{u_i(z)}.$$

4. *Fix $z \in \mathcal{D}$, and let $g_1, \ldots, g_k \in G$ be such that $g_i = \infty$ when $u_i(z) = 0$, and $g_i = (p_i, q_i)$ with $p_i \cdot u_i(z) + q_i \cdot v_i(z) = 2\pi\sqrt{-1}$ otherwise. Then the completion of M with respect to $\mathfrak{h}(z)$ is homeomorphic to $M_{c(g_1)\ldots c(g_k)}$, and the structure of M extends to a hyperbolic cone manifold structure as described in Theorem 3.1.*

Partially flat triangulations of the torus. Let us consider a triangulation $\mathcal{T}$ of the torus T (all notation overlaps between this paragraph and the previous section are intentional, and their motivation should be clear to the reader). The combinatorics of $\mathcal{T}$ allows to write down systems $\mathcal{C}^*_{\mathcal{T}}$ and $\mathcal{M}_{\mathcal{T}}$, the latter requiring the choice of a basis λ, μ of $H_1(T)$. For $z_1, \ldots, z_n \in \pi_+$, $\mathcal{C}_{\mathcal{T}}(z)$ holds if and only if there is on T a similarity structure inducing on the j-th triangle the structure with modulus z_j. Moreover, also $\mathcal{M}_{\mathcal{T}}(z)$ holds if and only if this structure is compatible with a Euclidean structure. Let us fix now a solution $z^{(0)}$ of $\mathcal{C}^*_{\mathcal{T}}$ and $\mathcal{M}_{\mathcal{T}}$ which is only *partially* (but not totally) *flat*. It was shown in [21] that $z^{(0)}$ still yields a Euclidean structure (up to scaling) on T. However, being only interested in the situations arising on $\partial\overline{M}$ when subdividing an Epstein-Penner decomposition, we may take as an *assumption* that there is on T a Euclidean structure inducing on the j-th triangle of $\mathcal{T}$ the structure with modulus $z_j^{(0)}$. Of course when $z_j^{(0)}$ is real this means that the triangle has been collapsed to a segment. We will use foliations to signify collapse, as in Fig. 3. We will also assume that foliated components of T are bigons intersecting at their ends only, as in Fig. 4.

Before proceeding, we need to recall that for a solution $z \in (\mathbb{C} \setminus \{0, 1\})^n$ of $\mathcal{C}^*_{\mathcal{T}}$, a representation $h(z) : H_1(T) \to \mathbb{C}^*$ can be defined as explained above. Moreover $\mathcal{M}_{\mathcal{T}}(z)$ is the system $h(z)(\lambda) = h(z)(\mu) = 1$. (It follows from this that all systems $\mathcal{M}_{\mathcal{T}}(z)$ arising from different choices of the basis of $H_1(T)$ are equivalent to each other. However, we will not need to change basis.)

Proposition 3.3. *There exist a neighbourhood $\mathcal{U}$ of $z^{(0)}$ in $(\mathbb{C}\setminus\{0, 1\})^n$ such that:*

1. *For $z \in \mathcal{U}$, $\mathcal{C}_{\mathcal{T}}(z)$ is equivalent to $\mathcal{C}^*_{\mathcal{T}}(z)$.*

2. *If $\mathcal{D} := \{z \in \mathcal{U} : \mathcal{C}_{\mathcal{T}}(z)\}$ and $z \in \mathcal{D}$, then $h(z)(\lambda) = 1$ if and only if $h(z)(\mu) = 1$.*

3. *If $u(z) = \log(h(z)(\lambda))$ and $v(z) = \log(h(z)(\mu))$, where $\log$ is holomorphic near $1 \in \mathbb{C}$ and $\log(1) = 0$, then the limit of $v(z)/u(z)$, as z tends to $z^{(0)}$ in $\mathcal{D}$ and $u(z) \neq 0$, exists and is a non-real number τ.*

4. *Each $z \in \mathcal{D}$ defines on T a similarity structure $\mathfrak{s}(z)$.*

5. *For $z \in \mathcal{D}$, $\mathfrak{s}(z)$ is compatible with a Euclidean structure on T if and only if $h(z)(\lambda) = 1$.*

6. *If $h(z)(\lambda) \neq 1$, a developing map for $\mathfrak{s}(z)$ is given by*

$$\mathbb{R}^2 \ni (x, y) \mapsto \exp(u(z)x + v(z)y) \in \mathbb{C}$$

where $\mathbb{R}^2$ is the universal cover of T, with deck transformation group $\mathbb{Z}^2$.

Remark 3.4. Oriented similarity structures in dimension two are equivalent to complex affine structures in dimension one, and we will use both indistinctly.

Proof of 3.3. Point 1 is clear from continuity. To prove the other points we start with the Euclidean structure on T. Let $\sigma_1, \ldots, \sigma_r$ be the triangles of $\mathcal{T}$ that have non-zero area for this Euclidean structure. The remaining triangles are flat, so they have a longest edge (the one with angle zero at each endpoint), and we abstractly glue each one of these triangles to its neighbour along the longest edge. Since we assume that the foliated components of T are bigons as described in Fig. 4, each flat triangle is glued to either a fat one or to a family of flat triangles glued to a fat one. The result of this gluing process is a family of abstract triangulated polygons $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_r$, such that each $\tilde{\sigma}_i$ contains exactly one triangle that is fat for the Euclidean structure. Moreover we have a family of pairings between the edges of the $\tilde{\sigma}_i$'s, yielding T as identification space.

The parameters $z \in \mathcal{U}$ define a complex affine structure on the triangles σ_i that we denote by $\sigma_i(z)$. Now we define the induced structures on the $\tilde{\sigma}_i$'s. We first repeat geometrically the combinatorial construction of $\tilde{\sigma}_i$, namely we add the triangles with parameter in π_+ and we remove the triangles with parameter in $-\pi_+$. The triangles with real parameter are the flat ones, and for them we add a new vertex in the interior of the edge they represent, according to the real parameter. This process is only possible when z is close to $z^{(0)}$. We denote by $\tilde{\sigma}_i(z)$ the complex affine polygon obtained in this way. The next lemma proves point 4 of Proposition 3.3.

Lemma 3.5. *For $z \in \mathcal{D}$ and for $i = 1, \ldots, r$, $\tilde{\sigma}_i(z)$ defines a complex affine structure on the polygon $\tilde{\sigma}_i$. These structures match under the edge-pairings and induce a complex affine structure $\mathfrak{s}(z)$ on T.*

Proof of 3.5. For the first assertion we have to show that there is a natural combinatorial equivalence between $\tilde{\sigma}_i$ and $\tilde{\sigma}_i(z)$. We view $\partial\tilde{\sigma}_i(z^{(0)})$ not as a triangle but as a polygon combinatorially equivalent to $\partial\tilde{\sigma}_i$, because each time we glue a flat triangle we are adding a new vertex. Hence $\tilde{\sigma}_i(z^{(0)})$ is a polygon in $\mathbb{C}$, with every angle but three equal to π. Now the polygon $\partial\tilde{\sigma}_i(z^{(0)})$ is combinatorially isomorphic to the abstract polygon $\partial\tilde{\sigma}_i$, and $\partial\tilde{\sigma}_i(z)$ is isomorphic to $\partial\tilde{\sigma}_i(z^{(0)})$ for $z \in \mathcal{U}$, because the vertices depend continuously on z, so $\partial\tilde{\sigma}_i(z)$ is equivalent to $\partial\tilde{\sigma}_i$.

Having shown that the $\tilde{\sigma}_i(z)$'s are equivalent to the $\tilde{\sigma}_i$'s, we can now realize the edge-pairings by similarities. Consistency equations $\mathcal{C}_{\mathcal{T}}(z)$ are readily seen to imply that the similarity structure defined on T minus the vertices extends to the vertices, whence the conclusion. $\boxed{3.5}$

We next consider the holonomy of $\mathfrak{s}(z)$. This is a homomorphism $\pi_1(T) \to \mathrm{Aff}(\mathbb{C})$ well-defined up to conjugation. Speaking of holonomy we need to refer to $\pi_1(T)$, but we will freely use the canonical isomorphism with $H_1(T)$. Given $f \in \mathrm{Aff}(\mathbb{C})$, if $f(w) = \alpha w + \beta$ we call α the linear part of f. Note that α is invariant under conjugation, so the linear part of the holonomy is a well-defined homomorphism $\pi_1(T) \to \mathbb{C}^*$, which depends only on the complex affine structure.

Lemma 3.6. *Given $g \in \pi_1(T)$, the linear part of the holonomy of g corresponding to $\mathfrak{s}(z)$ is $h(z)(g)$, where h is defined as above. In addition, there exists a representative $\rho(z)$ of the holonomy such that $\rho(z)(\lambda)$ and $\rho(z)(\mu)$ are respectively given by*

$$w \mapsto \mathrm{e}^{u(z)}w + a(z) \qquad \textit{and} \qquad w \mapsto \mathrm{e}^{v(z)}w + b(z),$$

where $a, b : \mathcal{D} \to \mathbb{C}$ are restrictions to $\mathcal{D}$ of global rational functions with denominators not vanishing in $\mathcal{D}$.

Proof of 3.6. To prove the first assertion we recall the general recipe to compute the linear part of the holonomy. We consider the CW-decomposition of T where the 2-cells are the polygons $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_r$ and the 1-skeleton is the union of the boundaries of these polygons. Given an element in $\pi_1(T)$, we represent it by a path γ in the 1-skeleton. Since the 2-cells are polygons of $\mathbb{C}$ defined up to similarity, the complex ratio between two 1-cells with a common end is an invariant of the similarity structure. The linear part of the holonomy of the oriented path γ is the product of the ratios between each pair of consecutive 1-cells of γ, taking care of the orientations. In our situation, each ratio between consecutive 1-cells is a product of parameters z_j, $1-1/z_j$ or $1/(1-z_j)$, and one can easily check that the linear part of the holonomy of γ is precisely $h(\gamma)(z)$ as defined above.

To prove the second assertion, we fix a polygon $\tilde{\sigma}_i$ and one of its edges. We normalize the developing map $D(z)$ so that it maps this edge to the segment $[0, 1]$ in $\mathbb{C}$. In the fundamental group, we choose the basepoint to be the initial point of the edge we have fixed, and consider the holonomy $\rho(z)$ corresponding to $D(z)$. The first assertion of the lemma and the definition of u imply that the linear part of $\rho(z)(\lambda)$ is indeed $\mathrm{e}^{u(z)}$. Moreover, by our choices, $a(z) = D(z)(\tilde{\lambda}(1))$, where $\tilde{\lambda}$ is a lift of λ to the universal covering such that $\tilde{\lambda}(0) \in D(z)^{-1}(0)$. Now, $D(z)$ is constructed by patching together in $\mathbb{C}$ triangles with moduli z_j, with one triangle having vertices 0 and 1. All resulting vertices, in particular $a(z) = D(z)(\tilde{\lambda}(1))$, are therefore polynomials in the z_j, $1-1/z_j$ and $1/(1-z_j)$. This implies the conclusion for $\rho(z)(\lambda)$, and the same argument applies to μ. $\boxed{3.6}$

Since the complex affine structure $\mathfrak{s}(z^{(0)})$ is compatible with a Euclidean structure, $u(z^{(0)}) = v(z^{(0)}) = 0$ and $\langle a(z^{(0)}), b(z^{(0)})\rangle$ is a lattice in $\mathbb{C}$. In particular $a(z^{(0)}), b(z^{(0)}) \in \mathbb{C} \setminus \{0\}$ and $\tau = b(z^{(0)})/a(z^{(0)}) \in \mathbb{C} \setminus \mathbb{R}$. Moreover it follows from the commutativity between λ and μ that:

$$a(z)(\mathrm{e}^{v(z)} - 1) = b(z)(\mathrm{e}^{u(z)} - 1).$$

Points 2, 3 and 5 in Proposition 3.3 follow directly from this equality. We are left to prove point 6. We start with $\mathfrak{s}(z^{(0)})$. Since this structure is compatible with a Euclidean one, it is complete, because every Riemannian structure on a compact manifold is complete. This means that the structure $\mathfrak{s}(z^{(0)})$ is realized by the quotient $\mathbb{C}/\Gamma$, where the lattice $\Gamma = \langle a(z^{(0)}), b(z^{(0)})\rangle < \mathbb{C}$ is the image of $\pi_1(T)$ under the holonomy. Using the fact that the isotopy class of a homeomorphism of the torus is determined by its action on the fundamental group, it follows that a developing map for $\mathfrak{s}(z^{(0)})$ is given by any equivariant homeomorphism between $\mathbb{R}^2$ and $\mathbb{C}$. Hence a developing map for $\mathfrak{s}(z^{(0)})$, normalized as in the proof of the previous lemma, is given by

$$\mathbb{R}^2 \ni (x,y) \mapsto a(z^{(0)})x + b(z^{(0)})y \in \mathbb{C}.$$

By [32] or §1.7 of [3], for $z \in \mathcal{D}$, to give a developing map of $\mathfrak{s}(z)$ it suffices to deform the developing map of $\mathfrak{s}(z^{(0)})$ to a local embedding equivariant with the holonomy. The following family of maps has the required properties:

$$\mathbb{R}^2 \ni (x,y) \mapsto \begin{cases} a(z) \cdot \frac{\exp((u(z)x+v(z)y))-1}{\exp(u(z))-1} & \text{if } u(z) \neq 0 \\ a(z)x + b(z)y & \text{otherwise} \end{cases} \in \mathbb{C}. \qquad (1)$$

More precisely, this is a family of maps from $\mathbb{R}^2$ to $\mathbb{C}$ parametrized by $z \in \mathcal{D}$. This family depends continuously on the parameter $z \in \mathcal{D}$, in the sense that if we have a convergent sequence in $\mathcal{D}$, then the corresponding sequence of maps converges uniformly on compact subsets of $\mathbb{R}^2$ for the C^1 topology. In addition, the map corresponding to $z \in \mathcal{D}$ is equivariant with the holonomy of $\mathfrak{s}(z)$ in Lemma 3.6. Hence it is a developing map for $\mathfrak{s}(z)$ when $z \in \mathcal{D}$. When $u(z) \neq 0$, if we compose the map in (1) with a suitable complex affine transformation, we obtain the map in point 6 of the proposition. $\boxed{3.3}$

3-dimensional developing map. Points 1 and 3 of Theorem 3.2 follow directly from Proposition 3.3, considering the various T_i's. To establish the other points, we go back now to the setting of Section 2. We know that each $z \in \mathcal{D}$ defines on M a hyperbolic structure $\mathfrak{h}(z)$, and our plan here is to develop it to analyze its completion. We will cut M along a collection of disjoint boundary-parallel tori, getting a compact manifold M_0 with boundary, together with cusps $C_1, \ldots, C_k$, with $C_i \cong T_i \times [0, \infty)$ and T_i corresponding to $T_i \times \{\infty\}$. We will allow ourself to isotope the cutting tori without changing notation. Since M is ∂-incompressible, if we take a developing map of M and restrict it to a component of the preimage (under the universal covering) of C_i, we get a developing map for the restriction $\mathfrak{h}_i(z)$ of $\mathfrak{h}(z)$ to C_i. Therefore the completion of M is obtained by completing the various C_i's separately and then glueing back to M_0 along the tori.

Proposition 3.7. *If $z \in \mathcal{D}$ then $\mathfrak{h}_i(z)$ is complete if and only if $u_i(z) = 0$. If $u_i(z) \neq 0$ then a developing map for $\mathfrak{h}_i(z)$ is given by:*

$$\begin{array}{rcl} \mathbb{R}^2 \times [0,\infty) & \to & \mathbb{H}^3 \cong \mathbb{C} \times (0,\infty) \\ (x,y,t) & \mapsto & (\exp(u(z)x + v(z)y), \exp(t + \Re(u(z)x + v(z)y))). \end{array}$$

If $p_i \cdot u_i(z) + q_i \cdot v_i(z) = 2\pi\sqrt{-1}/r_i$ for some coprime pair of integers (p_i, q_i) and a real number $r_i > 0$, then the completion of C_i is obtained by attaching $D^2 \times S^1$ to $T_i \times [0,\infty]$ along $T_i \times \{\infty\}$, with $S^1 \times \{*\}$ glued to $(p_i\lambda_i + q_i\mu_i) \times \{\infty\}$, and the result has the structure of a hyperbolic cone manifold with boundary, with cone locus $\{0\} \times S^1$ and angle $2\pi/r_i$.

Proof of 3.7. We will use both the statement and the proof of Proposition 3.3, denoting by $\mathfrak{s}_i(z)$ the similarity structure defined on T_i according to that proposition. Now, the hyperbolic structure $\mathfrak{h}(z)$ on the open manifold M induces another similarity structure on T_i, which we denote by $\mathfrak{s}_i^*(z)$. We have the following:

Lemma 3.8. *$\mathfrak{s}_i(z) = \mathfrak{s}_i^*(z)$ for all $z \in \mathcal{D}$.*

Proof of 3.8. Recall first that $\mathfrak{s}_i(z)$ is obtained by glueing together polygons $\tilde{\sigma}_j(z)$ as in Lemma 3.5. Moreover, since $\mathfrak{h}(z)$ is obtained by glueing together the polyhedra $\tilde{P}_\alpha(z)$ of Section 2, to get $\mathfrak{s}_i^*(z)$ one has to intersect the $\tilde{P}_\alpha(z)$ with horospheres centered at ideal vertices corresponding to the i-th cusp, and patch together the resulting affine polyhedra, which we denote by $Q_l(z)$.

Both the $\tilde{\sigma}_j(z)$ and the $Q_l(z)$ are obtained by grouping together some of the triangles with moduli $z_1, \ldots, z_n$, in such a way that each flat or negative triangle gets grouped with at least one fat triangle. The grouping rules, however, are different, so indeed we have something to prove. We first remark that $\mathfrak{s}_i(z^{(0)}) = \mathfrak{s}_i^*(z^{(0)})$, because geometrically (even if not combinatorially) each $Q_l(z^{(0)})$ is obtained by glueing together some $\tilde{\sigma}_j(z^{(0)})$.

We will now show that $\mathfrak{s}_i(z)$ and $\mathfrak{s}_i^*(z)$ have the same holonomy for $z \in \mathcal{D}$. Since this holonomy depends analytically on $\in \mathcal{D}$, by Lemma 3.6, knowing that $\mathfrak{s}_i(z^{(0)}) = \mathfrak{s}_i^*(z^{(0)})$, it follows from Theorem 1.7.1 of [3] or from [32] that $\mathfrak{s}_i(z) = \mathfrak{s}_i^*(z)$ for $z \in \mathcal{D}$. Using the recipe (based on ratios of segments) mentioned in Lemma 3.6, one gets combinatorial rules for the holonomies of $\mathfrak{s}_i(z)$ and $\mathfrak{s}_i^*(z)$. These rules involve only the moduli $z_1, \ldots, z_n$ and apply to loops which are simplicial in the CW-structures on T_i induced respectively by the

$\tilde{\sigma}_j$'s and by the Q_l's. These CW-structures have, as a common subdivision, the triangulation $\mathcal{T}_i$ induced by $\mathcal{T}$ on T_i. Using the consistency relations $\mathcal{C}^*_{\mathcal{T}_i}$ one easily sees that the two rules extend to one and the same combinatorial rule which applies to loops which are simplicial in $\mathcal{T}_i$. This shows that the holonomies are the same, whence the conclusion. $\boxed{3.8}$

Since $\mathfrak{h}_i(z^{(0)})$ is a complete cusp, it is isometric to the quotient of a horoball under the action of $\pi_1(C_i)$ via the holonomy representation (see Chapter D in [1] for instance). Hence, if we assume that the horoball is centered at $\infty \in \mathbb{C} \cup \{\infty\} \cong \partial\mathbb{H}^3$, the complete cusp has a developing map of the following form:

$$\begin{array}{rcl} \mathbb{R}^2 \times [0,\infty) & \to & \mathbb{H}^3 \cong \mathbb{C} \times (0,\infty) \\ (x,y,t) & \mapsto & (a_i(z^{(0)})x + b_i(z^{(0)})y, \exp(t)), \end{array}$$

where $a_i(z^{(0)})$ and $b_i(z^{(0)})$ are as in Lemma 3.6.

We will apply [3] as in the proof of Proposition 3.3. To do this, we shall describe the holonomy representation of $\mathfrak{h}_i(z)$ for $z \in \mathcal{D}$ using the similarity structure on T_i induced by $\mathfrak{h}(z)$. By Lemma 3.8, this structure is $\mathfrak{s}_i(z)$, which is defined as in Proposition 3.3. Hence, a holonomy representation for $\mathfrak{h}_i(z)$ can be recovered from the holonomy representation of $\mathfrak{s}_i(z)$ as in Lemma 3.6, because the hyperbolic holonomy is the conformal extension of the similarity holonomy.

Then, using [3] as in the proof of Proposition 3.3, after composing with a hyperbolic isometry we deduce that the following is a developing map of $\mathfrak{h}_i(z)$ on the cusp C_i:

$$\mathbb{R}^2 \times [0,\infty) \to \mathbb{H}^3 \cong \mathbb{C} \times (0,\infty)$$

$$(x,y,t) \mapsto \begin{cases} (\exp(u_i(z)x + v_i(z)y), \exp(t + \Re(u_i(z)x + v_i(z)y))) & \text{if } u_i(z) \neq 0 \\ (a_i(z)x + b_i(z)y, \exp(t)) & \text{otherwise.} \end{cases}$$

Since the argument of [3] applies only to compact manifolds, we apply it to $\mathbb{R}^2 \times [0,t_n]$ and we consider the limit when $t_n \to \infty$. This proves the first assertion of the proposition.

When $u_i(z) = 0$, it follows from the expression of this developing map that the end is complete, as proved in [1], [22] or [27].

Assume from now to the end of the proof that $u_i(z) \neq 0$. The image of $\mathbb{R}^2 \times \{t\}$ is precisely the set of points that are at a fixed distance from the geodesic γ with endpoints 0 and ∞, which is the geodesic fixed by the holonomy representation. Actually, this distance tends to 0 as t goes to ∞. More precisely, the image of $\mathbb{R}^2 \times [t, \infty)$ is exactly $U_{r(t)}(\gamma) \setminus \gamma$, where U_r denotes the tubular r-neighbourhood, and $r(t) \to 0$ as $t \to \infty$.

Let $n_i, m_i \in \mathbb{Z}$ be such that $p_i \cdot n_i - q_i \cdot m_i = 1$. The quadrilateral $Q \subset \mathbb{R}^2$ with vertices $(0,0)$, (p_i, q_i), $(p_i + m_i, q_i + n_i)$ and (m_i, n_i) is a fundamental domain for the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$. We can also describe Q as:

$$Q = \{(x,y) \in \mathbb{R}^2 : \ 0 \leq n_i x - m_i y \leq 1, \ 0 \leq -q_i x + p_i y \leq 1\}.$$

The orbit of Q under the action of the cyclic group generated by (m_i, n_i), which corresponds to $m_i \lambda_i + n_i \mu_i$ in $\pi_1(T_i)$, is the strip $S = \{(x,y) \in \mathbb{R}^2 : 0 \leq n_i x - m_i y \leq 1\}$.

First we deal with the case where the relation $p_i \cdot u_i(z) + q_i \cdot v_i(z) = 2\pi\sqrt{-1}$ is satisfied. For fixed $t \in [0, \infty)$, the restriction of the developing map to $S \times \{t\}$ glues one side of S to the other one, and its image is precisely $\partial U_{r(t)}(\gamma)$, *i.e.* the set of points at distance $r(t)$ from γ. In other words, the developing map restricted to $\mathbb{R}^2 \times \{t\}$ induces the universal covering of the cylinder $\partial U_{r(t)}(\gamma)$, and the deck transformation group is the cyclic group generated by (p_i, q_i), which corresponds to $p_i \lambda_i + q_i \mu_i$ in $\pi_1(T_i)$. This description implies that C_i is isometric to the quotient of $U_{r(0)}(\gamma) \setminus \gamma$ under the action of the holonomy of $m_i \lambda_i + n_i \mu_i$. This action extends to a discrete and free action on the whole of $U_{r(0)}(\gamma)$, so the completion of C_i is obtained by adding the quotient of γ, and the result is a genuine hyperbolic manifold. Topologically, this manifold is precisely the Dehn filling with meridian $p_i \lambda_i + q_i \mu_i$.

In the general case we have $p_i \cdot u_i(z) + q_i \cdot v_i(z) = 2\pi\sqrt{-1}/r_i$, and we replace $\mathbb{H}^3$ by a singular space denoted by $\mathbb{H}^3_{\alpha_i}$, where $\alpha_i = 2\pi/r_i$. The space $\mathbb{H}^3_{\alpha_i}$ has a singular line $\Sigma \cong \mathbb{R}$, $\mathbb{H}^3_{\alpha_i} \setminus \Sigma$ has a non-complete hyperbolic metric and the singularity on Σ is conical with angle $\alpha_i = 2\pi/r_i$. In cylindrical coordinates the metric on $\mathbb{H}^3_{\alpha_i} \setminus \Sigma$ has the form:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \left(\frac{\alpha_i}{2\pi}\right)^2 \sinh^2(r)\mathrm{d}\vartheta^2 + \cosh^2(r)\mathrm{d}h^2$$

where $r \in (0, +\infty)$ is the distance to Σ, $\vartheta \in [0, 2\pi)$ is the angular parameter and $h \in \mathbb{R}$ if the height.

The developing map $\widetilde{C}_i \to \mathbb{H}^3 \setminus \gamma$ induces a developing map $\widetilde{C}_i \to \mathbb{H}^3_{\alpha_i} \setminus \Sigma$, because the universal coverings of $\mathbb{H}^3 \setminus \gamma$ and of $\mathbb{H}^3_{\alpha_i} \setminus \Sigma$ are isometric. Then the argument in the non-singular case above (where $r_i = 1$) applies to the singular case after replacing the pair $(\mathbb{H}^3, \gamma)$ by $(\mathbb{H}^3_{\alpha_i}, \Sigma)$. The completion is of course in this case a cone manifold with cone angle α_i along the loop added. $\boxed{3.7}$

Proposition 3.7 and the discussion preceding it imply point 4 in Theorem 3.2. We are only left to establish point 2, which we do now.

Proposition 3.9. *If $z \in \mathcal{D}$ and $u_1(z) = \cdots = u_k(z) = 0$ then $z = z^{(0)}$.*

Proof of 3.9. Having already established point 1 in Theorem 3.2, we can rephrase the statement as follows: *if $\mathcal{D}_0$ is the set of solutions z in $\mathcal{U}$ of both $\mathcal{C}_\mathcal{T}$ and $\mathcal{M}_\mathcal{T}$, then $z^{(0)}$ is an isolated point of $\mathcal{D}_0$.* Assume this is not the case. Since $\mathcal{D}_0$ is an analytic space, we can find a non-constant curve in $\mathcal{D}_0$ starting at $z^{(0)}$. Therefore, at least one of the coordinates z_j assumes uncountably many different values on $\mathcal{D}_0$.

Now, by Proposition 3.7, every $z \in \mathcal{D}_0$ defines on M a complete finite-volume hyperbolic structure, which must be isometric to the original structure by Mostow rigidity. It follows that for all $z \in \mathcal{D}_0$ the original manifold M contains a geodesic ideal tetrahedron, possibly flat and with some paired faces, of modulus z_j or $\overline{z_j}$, depending on whether $\Im(z_j)$ is non-negative or non-positive. In particular, under the assumption that $z^{(0)}$ is not isolated, M contains uncountably many pairwise non-isometric (possibly flat) geodesic ideal tetrahedra.

Let us consider now the universal covering $\mathbb{H}^3 \to M$, on which the group of deck transformations acts as a subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$ identified to $\pi_1(M)$. It is very easy to see that each geodesic ideal tetrahedron contained in M is actually the projection of the convex hull of 4 points of $\partial\mathbb{H}^3$ which are fixed points of parabolic elements of $\pi_1(M)$. Since $\pi_1(M)$ is countable and each parabolic element has one fixed point, we see that in M there are at most countably many pairwise non-isometric (possibly flat) geodesic ideal tetrahedra. This gives a contradiction and concludes the proof. $\boxed{3.9}$

4 Hyperbolic filling parameters

The aim of this section is to show that the set of parameters $(c_1, \ldots, c_k)$ arising as in Theorem 3.2(4) covers a neighbourhood of $(\infty, \ldots, \infty)$ in $(\mathbb{Z}^2 \sqcup$

$\{\infty\})^k$. This will imply the conclusion of the proof. We will start with a combinatorial argument due to Neumann and Zagier [20], which shows that the space $\mathcal{D}$ of deformed structures is sufficiently big (*i.e.* it has (complex) dimension exactly k). Later we will modify the approach of [20] to avoid the assumption that $z^{(0)}$ is a smooth point of $\mathcal{D}$.

Note first that the expressions z, $1/(1-z)$ and $1-1/z$ can all be rewritten as $\delta_0 \cdot z^{\delta_1} \cdot (1-z)^{\delta_2}$ for suitable $\delta_0, \delta_1, \delta_2 \in \{\pm 1\}$. Recall that our ideal triangulation $\mathcal{T}$ of M consists of tetrahedra Δ_j, $j = 1, \ldots, n$, and ∂M consists of tori T_i, $i = 1, \ldots, k$.

Lemma 4.1. *$\mathcal{T}$ contains n edges.*

Proof of 4.1. Since $\partial \overline{M}$ is made of tori, $\chi(\overline{M}) = 0$. Hence $\chi(\widehat{M}) = k$, because each torus is collapsed to a point. In $\mathcal{T}$ there are twice as many faces as tetrahedra, so $k = k - (\#\text{edges}) + 2n - n$, and the conclusion readily follows. $\boxed{4.1}$

Let us list the edges in $\mathcal{T}$ as e_m, $m = 1, \ldots, n$. For $m, j \in \{1, \ldots, n\}$ let us define $(\theta_1(m,j), \theta_2(m,j))$ as the sum of all pairs (δ_1, δ_2) over the edges e of Δ_j which get identified to e_m, where the modulus of Δ_j along e is $\pm z_j^{\delta_1}(1 - z_j)^{\delta_2}$. For suitable $\varepsilon_m \in \{\pm 1\}$, $m = 1, \ldots, n$, we can therefore write $\mathcal{C}_{\mathcal{T}}(z)$ as

$$\prod_{j=1}^{n} z_j^{\theta_1(m,j)} \cdot (1 - z_j)^{\theta_2(m,j)} = \varepsilon_m, \qquad m = 1, \ldots, n. \tag{2}$$

Let us denote now by v_i the vertex of $\widehat{M}$ obtained by collapsing $T_i \subset \partial \overline{M}$. For $i \in \{1, \ldots, k\}$ and $m \in \{1, \ldots, n\}$ we define $x(i,m) \in \{0, 1, 2\}$ as the number of ends of e_m which get identified to v_i in $\overline{M}$. We have now two matrices $X \in \mathcal{M}(k \times n, \mathbb{C})$ and $\Theta = (\Theta_1, \Theta_2) \in \mathcal{M}(n \times 2n, \mathbb{C})$. The entries are actually integers, but it will be convenient to view X and Θ as complex matrices. The next two combinatorial results are due to Neumann and Zagier [20] and show that $\mathcal{D}$ is an open portion of a complex algebraic variety of dimension at least k. We note that this result in [1] was deduced from a much harder combinatorial lemma from [20].

Lemma 4.2. $X \cdot \Theta = 0$.

Proof of 4.2. We must check that for all i and j

$$\sum_{m=1}^{n} x(i,m)\cdot\theta_1(m,j) = \sum_{m=1}^{n} x(i,m)\cdot\theta_2(m,j) = 0$$

$$i.e. \qquad \sum_{m=1}^{n} x(i,m)\cdot(\theta_1(m,j),\theta_2(m,j)) = 0.$$

We can rewrite the last sum as

$$\sum_{m=1}^{n} \sum_{\substack{v \text{ endpoint of } e_m \\ v \text{ identified to } v_i}} \sum_{\substack{e \text{ edge of } \Delta_j \\ e \text{ identified to } e_m \\ \operatorname{mod}(\Delta_j|e) = \pm z_j^{\delta_1}(1-z_j)^{\delta_2}}} (\delta_1,\delta_2)$$

$$= \sum_{\substack{v \text{ vertex of } \Delta_j \\ v \text{ identified to } v_i}} \sum_{\substack{e \text{ edge of } \Delta_j \\ e \text{ contains } v \text{ as endpoint} \\ \operatorname{mod}(\Delta_j|e) = \pm z_j^{\delta_1}(1-z_j)^{\delta_2}}} (\delta_1,\delta_2)$$

$$= \sum_{\substack{v \text{ vertex of } \Delta_j \\ v \text{ identified to } v_i}} \Big((1,0)+(0,-1)+(-1,1)\Big) = 0.$$

This concludes the proof. $\boxed{4.2}$

Lemma 4.3. $\operatorname{rank}_{\mathbb{C}}(X) = k$.

Proof of 4.3. Let $a_1,\ldots,a_k \in \mathbb{C}$ be such that $(a_1,\ldots,a_k)\cdot X = 0$, *i.e.*

$$\sum_{i=1}^{k} a_i \cdot x(i,m) = 0, \qquad m = 1,\ldots,n.$$

Using the definition of X, this means that $a_{i_0} + a_{i_1} = 0$ whenever v_{i_0} and v_{i_1} are the ends of some edge in $\widehat{M}$. If we examine a face of some Δ_j having vertices v_{i_0}, v_{i_1} and v_{i_2}, the three edges of the face yield respectively the relations

$$a_{i_0} + a_{i_1} = 0, \qquad a_{i_0} + a_{i_2} = 0, \qquad a_{i_1} + a_{i_2} = 0.$$

Therefore $a_i = 0$ for $i = 1,\ldots,k$, and the conclusion follows. $\boxed{4.3}$

Corollary 4.4. $\operatorname{rank}_{\mathbb{C}}(\Theta) \le n-k$, in particular $k \le n$.

Going back to the system $\mathcal{C}_{\mathcal{T}}$ written as in formula (2), we can now show that it can be replaced by a system of $n-k$ equations only. This fact, even if not explicitly stated in [20], was certainly known to the authors. We reproduce here with minor improvements the proof given in [1]. For the sake of simplicity we rearrange the edges $e_1,\ldots,e_n$ in such a way that the last k rows of Θ are linearly dependent on the first $n-k$.

Proposition 4.5.

$$\mathcal{D}=\left\{z\in\mathcal{U}:\prod_{j=1}^{n} z_j^{\theta_1(m,j)}\cdot(1-z_j)^{\theta_2(m,j)}=\varepsilon_m,\ m=1,\ldots,n-k\right\}.$$

Proof of 4.5. We can choose continuous branches of the logarithm function near $z_j^{(0)}$ and $(1-z_j^{(0)})$, $j=1,\ldots,n$, and assume that the neighbourhood $\mathcal{U}$ of $z^{(0)}$ used to define $\mathcal{D}$ is small enough that $\log(z_j)$ and $\log(1-z_j)$ are defined for $z\in\mathcal{U}$. By the properties of the exponential map there exist constants $r_m\in\mathbb{Z}$, $m=1,\ldots,n$, such that

$$\sum_{j=1}^{n}\Big(\theta_1(m,j)\log(z_j^{(0)})+\theta_2(m,j)\log(1-z_j^{(0)})\Big)=\sqrt{-1}\,\pi(2r_m+(1-\varepsilon_m)/2).$$

By continuity, if $\mathcal{U}$ is small enough, for $z\in\mathcal{U}$ and $m\in\{1,\ldots,n\}$ the next two equations are equivalent:

$$\prod_{j=1}^{n} z_j^{\theta_1(m,j)}\cdot(1-z_j)^{\theta_2(m,j)}=\varepsilon_m, \tag{3}$$

$$\sum_{j=1}^{n}\Big(\theta_1(m,j)\log(z_j)+\theta_2(m,j)\log(1-z_j)\Big)=\sqrt{-1}\,\pi(2r_m+(1-\varepsilon_m)/2). \tag{4}$$

We have to show that the first $n-k$ of these equations imply the last k of them. We will use the logarithm form (4) of the equations. By assumption, for $m>n-k$ there exist $a_m^1,\ldots,a_m^{n-k}\in\mathbb{C}$ such that

$$\theta_t(m,j)=\sum_{l=1}^{n-k}a_m^l\cdot\theta_t(l,j),\qquad t=1,2,\quad j=1,\ldots,n.$$

Therefore if $z \in \mathcal{U}$ solves the first $n-k$ equations we have for $m > n-k$

$$\begin{aligned} & \sum_{j=1}^{n} \Big(\theta_1(m,j)\log(z_j) + \theta_2(m,j)\log(1-z_j)\Big) \\ = & \sum_{j=1}^{n}\sum_{l=1}^{n-k} a_m^l \Big(\theta_1(l,j)\log(z_j) + \theta_2(l,j)\log(1-z_j)\Big) \\ = & \sum_{l=1}^{n-k} a_m^l \sqrt{-1}\,\pi(2r_l + (1-\varepsilon_l)/2). \end{aligned}$$

For $z = z^{(0)}$ the first line equals $\sqrt{-1}\,\pi(2r_m + (1-\varepsilon_m)/2)$ so the last line has the same (constant) value, and the conclusion follows. $\boxed{4.5}$

We note now that by Theorem 3.2(1,3) for $i = 1, \ldots, k$ we can define a function $g_i : \mathcal{D} \to S^2 = \mathbb{R}^2 \sqcup \infty$ as $g_i(z) = \infty$ if $u_i(z) = 0$, and $g_i(z)$ as the only pair (p,q) of real numbers such that $p \cdot u_i(z) + q \cdot v_i(z) = 2\pi\sqrt{-1}$ otherwise. The rest of this section is devoted to establishing the following result, which, together with Theorem 3.2(4), implies Theorem 3.1 and hence Theorem 0.1.

Proposition 4.6. *The image of* $g = (g_1, \ldots, g_k) : \mathcal{D} \to (S^2)^k$ *covers a neighbourhood of* $(\infty, \ldots, \infty)$.

Proof of 4.6. Let us consider the homeomorphism $\varphi_i : S^2 \to S^2$ defined by

$$\varphi_i(p,q) = \frac{2\pi\sqrt{-1}}{p + \tau_i q}$$

(we are viewing the first S^2 as $\mathbb{R}^2 \sqcup \{\infty\}$ and the second one as $\mathbb{C} \sqcup \{\infty\}$, and as usual $1/0 = \infty$, $1/\infty = 0$). We define now $\tilde{u}_i : \mathcal{D} \to \mathbb{C}$ as $\varphi_i \circ g_i$. To conclude it is sufficient to show that the image of $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_k) : \mathcal{D} \to \mathbb{C}^k$ covers a neighbourhood of 0. Recall first the following two essential properties of $\mathcal{D}$ already established:

1. $\mathcal{D}$ is a (germ of) analytic variety, defined in $\mathbb{C}^n$ as the zero set of $n-k$ holomorphic functions.

2. There is a map $u : \mathcal{D} \to \mathbb{C}^k$ which is the restriction of a holomorphic function on an open subset of $\mathbb{C}^n$, such that $u^{-1}(\{0\}) = \{z^{(0)}\}$.

Under these assumptions, the preparation theorem of Weierstrass [18] implies that $u : \mathcal{D} \to \mathbb{C}^k$ is an open map (more precisely, it is a covering branched over a *real* codimension-2 set). We denote now by $\| \, . \, \|$ the usual Euclidean norm on $\mathbb{C}^k$, and claim that

$$\lim_{z \in \mathcal{D}, z \to z^{(0)}} \frac{\|\tilde{u} - u\|}{\|u\|} = 0. \tag{5}$$

Of course it is sufficient to show that for all i

$$\lim_{z \in \mathcal{D}, u_i(z) \neq 0, z \to z^{(0)}} \frac{\tilde{u}_i - u_i}{u_i} = 0.$$

Using the relations

$$p_i \cdot u_i + q_i \cdot v_i = 2\pi\sqrt{-1}\,, \qquad \tilde{u}_i = \frac{2\pi\sqrt{-1}}{p_i + \tau_i q_i}, \qquad \frac{v_i}{u_i} \longrightarrow \tau_i, \qquad \Im(\tau_i) \neq 0$$

we see that

$$\frac{\tilde{u}_i - u_i}{u_i} = \tau_i \cdot \frac{q_i}{p_i + \tau_i q_i} \cdot \left(\frac{v_i}{\tau_i u_i} - 1 \right) \longrightarrow 0$$

because $|q_i/(p_i + \tau_i q_i)|$ is bounded from above by $1/|\Im(\tau_i)|$. Formula (5) is proved.

Let us consider now the function $\|u\| : \mathcal{D} \to \mathbb{R}_+$, denoted by f. Note that $f^{-1}(\{0\}) = \{z^{(0)}\}$ and that f is the restriction to $\mathcal{D}$ of an ambient map whose square is real-analytic. Since $u : \mathcal{D} \to \mathbb{C}^k$ is open and $u^{-1}(\{0\}) = \{z^{(0)}\}$, we can choose a small $R > 0$ and restrict $\mathcal{D}$ so that $u : \mathcal{D} \to B_R(0)$ is proper and surjective. Here $B_R(0)$ is the open R-ball centered at 0 in $\mathbb{C}^k$. In the sequel $S_r(0)$ will denote the R-sphere. For $0 < r < R$ we also set $\mathcal{D}_{\leq r} = f^{-1}([0, r])$, and $\mathcal{D}_{=r} = f^{-1}(\{r\})$.

Using the general theory of analytic spaces [33] we can now choose a good stratification of $\mathcal{D}$ (with respect to singularity), and assume that, away from $z^{(0)}$, f is transversal to all strata. Since $\mathcal{D}$ is defined by complex-analytic functions, its top real $2k$-dimensional strata are naturally oriented, and there are no strata of real dimension $2k - 1$. Now, for $0 < r < R$ we consider the induced stratification of $\mathcal{D}_{=r}$, and orient the top real $(2k-1)$-dimensional strata using f and the previous orientation. ($\mathcal{D}_{\leq r}$ actually has a (stratified) conic structure with basis $\mathcal{D}_{=r}$, vertex $z^{(0)}$ and height function f, but we will not need all this information.) Since in $\mathcal{D}_{=r}$ there are no strata of real dimension $2k-2$, we can view it as a geometric $(2k-1)$-cycle. Similarly, $\mathcal{D}_{\le r}$ can be regarded as a $2k$-dimensional geometric $\mathbb{Z}$-chain with boundary $\mathcal{D}_{=r}$.

Using Sard's lemma we see that there exist arbitrarily small $r>0$ and regular values $w \in S_r(0)$ for the restriction of u to all strata of both $\mathcal{D}$ and $\mathcal{D}_{=r}$. Since u is complex-analytic, each preimage of w has index $+1$ with respect to u. Orientation conventions imply that the same is true with respect to $u|_{\mathcal{D}_{=r}}$. Moreover $u|_{\mathcal{D}_{=r}}$ is surjective onto $S_r(0)$. We deduce that $u|_{\mathcal{D}_{=r}} : \mathcal{D}_{=r} \to S_r(0)$, as a geometric cycle in $S_r(0)$, represents a *strictly positive* (in particular, non-zero) multiple of the canonical generator of $H_{2k-1}(S_r(0)) \cong H_{2k-1}(\mathbb{C}^k \setminus \{0\})$ (we will take all homology groups with integer coefficients).

Using formula (5) we can now assume that r is small enough that $\|\tilde{u}-u\| < \|u\|/2$ on $\mathcal{D}_{\le r}$, in particular $\|\tilde{u}-u\| < r/2$. This implies that $\tilde{u}(\mathcal{D}_{=r}) \subset \mathbb{C}^k \setminus \{0\}$, moreover $\tilde{u} : \mathcal{D}_{=r} \to \mathbb{C}^k \setminus \{0\}$ is homotopic to $u : \mathcal{D}_{=r} \to \mathbb{C}^k \setminus \{0\}$, whence it represents the same non-zero element of $H_{2k-1}(\mathbb{C}^k \setminus \{0\})$.

We claim now that $\tilde{u}(\mathcal{D}_{\le r})$ contains $D_{r/2}(0)$. Assume by contradiction that there exists $w^{(0)} \in D_{r/2}(0) \setminus \tilde{u}(\mathcal{D}_{\le r})$. Note that each half-line in $\mathbb{C}^k$ with origin in $w^{(0)}$ meets $S_r(0)$ exactly once, so we have a natural "radial" projection $p : \mathbb{C}^k \setminus \{w^{(0)}\} \to S_r(0)$. We can now consider the $2k$-dimensional geometric chain $p \circ \tilde{u} : \mathcal{D}_{\le r} \to S_r(0)$, whose boundary $p \circ \tilde{u} : \mathcal{D}_{=r} \to S_r(0)$ is therefore zero in $H_{2k-1}(S_r(0)) \cong H_{2k-1}(\mathbb{C}^k \setminus \{0\})$. Now, for $z \in \mathcal{D}_{=r}$ we have $\|\tilde{u}(z)\| > r/2$. Since $\|w^{(0)}\| \le r/2$, by the definition of p, the segment joining $\tilde{u}(z)$ and $p(\tilde{u}(z))$ does not contain 0. In particular, the geometric $(2k-1)$-chains $p \circ \tilde{u} : \mathcal{D}_{=r} \to \mathbb{C}^k \setminus \{0\}$ and $\tilde{u} : \mathcal{D}_{=r} \to \mathbb{C}^k \setminus \{0\}$ are homotopic to each other in $\mathbb{C}^k \setminus \{0\}$. This is a contradiction, because the former is zero in $H_{2k-1}(\mathbb{C}^k \setminus \{0\})$ and the latter is not. Our claim is established and the proof is complete. $\boxed{4.6}$

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