Dehn filling of the "magic" 3-manifold

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Abstract

We classify all the non-hyperbolic Dehn fillings of the complement of the chain-link with 3 components, deducing the classification of all non-hyperbolic Dehn fillings of infinitely many 1-cusped and 2-cusped hyperbolic manifolds, including many of the smallest ones known.

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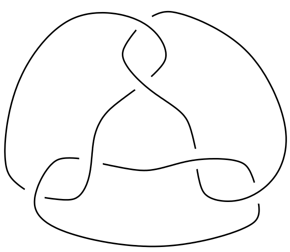
We study in this paper the Dehn fillings of the complement N of the chain-link with 3 components in S^3, shown in Fig. 1. The relevance of N to 3-dimensional topology comes from the fact that filling N one gets most of the hyperbolic manifolds known and most of the interesting non-hyperbolic fillings of cusped hyperbolic manifolds. We also note that N actually is the so-called "magic manifold" of [10, 9], and appears as $M6_1^3$ in [5].

The main results of this paper, stated in Section 1, provide a complete classification (with prime and JSJ decomposition) of all non-hyperbolic Dehn fillings of N, including partial ones. Quite some energy has been devoted in the literature to the understanding of *exceptional* slopes on hyperbolic manifolds, *i.e.* slopes giving non-hyperbolic fillings, but a complete description of the actual filled manifolds is seldom available. For this reason we hope that our results will prove useful.

In Section 2 we concentrate on the hyperbolic Dehn fillings of N, showing that many notable examples arise like this. Then, using the results of Section 1, we



The complement of the chain-link is a hyperbolic manifold. Among hyperbolic manifolds with 3 cusps, it is the one having smallest known volume 5.3334... and smallest complexity 6.

Figure 1: The chain-link with 3 components

analyze the non-hyperbolic fillings of many such examples. In particular, we realize via our N most known maxima for the number of exceptional slopes on 1- and 2-cusped manifolds, carefully describing the corresponding fillings. Similarly, we realize most known maxima for the distance between different types of exceptional slopes. Section 3 contains all the proofs.

We warmly thank Jeff Weeks for very helpful suggestions on how to use SnapPea for the last step of the proof of Theorem 1.3.

1 Non-hyperbolic fillings of the chain-link complement

We state here the main results of the paper, namely Theorems 1.1, 1.2, and 1.3 which classify all non-hyperbolic fillings of the chain-link complement N. We also use these results to collect some informations on chirality and cosmetic surgery on hyperbolic fillings of N.

1.1 Main results

To give our statements, we first make some conventions.

Homology bases We fix the standard (meridian, longitude) homology basis on each component of ∂N. Using it, we identify a slope α with a number in $\mathbb{Q} \cup \{\infty\}$, and we denote by $N(\alpha)$ the Dehn filling of N along α. Any two components of ∂N can be interchanged by an automorphism which preserves the bases, so $N(\alpha)$ depends on α only. On each partial filling $N(\alpha)$ or $N(\alpha, \beta)$ we take the induced homology bases, and we note that the two components of $N(\alpha)$ are still symmetric.

Lens spaces For simplicity we employ a more flexible notation for lens spaces than the usual one. We write $L(2,q)$ for $\mathbb{RP}^3$, $L(1,q)$ for S^3, $L(0,q)$ for $S^2 \times S^1$, and $L(p,q)$ for $L(|p|,q')$ with $q \equiv q' \pmod{p}$ and $0 < q' < |p|$, for any coprime p, q.

Torus bundles We denote by T_X the torus bundle with monodromy $X \in \mathrm{GL}_2(\mathbb{Z})$.

Seifert manifolds If F is a surface, $k \in \mathbb{N}$, $b \in \mathbb{Z}$ and $\{(p_i, q_i)\}_{i=1}^k$ are coprime pairs with $|p_i| \geqslant 2$, we define a Seifert manifold $\big(F, (p_1, q_1), \ldots, (p_k, q_k), b\big)$ as follows. Let Σ be F minus $k+1$ open discs and let W be the orientable S^1-bundle over Σ. Give W any orientation, pick a section σ of W, and choose positive homology bases $\{(\mu_i, \lambda_i)\}_{i=0}^k$ on the components of ∂W arising from the punctures of F, with $\mu_i \subset \partial\sigma$ and a fibre as λ_i. Then $\big(F, (p_1, q_1), \ldots, (p_k, q_k), b\big)$ is the Dehn filling of W along $p_1\mu_1 + q_1\lambda_1, \ldots, p_k\mu_k + q_k\lambda_k, \mu_0 + b\lambda_0$.

We say the parameters are *normalized* if $p_i > q_i > 0$. In our statements we will allow parameters to be non-normalized only when they depend on some variable. Note that b can be dismissed when F has boundary and that our "filling parameters" (p_i, q_i) are not the usual Seifert orbital parameters (a_i, b_i), which are computed as $a_i = |p_i|$ and $b_i \equiv q_1^{-1} \cdot p_i/|p_i| \pmod{|p_i|}$.

We denote by D, A, S, P, and T respectively the disc, the annulus, the Möbius strip, the pair of pants, and the torus.

Graph manifolds If F is an orientable surface with $h \geqslant 1$ boundary components and Σ is F minus k discs, we proceed as above to construct homology bases $\{(\mu_i, \lambda_i)\}_{i=1}^{h+k}$ on $\partial(\Sigma \times S^1)$, with the extra requirement that the μ_i's should be oriented as components of $\partial\Sigma$. For coprime pairs $\{(p_i, q_i)\}_{i=1}^{k}$ with $|p_i| \geqslant 2$ we then get a Seifert manifold $\big(F, (p_1, q_1), \ldots, (p_k, q_k)\big)$ with fixed homology bases on its h boundary components. If M, M' are Seifert manifolds with boundary and orientable base surfaces, and $X \in \mathrm{GL}_2(\mathbb{Z})$, we can then define without ambiguity $M \cup_X M'$ to mean $M \cup_f M'$ with $f : T \to T'$, where T and T' are arbitrary boundary tori of M and M', respectively, and f acts on homology as X with respect to the fixed bases. Similarly we can define $M/_X$ when M has at least 2 components.

We are now ready to give our statements, which correspond to the filling of 1, 2, or all 3 components of ∂N, respectively. Throughout this paper 'hyperbolic' means 'finite-volume complete hyperbolic,' and the symbol N is only used to denote the complement of the 3-components chain-link in S^3, shown in Fig. 1.

Theorem 1.1. *$N(\alpha)$ is hyperbolic unless $\alpha \in \{\infty, -3, -2, -1, 0\}$, and:*

- $N(\infty) = T \times I$;
- $N(-3) = \big(A, (2,1)\big) \bigcup_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} \big(A, (2,1)\big)$;
- $N(-2) = \big(P, (3,1)\big)$;
- $N(-1) = \big(P, (2,1)\big)$;
- $N(0) = \big(D, (2,1), (3,1)\big) \bigcup_{\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)} \big(P \times S^1\big)$.

Theorem 1.2. *$M = N(p/q, r/s)$ is hyperbolic except if one of the following occurs up to permutation:*

- $p/q \in \{\infty, -3, -2, -1, 0\}$;

- $(p/q, r/s) \in \{(1,1), (-4,-1/2), (-3/2,-5/2)\}$.

In all cases M is as shown in Table 1, where n stands for a generic integer and $r/s \neq \infty$ unless $p/q = \infty$.

Theorem 1.3. *$M = N(p/q, r/s, t/u)$ is hyperbolic except if one the following occurs up to permutation:*

- *$p/q = \infty$, in which case $M = L(tr-us, tr'-us')$, where $r', s' \in \mathbb{Z}$ are such that $rs' - sr' = \pm 1$;*
- *$p/q \in \{-3,-2,-1,0\}$, in which case M is as described in Table 2 (with n and m standing for generic integers);*
- *$(p/q, r/s) \in \{(1,1), (-4,-1/2), (-3/2,-5/2)\}$, in which case M is as described in Table 3;*
- *$(p/q, r/s, t/u)$ is one of the 14 triples listed in Table 4, together with a description of the corresponding M.*

The next remark is important to understand the statement of Theorem 1.3.

Remark 1.4. We have compiled Tables 2 to 4 trying to keep them concise but easily usable. In particular:

- we have always excluded the case where one of the filling coefficients is ∞, because the first point of Theorem 1.3 is already sufficient in this case to determine which lens space the result is;
- whenever a certain value $p/q \in \{-3,-2,-1,0\}$ is chosen, the corresponding set of lines in Table 2 contains a complete classification of the resulting M, but one may have to permute r/s and t/u to get the result;
- the tables always provide the decomposition of M into prime connected summands and the JSJ decomposition of these summands, except when a connected sum involves a lens space and the parameters are such that the lens space is S^3;
- the triples listed in Table 4 do not appear in Tables 2 and 3;
- there is some overlapping between the cases covered in Tables 2 and 3. The recognition of M is of course coherent (note for instance that $N(1,1,n) = T_{\left(\begin{smallmatrix} n+1 & 1 \\ -1 & 0 \end{smallmatrix}\right)}$ is also a Seifert manifold for $n \in \{-3,-2,-1,0\}$, see Lemma 3.1);

p/q	r/s	M
∞	any	$D \times S^1$
-3	-1	$\mathbb{RP}^3 \# (D \times S^1)$
	-2	$D \times S^1$
	-3	$\big(S, (2,1)\big)$
	$-1 + \frac{1}{n}, \neq -2$	$\big(D, (2,1), (2n{+}1, 2)\big)$
	$\neq -1, \neq -1 + \frac{1}{n}$	$\big(D, (2,1), (r{+}s, s)\big) \bigcup_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} \big(A, (2,1)\big)$
-2	-2	$L(3,1) \# (D \times S^1)$
	$-2 + \frac{1}{n}$	$D \times S^1$
	$\neq -2, \neq -2 + \frac{1}{n}$	$\big(D, (3,1), (r{+}2s, s)\big)$
-1	-3	$\mathbb{RP}^3 \# (D \times S^1)$
	$-3 + \frac{1}{n}$	$D \times S^1$
	$\neq -3, \neq -3 + \frac{1}{n}$	$\big(D, (2,1), (r{+}3s, s)\big)$
0	$\in \mathbb{Z}$	$\big(D, (2,1), (3,1)\big)$
	$\notin \mathbb{Z}$	$\big(D, (2,1), (3,1)\big) \bigcup_{\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)} \big(A, (s,r)\big)$
1	1	$\big(P \times S^1\big) / _{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)}$
$-\frac{3}{2}$	$-\frac{5}{2}$	$\big(D, (2,1), (3,1)\big) \bigcup_{\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)} \big(A, (2,1)\big)$
-4	$-\frac{1}{2}$	$\big(D, (2,1), (3,1)\big) \bigcup_{\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)} \big(A, (2,1)\big)$

Table 1: Classification of non-hyperbolic fillings of N with 1 boundary component

p/q	r/s	t/u	M
-3	1 or $-\frac{5}{3}$	r/s	$(K,1)$
	-1	any	$\mathbb{RP}^3\#L(t{+}3u,u)$
	-2	any	$L(5t{+}7u,2t{+}3u)$
	$-1+\frac{1}{n}$ $\neq -2$	$-1+\frac{1}{m}$	$L\big((2n{+}1)(2m{+}1){-}4,(2n{+}1)m{-}2\big)$
		$\neq -1,\ \neq -1+\frac{1}{m}$	$\big(S^2,(2,1),(2n{+}1,-2),(t{+}u,u)\big)$
	-3	$\neq -1,\ \neq -1+\frac{1}{m}$	$\big(\mathbb{RP}^2,(2,1),(t{+}u,u)\big)$
	$\neq -1,\ \neq -3$ $\neq -1+\frac{1}{n}$	$\neq -1,\ \neq -3$ $\neq -1+\frac{1}{m}$ $\neq \frac{r}{s}$ if $\frac{r}{s}\in\{1,-\frac{5}{3}\}$	$\big(D,(2,1),(r{+}s,s)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}$ $\big(D,(2,1),(t{+}u,u)\big)$
-2	-2	any	$L(3,1)\#L(t{+}2u,u)$
	$-2+\frac{1}{n}$	any	$L\big(3n(t{+}2u){-}2t{-}u,n(t{+}2u){-}t{-}u\big)$
	$\neq -2$ $\neq -2+\frac{1}{n}$	$\neq -2$ $\neq -2+\frac{1}{m}$	$\big(S^2,(3,2),(r{+}2s,-s),(t{+}2u,-u)\big)$
-1	-3	any	$\mathbb{RP}^3\#L(t{+}3u,u)$
	$-3+\frac{1}{n}$	any	$L\big(2n(t{+}3u){-}t{-}u,n(t{+}3u){+}u\big)$
	$\neq -3$ $\neq -3+\frac{1}{n}$	$\neq -3$ $\neq -3+\frac{1}{m}$	$\big(S^2,(2,1),(r{+}3s,-s),(t{+}3u,-u)\big)$
0	n	$-4-n$	$\mathbb{RP}^3\#L(3,1)$
		$-4-n+\frac{1}{m}$	$L(6m{-}1,2m{-}1)$
		$\neq -4-n$ $\neq -4-n+\frac{1}{m}$	$\big(S^2,(2,-1),(3,1),(t{+}(n{+}4)u,u)\big)$
	$\frac{1}{2}+n$	$-\frac{9}{2}-n$	$\big(\mathbb{RP}^2,(2,1),(3,1),-1\big)$
	$\notin\mathbb{Z}$	$\notin\mathbb{Z}$ $\neq -\frac{r}{s}-4$ if $\lvert s\rvert=2$	$\big(D,(s,r{+}2s),(u,t{+}2u)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\-1&-1\end{smallmatrix}\right)}$ $\big(D,(2,1),(3,1)\big)$

Table 2: Classification of closed non-hyperbolic fillings of N

p/q	r/s	t/u	M
1	1	n	$T_{\left(\begin{smallmatrix} n+1 & 1 \\ -1 & 0 \end{smallmatrix}\right)}$
		$\notin \mathbb{Z}$	$\big(A, (u, t{+}u)\big) / \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$
$-3/2$	$-5/2$	-2	$\mathbb{RP}^3$
		-1	$L(13, 5)$
		0	$\big(\mathbb{RP}^2, (2, 1), (3, 1), -1\big)$
		$-2 + \frac{1}{n}$ $\neq -1$	$\big(S^2, (2, 1), (3, -1), (2n{-}1, 2)\big)$
		$\neq 0$ $\neq -2$ $\neq -2 + \frac{1}{n}$	$\big(D, (2, 1), (3, 1)\big) \bigcup_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)}$ $\big(D, (2, 1), (t{+}2u, u)\big)$
-4	$-1/2$	-1	$L(11, 3)$
		$-\frac{1}{2}$	$\big(\mathbb{RP}^2, (2, 1), (3, 1), -1\big)$
		0	$L(13, 5)$
		n $\neq -1$ $\neq 0$	$\big(S^2, (2, 1), (3, -1), (2n{+}1, 2)\big)$
		$\notin \mathbb{Z} \cup \{-\frac{1}{2}\}$	$\big(D, (2, 1), (3, 1)\big) \bigcup_{\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)}$ $\big(D, (2, 1), (u, t)\big)$

Table 3: Classification of closed non-hyperbolic fillings of N (continued)

p/q	r/s	t/u	M
-5	-5	$-1/2$	$\big(A,(2,1)\big)/_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}$
-4	-4	$-2/3$	$\big(A,(2,1)\big)/_{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)}$
-4	$-3/2$	$-3/2$	$T_{\left(\begin{smallmatrix}-3&1\\-1&0\end{smallmatrix}\right)}$
-4	$-1/3$	1	$\big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\-1&-1\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$
$-8/3$	$-3/2$	$-3/2$	$\big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\-1&-1\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$
$-5/2$	$-5/2$	$-4/3$	$\big(A,(2,1)\big)/_{\left(\begin{smallmatrix}2&1\\1&0\end{smallmatrix}\right)}$
$-5/2$	$-5/3$	$-5/3$	$\big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}-1&1\\0&-1\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$
$-7/3$	$-7/3$	$-3/2$	$\big(A,(2,1)\big)/_{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)}$
1	2	2	$\big(S^2,(2,1),(3,1),(7,1),-1\big)$
1	2	3	$\big(S^2,(2,1),(4,1),(5,1),-1\big)$
1	2	4	$\big(S^2,(3,1),(3,1),(4,1),-1\big)$
1	2	5	$\big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$
1	3	3	$\big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}1&2\\0&-1\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$
2	2	2	$\big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}2&3\\-1&-2\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$

Table 4: Classification of closed non-hyperbolic fillings of N (continued)

- in Table 3 the elements of each triple are arranged in increasing order, and the triples themselves are arranged in a lexicographically increasing order.

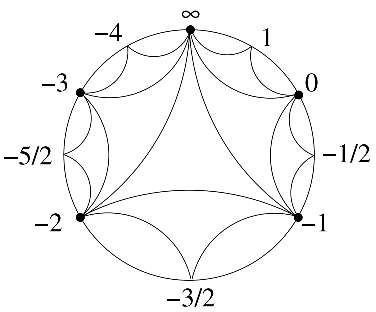
The proof of Theorems 1.1 to 1.3 goes as follows: we first prove Theorem 1.1 using spines. Actually, we prove a more refined version of it, in which we determine the image in $N(\alpha)$ of the bases of the non-filled boundary tori. Using spines, we also recognize $N(1,1)$, $N(2,2,2)$, and $N(1,3,3)$. We then use Proposition 1.5 below (discussed in Section 3) to recognize all the other fillings listed in Tables 1 to 4. Next, we use the Gromov-Thurston 2π theorem to show that if M is a filling of N, but not one of those already recognized to be non-hyperbolic, then M admits a complete metric with negative sectional curvature. An application of the orbifold theorem [3] then implies that M is actually hyperbolic. This is because our chain-link complement N admits an orientation-preserving order-2 involution, given by a rotation of angle π around a horizontal circle in Fig. 1. Its effect on the homology of each cusp is given by multiplication by -1, therefore the involution extends to every Dehn filling of N. The orbifold theorem then guarantees that each filling of N satisfies the geometrization conjecture.

Proposition 1.5. *$N(-3/2) = N(-4)$ and the following holds for all $\alpha,\beta \in \mathbb{Q}\cup\{\infty\}$:*

$$\begin{aligned}
N\left(-\frac{3}{2},\alpha,\beta\right) &= N\left(-4,-\frac{\alpha+1}{\alpha+2},-\beta-3\right) &\quad (1)\\
N\left(-\frac{5}{2},\alpha,\beta\right) &= N\left(-\frac{5}{2},-\frac{\alpha+3}{\alpha+2},-\frac{2\beta+3}{\beta+1}\right) &\quad (2)\\
N\left(-\frac{3}{2},\alpha,\beta\right) &= N\left(-\frac{3}{2},-\frac{2\alpha+5}{\alpha+2},-\frac{2\beta+5}{\beta+2}\right) &\quad (3)\\
N\left(-\frac{1}{2},\alpha,\beta\right) &= N\left(-\frac{1}{2},-\alpha-4,-\beta-4\right) &\quad (4)\\
N(1,2,\alpha) &= N(1,2,-\alpha+2) &\quad (5)\\
N(1,-4,\alpha) &= N\left(1,-4,\frac{1}{\alpha}\right) &\quad (6)
\end{aligned}$$

Relations (2) to (6) are induced by automorphisms of $N(-5/2)$, $N(-3/2)$, $N(-1/2)$, $N(1,2)$, and $N(1,-4)$, respectively. Moreover (2) has order 3, while (3)-(6) have order 2.

We note that N and the manifolds involved in Proposition 1.5 are well-known hyperbolic ones. Using SnapPea one sees that N is the manifold called the "magic" one in [10] and listed as $M6^3_1$ in [5]. The notable partial fillings of N are as described in Fig. 2 and Section 2. (For the precise meaning of Fig. 2, see after the statement of Corollary 2.2; solid dots mark exceptional slopes on N).



The 2-cusped manifolds of Proposition 1.5 are link complements in the 3-sphere: $N(-4) = N(-3/2)$ of the Whitehead sister link, $N(-5/2)$ of the Berge link, $N(-1/2)$ of the 3/10-bridge link, $N(1)$ of the Whitehead link. In addition, $N(1,2)$ is the figure-8 knot complement and $N(1,-4)$ is the figure-8 knot sister.

Figure 2: Important fillings of the chain-link complement

1.2 Some consequences

Non-hyperbolic closed fillings The following result is easily proved as a consequence of Theorem 1.3 and Lemma 3.1:

Corollary 1.6. *The set of all non-hyperbolic closed manifolds obtained by Dehn surgery on the chain-link consists of:*

- S^3, $S^2 \times S^1$, $L(p,q)$,
- $\mathbb{RP}^3 \# L(p,q)$, $L(3,1) \# L(p,q)$, $\mathbb{RP}^3 \# (S^2 \times S^1)$, $L(3,1) \# (S^2 \times S^1)$,
- $\big(S^2, (2,1), (p,q), (r,s), b\big)$, $\big(S^2, (3,1), (p,q), (r,s), b\big)$, $\big(\mathbb{RP}^2, (2,1), (p,q), b\big)$,
- $\big(D, (2,1), (p,q)\big) \bigcup_X \big(D, (2,1), (r,s)\big)$,
- $\big(D, (2,1), (3,1)\big) \bigcup_{\left(\begin{smallmatrix} 1 & 1 \\ n & n\pm 1 \end{smallmatrix}\right)} \big(D, (p,q), (r,s)\big)$,
- $T_{\left(\begin{smallmatrix} n & 1 \\ -1 & 0 \end{smallmatrix}\right)}$,
- $\big(A, (p,q)\big)/_Y$,
- $\big(D, (2,1), (2,1)\big) \bigcup_{\left(\begin{smallmatrix} 1 & 2 \\ 0 & -1 \end{smallmatrix}\right)} \big(D, (2,1), (3,1)\big)$,
- $\big(D, (2,1), (2,1)\big) \bigcup_{\left(\begin{smallmatrix} 2 & 3 \\ -1 & -2 \end{smallmatrix}\right)} \big(D, (2,1), (3,1)\big)$,

where (p,q) and (r,s) are any coprime pairs with $p > q > 0$ and $r > s > 0$, b, n are any integers, and $X, Y \in \mathrm{GL}_2(\mathbb{Z})$ are any matrices whose top-right entry equals ± 1, with $\det(Y) = -1$.

Concerning this corollary, note that the top-right entry of a matrix X or Y as in the statement is the geometric intersection number of the fibres of the Seifert fibrations that are glued together. The only graph manifolds of the list for which this intersection number is not ± 1 are $N(2,2,2)$ and $N(1,3,3)$. We also note that not all the manifolds listed can be obtained as fillings of a *hyperbolic* 1-cusped $N(\alpha,\beta)$. For instance, the only reducible manifolds thus obtained are $S^2 \times S^1$ (with $\alpha = 1/\beta$ and $\alpha, \beta \in \mathbb{Q} \setminus \{-3,-2,-1,1\}$) and $\mathbb{RP}^3 \# L(3,1)$ (with $\alpha = n$, $\beta = -4-n$ and $n \in \mathbb{Z} \setminus \{-4,-3,-2,-1,0\}$).

Chirality The following result is proved in Subsection 2.1.

Theorem 1.7. *Let M be a hyperbolic $N(\alpha)$ or $N(\alpha,\beta)$ and let f be an automorphism of M which leaves a component T of ∂M invariant. Then f acts on $H_1(T)$ as the multiplication by ± 1, except when M and f are as described in Proposition 1.5.*

Remark 1.8. If $M = N(\alpha,\beta)$ then the assumption $f(T) = T$ is empty. If $M = N(\alpha)$ there exists $\tau : M \to M$ switching the components of ∂M, so either f or $\tau \circ f$ satisfies the assumption.

Corollary 1.9. *The only amphichiral hyperbolic manifolds with boundary obtained as fillings of N are $N(-1/2)$, $N(1,2)$, and $N(1,-4)$.*

Proof. Let M be amphichiral and let $\tau : M \to M$ be orientation-reversing. Since the components of ∂M are symmetric, we can assume τ leaves them invariant. If $T \subset \partial M$ then τ acts on $H_1(T)$ as a matrix with determinant -1, in particular not as multiplication by ± 1. Then M and τ appear in Proposition 1.5. And the only orientation-reversing automorphisms in this proposition are those giving (4), (5), and (6). $\square$

Cosmetic fillings We recall some definitions from [4]. Let M be an orientable hyperbolic manifold and T be a boundary torus of M. Two distinct slopes α, β on T are *equivalent* if there is an automorphism of M sending α to β. They are *cosmetic* if $M(\alpha) = M(\beta)$. They are *truly cosmetic* if there is a homeomorphism from $M(\alpha)$ to $M(\beta)$ preserving an orientation on M, and *reflectively cosmetic* if there is one reversing it (both possibilities can hold).

The only known 1-cusped M with a pair of non-equivalent cosmetic slopes was found in [4] and it is $M = N(-5/2, -2/5)$ which admits the cosmetic pair of slopes $-2, -1$ giving $L(49,18)$, as one sees from Table 2 or Table 12 below. The two slopes are not truly cosmetic. In fact, it is conjectured [4, 12] that non-equivalent slopes on a 1-cusped M are never truly cosmetic.

M	α	β	$M(\alpha)$	truly	reflectively
N	-4	$-3/2$	Whitehead sister	no	yes
$N(-3+1/n)$ $n\in\mathbb{Z}\setminus\{0,\pm1,2\}$	∞	-1	$D\times S^1$	yes	yes
$N(-2+1/n)$ $n\in\mathbb{Z}\setminus\{0,\pm1,\pm2\}$	∞	-2	$D\times S^1$	yes	yes
$N(-12/5)$	-2	-1	$\big(D,(2,1),(3,1)\big)$	yes	no
$N(-6)$	-1	0	$\big(D,(2,1),(3,1)\big)$	no	yes
$N(-4/3)$	-3	-1	$\big(D,(2,1),(5,2)\big)$	no	yes

Table 5: Cosmetic slopes on N and $N(\gamma)$

Cosmetic phenomena seem to be more frequent on manifolds with more cusps. All the 2-cusped M's with distinct slopes α,β on a torus $T\subset\partial M$ such that $M(\alpha)=M(\beta)$ is a solid torus are described in [1, 8]. Infinitely many (but not all) such pairs of slopes are non-equivalent. This also follows from the next result, proved in Subsection 2.1.

Theorem 1.10. *Let M be N or a hyperbolic $N(\gamma)$, and let α,β be a cosmetic pair of non-equivalent slopes on M. Then M,α,β are as in Table 5.*

In particular, if M is the Whitehead link sister complement $N(-4)$, the solid torus, $\big(D,(2,1),(3,1)\big)$, or $\big(D,(2,1),(5,2)\big)$, there are two hyperbolic knots K,K' in M with homeomorphic complements, and no automorphism of M sending K to K'. Using SnapPea, one sees that the two knots in the Whitehead link sister are the two shortest geodesics, having length $0.96242\ldots$ and $1.06128\ldots$.

2 Non-hyperbolic fillings of 1- and 2-cusped manifolds

Theorems 1.1, 1.2, and 1.3 can be used to list all the non-hyperbolic Dehn fillings of infinitely many hyperbolic manifolds with 1 or 2 cusps. We study in this section the most notable of these manifolds. Recall that N always denotes the chain-link complement.

2.1 Number of exceptional slopes

If M is a 1-cusped hyperbolic manifold, following [12], we denote by $E(M)$ the set of exceptional slopes on ∂M (*i.e* the slopes giving non-hyperbolic filling), and we set

$e(M) = |E(M)|$, which is finite by [18]. If M has more than one cusp, we pick one T and define $E_T(M)$ as the set of exceptional slopes on T. Then we define $e(M)$ as the maximum of $|E_T(M)|$ over all cusps T. Theorem 1.1 readily implies the following:

Corollary 2.1. $E_T(N) = \{\infty, -3, -2, -1, 0\}$ *for all cusps* T*, whence* $e(N) = 5$.

We do not know if there is any other hyperbolic M with at least 3 cusps and $e(M) \geqslant 5$.

Exceptional slopes on 2-cusped manifolds Since no filling of a graph manifold is hyperbolic, Theorem 1.1 gives $e(M) \geqslant 5$ for all hyperbolic partial fillings M of N. We carefully describe now the M's such that $e(M) > 5$, starting from the 2-cusped ones. Theorems 1.1 and 1.2 and Proposition 1.5 imply the following:

Corollary 2.2. *Assume* $\alpha \notin \{\infty, -3, -2, -1, 0\}$*, so that* $N(\alpha)$ *is hyperbolic. Then* $e(N(\alpha)) = 5$ *except for* $\alpha = -4, -5/2, -3/2, -1/2, 1$*, in which case* $e(N(\alpha)) = 6$*. There are* 4 *distinct* $N(\alpha)$*'s corresponding to these* 5 *slopes, because* $N(-4) = N(-3/2)$.

Many informations on the 4 manifolds described in the previous proposition and their non-hyperbolic fillings are contained in Table 6. All 4 manifolds are actually well-known ones, being the Whitehead link exterior, the Whitehead sister (or $(-2, 3, 8)$ pretzel) link exterior, the Berge manifold and the exterior of the 2-bridge link with parameter $3/10$ (Remarks 2.3 and 2.4 below show how to recognize each of them by looking at its fillings). We provide for each manifold its label Mi_j^k in the Callahan-Hildebrand-Weeks census [5] (recall that Mi_j^k is the j-th manifold among those having k cusps and an ideal triangulation with i tetrahedra, ordered by increasing volume, and k is omitted if it is 1). We also arrange the exceptional slopes on $\partial\mathbb{H}^2 = \mathbb{Q} \cup \{\infty\}$ using the disc model of $\mathbb{H}^2$, and we connect by lines slopes having distance 1, thus showing a partial picture of the so-called Farey tessellation. Finally, we completely describe the non-hyperbolic fillings.

An important *caveat* concerning Table 6 is that, to express slopes on $\partial N(\alpha)$, we are still using the homology bases induced from those of N. In particular, when $N(\alpha)$ is a link complement, *we are not using the natural (meridian, longitude) bases.* In some cases switching from our basis to the natural one is easy (as an example, for the Whitehead link complement, using the natural basis each coefficient β in Table 6 should be replaced by $\beta - 1$), but in other cases it is harder. For $M4_2^2$ the list of exceptional slopes refers to its realization as $N(-3/2)$, not as $N(-4)$.

Remark 2.3. The Berge manifold $N(-5/2)$ is the unique manifold having 3 fillings giving the solid torus [1]. As shown in Table 6, it also has 3 annular and toroidal fillings. In [7] we have generalized this example, building an infinite class of hyperbolic manifolds M with non-empty geodesic boundary and $e(M) = 6$, with 3 fillings giving a handlebody and 3 other ones giving an annular manifold.

Remark 2.4. The manifold $N(-3/2)$ is determined as the only one with two annular fillings at distance 5, while $N(1)$ and $N(-1/2)$ are the only two manifolds with two annular fillings at distance 4, see [9, 10]. They can be further distinguished because $N(1)$ has one filling containing a non-separating torus, while $N(-3/2)$ has not.

Remark 2.5. The 4 manifolds of Table 6 are the 2-cusped ones with least known volume, and precisely the 4 ones having smallest complexity 4 (see [5]). The first 2 have canonical decomposition given by one regular ideal octahedron, the other 2 by 4 regular ideal tetrahedra. Repetitions in Table 6 are the effect of the relations listed in Proposition 1.5.

Exceptional slopes on 1-cusped manifolds We now concentrate on the hyperbolic $N(\alpha, \beta)$'s and describe those having more than 5 exceptional slopes. As above, we first provide a summarizing statement and then give many extra informations.

Corollary 2.6. *Assume α, β do not appear (up to permutation) in Table 1, so $M = N(\alpha, \beta)$ is hyperbolic. Then:*

(A) *$e(M) \geqslant 7$ for the 11 manifolds described in Tables 7 to 9 (where $e(M)$ attains once the value 10, twice the value 8, and 8 times the value 7);*

(B) *$e(M) = 6$ if $\alpha \in \{-4, -5/2, -3/2, -1/2, 1\}$ and β is arbitrary, provided (α, β) is not in Tables 7 to 9; the corresponding fillings are as in Tables 10 to 13; in all cases $E(N(\alpha, \beta)) = E(N(\alpha))$;*

(C) *$e(M) = 6$ for the 7 manifolds described in Table 14;*

(D) *$e(M) = 5$ in all other cases.*

No M is of both types (B) and (C); if it is of type (B) then it can be realized as $N(\alpha, \beta)$ for a unique $\alpha \in \{-5/2, -3/2, -1/2, 1\}$.

Proof. The only non-trivial fact to be proved is the last sentence. Suppose $e(M) = 6$ and $M = N(\alpha, \beta)$ is of type (B) with $\alpha \in \{-5/2, -3/2, -1/2, 1\}$. If M is also of type (C), its non-hyperbolic fillings are listed in Table 14. It is not difficult to see from Tables 10 to 13 that no M of type (B) can have these fillings, so M is not of type (C). Assume $M = N(\alpha', \beta')$ for some $\alpha' \in \{-5/2, -3/2, -1/2, 1\}$ distinct from α.

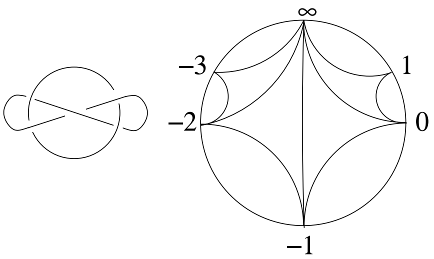
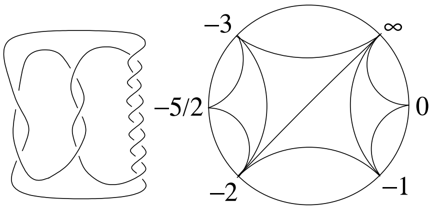
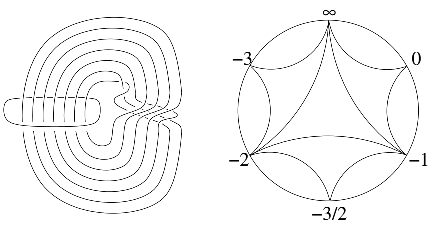
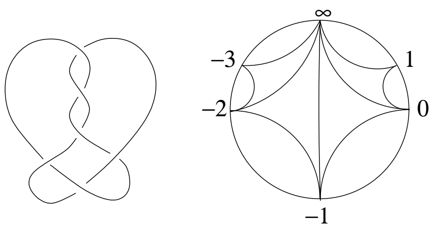
 The Whitehead link exterior $M4_1^2 = N(1)$	$\infty: \quad D \times S^1$ $-3: \quad \big(D,(2,1),(2,1)\big) \bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \big(A,(2,1)\big)$ $-2: \quad \big(D,(3,1),(3,1)\big)$ $-1: \quad \big(D,(2,1),(4,1)\big)$ $0: \quad \big(D,(2,1),(3,1)\big)$ $1: \quad \big(P \times S^1\big)/_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}$
 The Whitehead sister link exterior $M4_2^2 = N(-3/2) = N(-4)$	$\infty, -2: \quad D \times S^1$ $-3, -1: \quad \big(D,(2,1),(3,1)\big)$ $-5/2, 0: \quad \big(D,(2,1),(3,1)\big) \bigcup_{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)}$ $\big(A,(2,1)\big)$
 The Berge manifold $M4_3^2 = N(-5/2)$	$\infty, -2, -1: \quad D \times S^1$ $-3, -3/2, 0: \quad \big(D,(2,1),(3,1)\big) \bigcup_{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)}$ $\big(A,(2,1)\big)$
 The 3/10-bridge link exterior $M4_4^2 = N(-1/2)$	$\infty: \quad D \times S^1$ $-4, 0: \quad \big(D,(2,1),(3,1)\big) \bigcup_{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)} \big(A,(2,1)\big)$ $-3, -1: \quad \big(D,(2,1),(5,2)\big)$ $-2: \quad \big(D,(3,1),(3,2)\big)$

Table 6: Non-hyperbolic fillings of the 2-cusped $N(\alpha)$'s with $e(N(\alpha)) > 5$

Since $E(N(\alpha,\beta))$ and $E(N(\alpha',\beta'))$ are the same, we must have $\{\alpha,\alpha'\}=\{1,-1/2\}$. But $N(1,\beta,1)$ contains a non-separating torus, whereas $N(-1/2,\beta',\gamma)$ does not, whatever γ. □

We warn the reader that, as above, the filling coefficients of each $N(\alpha,\beta)$ in Tables 7 to 14 are expressed *using the homology bases induced from* N. When the same manifold admits several presentations as $N(\alpha,\beta)$, the coefficients refer to the first presentation listed. In Tables 7 to 9, we also provide for each manifold a surgery presentation along on a link in S^3, taken from [12]. Surgery coefficients on link diagrams always refer to the meridian-longitude homology bases.

Our next remark concerning Corollary 2.6 is that the 11 manifolds of point (A) are precisely the 11 known M's with $e(M)\geqslant 7$, and they are conjectured [12] to be the only ones. Moreover:

- precisely 2 of the 11 manifolds are knot complements, namely $N2_1=N(1,2)$, the complement of the figure-8 knot, and $M3_7$ (obtained in 6 different ways as a filling of N), the complement of the $(-2,3,7)$ pretzel knot;
- all the 11 manifolds except $M3_7$ and $M4_{10}=N(2,2)$ are fillings of the Whitehead link; $M3_7$ is a filling of any of the 4 manifolds in Table 6 except the Whitehead link; $M4_{10}$ is not a filling of any of these manifolds.

Turning to the M's with $e(M)=6$ in Corollary 2.6, we note that point (B) of course gives infinitely many examples, thoroughly described in Tables 10 to 13 according to which of the 4 manifolds in Table 6 they are a filling of. The captions of Tables 11 to 13 should be interpreted as shortened versions of the caption of Table 10.

Remark 2.7. There are 67 hyperbolic manifolds with complexity at most 4, see [5]. Using SnapPea one sees that 60 of them are fillings of N, while the other 7, namely $M4_i$ for $i\in\{33,34,40,41,45,51,52\}$, do not seem to be. Since $M4_i$ is amphichiral for $i\in\{33,34,51,52\}$, Corollary 1.9 shows it is certainly not a filling of N in these cases, but we do not know for $i\in\{40,41,45\}$.

We can now prove Theorems 1.7 and 1.10.

Proof of 1.7. We begin by noting that f acts on the slopes on T, and it leaves $E_T(M)$ invariant. Since $E_T(M)$ contains more than 2 slopes, the action on $E_T(M)$ is trivial if and only if the action on $H_1(T)$ is the multiplication by ±1. We suppose now that the action of f on $E_T(M)$ is not trivial and, examining the various possibilities for $E_T(M)$ and the number of cusps of M, we deduce either a contradiction or that the action of f is as described in Proposition 1.5.

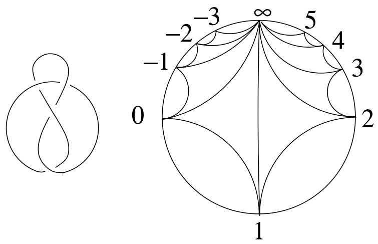
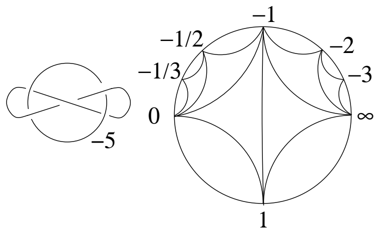
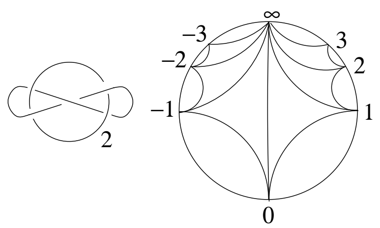
 The figure-8 knot $M2_1 = N(1,2)$	$\infty :\ S^3$ $-3,5 :\ \big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$ $-2,4 :\ \big(S^2,(3,1),(3,1),(4,1),-1\big)$ $-1,3 :\ \big(S^2,(2,1),(4,1),(5,1),-1\big)$ $0,2 :\ \big(S^2,(2,1),(3,1),(7,1),-1\big)$ $1 :\ T_{\left(\begin{smallmatrix}3&1\\-1&0\end{smallmatrix}\right)}$
 The figure-8 knot sister $M2_2 = N(1,-4) =$ $N(-4,-3/2) =$ $N(-3/2,-3/2)$	$-1 : L(10,3)$ $-1/2,-2 : \big(S^2,(2,1),(3,2),(3,2),-1\big)$ $-1/3,-3 : \big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\-1&-1\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$ $0,\infty : L(5,1)$ $1 : T_{\left(\begin{smallmatrix}-3&1\\-1&0\end{smallmatrix}\right)}$
 $M3_3 = N(1,3)$	$\infty : \mathbb{P}^3$ $-3 : \big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(2,1),(4,1)\big)$ $-2 : \big(S^2,(3,1),(3,1),(5,1),-1\big)$ $-1 : \big(S^2,(2,1),(4,1),(6,1),-1\big)$ $0 : \big(S^2,(2,1),(3,1),(8,1),-1\big)$ $1 : T_{\left(\begin{smallmatrix}4&1\\-1&0\end{smallmatrix}\right)}$ $2 : \big(S^2,(2,1),(4,1),(5,1),-1\big)$ $3 : \big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}1&2\\0&-1\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$

Table 7: Non-hyperbolic fillings of 1-cusped $M = N(\alpha,\beta)$'s with $e(M) \geqslant 8$

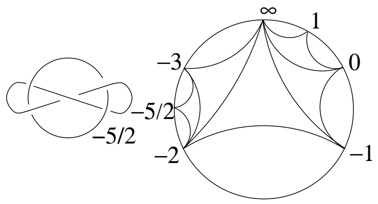
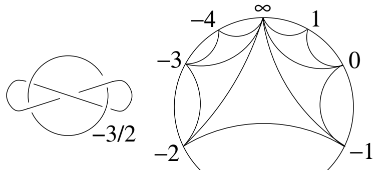
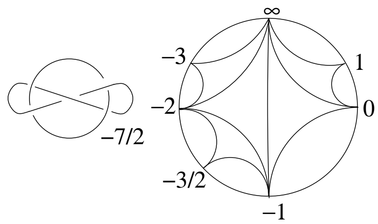
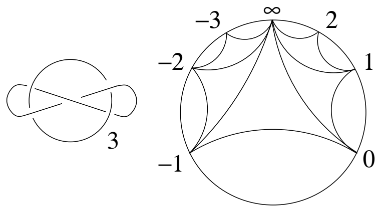
 $M3_1 = N(1,-3/2) =$ $N(-4,-4) = N(-4,-2/3)$ $= N(-3/2,-7/3)$	$\infty : L(5,2)$ $-3 : \left(S^2,(2,1),(2,1),(3,2)\right)$ $-2 : L(15,4)$ $-1 : \left(S^2,(2,1),(3,2),(4,1),-1\right)$ $0 : \left(S^2,(2,1),(3,1),(7,2),-1\right)$ $1 : \left(A,(2,1)\right)/\begin{pmatrix}1&1\\1&0\end{pmatrix}$ $-5/2 : \left(D,(2,1),(3,1)\right)\bigcup_{\begin{pmatrix}1&1\\0&-1\end{pmatrix}}\left(D,(2,1),(3,1)\right)$
 $M3_2 = N(1,-1/2) =$ $N(-5,-1/2)$	$\infty : L(3,1)$ $-4 : \left(S^2,(2,1),(3,2),(3,2),-1\right)$ $-3 : \left(S^2,(2,1),(2,1),(5,3),-1\right)$ $-2 : \left(S^2,(3,1),(3,1),(3,2),-1\right)$ $-1 : \left(S^2,(2,1),(4,1),(5,2),-1\right)$ $0 : \left(S^2,(2,1),(3,1),(9,2),-1\right)$ $1 : \left(A,(2,1)\right)/\begin{pmatrix}0&1\\1&0\end{pmatrix}$
 $M3_8 = N(1,-5/2) =$ $N(-5/2,-4/3) =$ $N(-5/2,-5/2)$	$\infty : L(7,2)$ $-3 : \left(D,(2,1),(2,1)\right)\bigcup_{\begin{pmatrix}0&1\\1&-1\end{pmatrix}}\left(D,(2,1),(3,1)\right)$ $-2 : L(21,8)$ $-3/2 : \left(D,(2,1),(3,1)\right)\bigcup_{\begin{pmatrix}1&1\\0&-1\end{pmatrix}}\left(D,(2,1),(3,1)\right)$ $-1 : L(14,3)$ $0 : \left(S^2,(2,1),(3,1),(5,2),-1\right)$ $1 : \left(A,(2,1)\right)/\begin{pmatrix}2&1\\1&0\end{pmatrix}$
 $M4_2 = N(1,4)$	$\infty : L(3,1)$ $-3 : \left(D,(2,1),(2,1)\right)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\left(D,(2,1),(5,1)\right)$ $-2 : \left(S^2,(3,1),(3,1),(6,1),-1\right)$ $-1 : \left(S^2,(2,1),(4,1),(7,1),-1\right)$ $0 : \left(S^2,(2,1),(3,1),(9,1),-1\right)$ $1 : T_{\begin{pmatrix}5&1\\-1&0\end{pmatrix}}$ $2 : \left(S^2,(3,1),(3,1),(4,1),-1\right)$

Table 8: Non-hyperbolic fillings of 1-cusped $M = N(\alpha,\beta)$'s with $e(M) = 7$

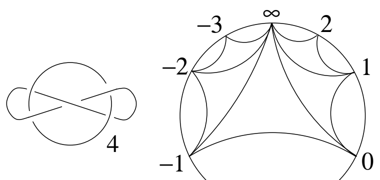
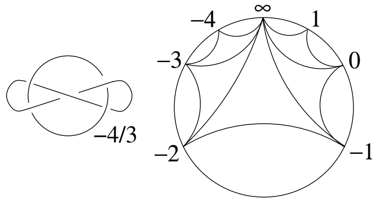
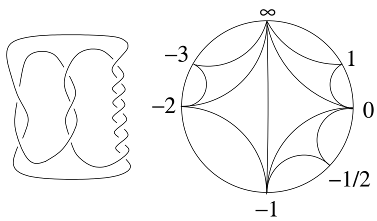
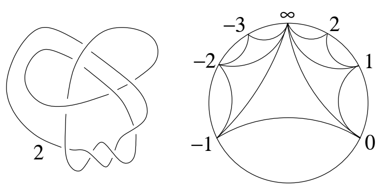
 $M5_1 = N(1,5)$	$\infty : L(4,1)$ $-3 : \big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(2,1),(6,1)\big)$ $-2 : \big(S^2,(3,1),(3,1),(7,1),-1\big)$ $-1 : \big(S^2,(2,1),(4,1),(8,1),-1\big)$ $0 : \big(S^2,(2,1),(3,1),(10,1),-1\big)$ $1 : T_{\left(\begin{smallmatrix}6&1\\-1&0\end{smallmatrix}\right)}$ $2 : \big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$
 $M4_{12} = N(1,-1/3)$	$\infty : L(4,1)$ $-4 : \big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\-1&-1\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$ $-3 : \big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)}\big(D,(2,1),(2,1)\big)$ $-2 : \big(S^2,(3,1),(3,1),(5,3),-1\big)$ $-1 : \big(S^2,(2,1),(4,1),(8,3),-1\big)$ $0 : \big(S^2,(2,1),(3,1),(14,3),-1\big)$ $1 : \big(A,(3,2)\big)/_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}$
 $(-2,3,7)$ pretzel knot $M3_7{=}N(-4,-1/3){=}N(-3/2,-1/2)$ $=N(-3/2,-8/3){=}N(-4,-5/2)$ $=N(-1/2,-5/2){=}N(-5/2,-5/3)$	$\infty : S^3$ $-3 : \big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}-1&1\\0&-1\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$ $-2 : \big(S^2,(2,1),(3,2),(5,2),-1\big)$ $-1 : L(18,5)$ $-1/2 : \big(D,(2,1),(3,1)\big)\bigcup_{\left(\begin{smallmatrix}1&1\\-2&-1\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$ $0 : L(19,7)$ $1 : \big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\-1&-1\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$
 $M4_{10} = N(2,2)$	$\infty : L(3,1)$ $-3 : \big(D,(2,1),(3,1)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$ $-2 : \big(S^2,(3,1),(4,1),(4,1),-1\big)$ $-1 : \big(S^2,(2,1),(5,1),(5,1),-1\big)$ $0 : \big(S^2,(2,1),(3,1),(8,1),-1\big)$ $1 : \big(S^2,(2,1),(3,1),(7,1),-1\big)$ $2 : \big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}2&3\\-1&-2\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$

Table 9: Non-hyperbolic fillings of 1-cusped $M = N(\alpha,\beta)$'s with $e(M) = 7$ (continued)

p/q	r/s	M
∞	any	$L(r{-}s, s)$
-3	$-1+\frac{1}{n}$	$\big(S^2, (2,1), (2,1), (2n{+}1, -2)\big)$
	$\neq -1+\frac{1}{n}$	$\big(D, (2,1), (2,1)\big) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \big(D, (2,1), (r{+}s, s)\big)$
-2	$-2+\frac{1}{n}$	$L(9n{-}3, 3n{-}2)$
	$\neq -2+\frac{1}{n}$	$\big(S^2, (3,-2), (3,1), (r{+}2s, s)\big)$
-1	$-3+\frac{1}{n}$	$L(8n{-}2, 4n{+}1)$
	$\neq -3+\frac{1}{n}$	$\big(S^2, (2,-1), (4,1), (r{+}3s, s)\big)$
0	-5	$\mathbb{RP}^3 \# L(3,1)$
	$-5+\frac{1}{n}$	$L(6n{-}1, 2n{-}1)$
	$\neq -5, -5+\frac{1}{n}$	$\big(S^2, (2,-1), (3,1), (r{+}5s, s)\big)$
1	n	$T_{\begin{pmatrix} n+1 & 1 \\ -1 & 0 \end{pmatrix}}$
	$\notin \mathbb{Z}$	$\big(A, (s, r{+}s)\big) / _{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$

Table 10: If $M4_1^2 = N(1)$ is the Whitehead link exterior and $M = M4_1^2(r/s) = N(1, r/s)$ then M is hyperbolic with $e(M) = 6$ if and only if $r/s \notin \{\infty, -4, -3, -5/2, -2, -3/2, -1, -1/2, -1/3, 0, 1/2, 0, 1, 2, 3, 4, 5\}$. Under this assumption we have $E(M) = \{\infty, -3, -2, -1, 0, 1\}$ in all cases, and the corresponding non-hyperbolic fillings are as described

p/q	r/s	M
∞	any	$L(3r+2s, r+s)$
-3	$-1+\frac{1}{n}$	$L(6n+7, 3n+2)$
	$\neq -1+\frac{1}{n}$	$\big(S^2, (2,1), (3,2), (r+s, s)\big)$
$-5/2$	$-2+\frac{1}{n}$	$\big(S^2, (2,1), (3,-1), (2n-1, 2)\big)$
	$\neq -2+\frac{1}{n}$	$\big(D, (2,1), (3,1)\big) \bigcup_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)} \big(D, (2,1), (r+2s, s)\big)$
-2	any	$L(4r+11s, r+3s)$
-1	$-3+\frac{1}{n}$	$L(6n+1, 3n+2)$
	$\neq -3+\frac{1}{n}$	$\big(S^2, (2,-1), (3,2), (r+3s, s)\big)$
0	n	$\big(S^2, (2,-1), (3,1), (2n+5, 2)\big)$
	$\notin \mathbb{Z}$	$\big(D, (2,1), (3,1)\big) \bigcup_{\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)} \big(D, (2,1), (s, r+2s)\big)$

Table 11: Fillings of $M4_2^2 = N(-3/2)$ for $r/s \notin \{\infty, -4, -3, -8/3, -5/2, -7/3, -2, -3/2, -1, -1/2, 0, 1\}$

p/q	r/s	M
∞	any	$L(5r+2s, 2r+s)$
-3	$-1+\frac{1}{n}$	$\big(S^2, (2,-1), (3,2), (2n+1, 2)\big)$
	$\neq -1+\frac{1}{n}$	$\big(D, (2,1), (3,1)\big) \bigcup_{\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)} \big(D, (2,1), (r+s, s)\big)$
-2	any	$L(8r+13s, 3r+5s)$
$-3/2$	$-2+\frac{1}{n}$	$\big(S^2, (2,-1), (3,2), (2n-1, 2)\big)$
	$\neq -2+\frac{1}{n}$	$\big(D, (2,1), (3,1)\big) \bigcup_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)} \big(D, (2,1), (r+2s, s)\big)$
-1	any	$L(3r+11s, 2r+7s)$
0	n	$\big(S^2, (2,-1), (3,1), (2n+3, 2)\big)$
	$\notin \mathbb{Z}$	$\big(D, (2,1), (3,1)\big) \bigcup_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)} \big(D, (2,1), (s, r)\big)$

Table 12: Fillings of $M4_3^2 = N(-5/2)$ for $r/s \notin \{\infty, -4, -3, -5/2, -2, -5/3, -3/2, -4/3, -1, -1/2, 0, 1\}$

p/q	r/s	M
∞	any	$L(r{+}2s, s)$
-4	$-\frac{1}{2}$	$\left(\mathbb{RP}^2, (2,1), (3,1), -1\right)$
	n	$\left(S^2, (2,-1), (3,2), (2n{+}1, 2)\right)$
	$\notin \mathbb{Z} \cup \{-\frac{1}{2}\}$	$\left(D, (2,1), (3,1)\right) \bigcup_{\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)} \left(D, (2,1), (s,r)\right)$
-3	$-1 + \frac{1}{n}$	$L(10n{+}1, 5n{-}2)$
	$\neq -1 + \frac{1}{n}$	$\left(S^2, (2,-1), (5,3), (r{+}s, s)\right)$
-2	$-2 + \frac{1}{n}$	$L(9n, 3n{-}1)$
	$\neq -2 + \frac{1}{n}$	$\left(S^2, (3,-2), (3,2), (r{+}2s, s)\right)$
-1	$-3 + \frac{1}{n}$	$L(10n{-}1, 5n{+}2)$
	$\neq -3 + \frac{1}{n}$	$\left(S^2, (2,-1), (5,2), (r{+}3s, s)\right)$
0	$-\frac{7}{2}$	$\left(\mathbb{RP}^2, (2,1), (3,1), -1\right)$
	n	$\left(S^2, (2,-1), (3,1), (2n{+}7, 2)\right)$
	$\notin \mathbb{Z} \cup \{-\frac{7}{2}\}$	$\left(D, (2,1), (3,1)\right) \bigcup_{\left(\begin{smallmatrix} 1 & 1 \\ 2 & 3 \end{smallmatrix}\right)} \left(D, (2,1), (s,r)\right)$

Table 13: Fillings of $M4_4^2 = N(-1/2)$ for $r/s \notin \{\infty, -5, -4, -3, -5/2, -2, -3/2, -1, 0, 1\}$

$m139 = M5_{48} = N(-5,-5)$	
$\infty : L(24,5)$	$-3 : \big(D,(2,1),(4,1)\big)\bigcup_{\left(\begin{smallmatrix}1 & 1\\ 0 & -1\end{smallmatrix}\right)}\big(D,(2,1),(4,1)\big)$
$-2 : \big(S^2,(3,1),(3,1),(3,2)\big)$	$-1 : \big(S^2,(2,1),(2,1),(2,1)\big)$
$-1/2 : \big(A,(2,1)\big)/_{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)}$	$0 : \big(S^2,(2,1),(3,2),(6,1),-1\big)$
$M4_{35} = N(-5/3,-5/3)$	
$\infty : L(16,7)$	$-5/2 : \big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}-1 & 1\\ 0 & -1\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$
$-3 : (K,1)$	$-2 : L(16,5)$
$-1 : \big(S^2,(2,1),(4,3),(4,3),-1\big)$	$0 : \big(D,(2,1),(3,1)\big)\bigcup_{\left(\begin{smallmatrix}1 & 1\\ -1 & 0\end{smallmatrix}\right)}\big(D,(3,1),(3,1)\big)$
$m208 = M5_{88} = N(-7/3,-7/3)$	
$\infty : L(40,11)$	$-3 : \big(D,(2,1),(4,1)\big)\bigcup_{\left(\begin{smallmatrix}1 & 1\\ 0 & -1\end{smallmatrix}\right)}\big(D,(2,1),(4,1)\big)$
$-2 : L(20,7)$	$-3/2 : \big(A,(2,1)\big)/_{\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)}$
$-1 : \big(S^2,(2,1),(2,1),(2,1),1\big)$	$0 : \big(D,(2,1),(3,1)\big)\bigcup_{\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)}\big(D,(3,1),(3,1)\big)$
$m120 = M5_{43} = N(2,3)$	
$\infty : L(5,2)$	$-3 : \big(D,(2,1),(3,1)\big)\bigcup_{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)}\big(D,(2,1),(4,1)\big)$
$-2 : \big(S^2,(3,1),(4,1),(5,1),-1\big)$	$-1 : \big(S^2,(2,1),(5,1),(6,1),-1\big)$
$0 : \big(S^2,(2,1),(3,1),(9,1),-1\big)$	$1 : \big(S^2,(2,1),(4,1),(5,1),-1\big)$
$s90 = M6_{91} = N(2,4)$	
$\infty : L(7,2)$	$-3 : \big(D,(2,1),(3,1)\big)\bigcup_{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)}\big(D,(2,1),(5,1)\big)$
$-2 : \big(S^2,(3,1),(4,1),(6,1),-1\big)$	$-1 : \big(S^2,(2,1),(5,1),(7,1),-1\big)$
$0 : \big(S^2,(2,1),(3,1),(10,1),-1\big)$	$1 : \big(S^2,(3,1),(3,1),(4,1),-1\big)$
$v224 = M7_{225} = N(2,5)$	
$\infty : L(9,2)$	$-3 : \big(D,(2,1),(3,1)\big)\bigcup_{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)}\big(D,(2,1),(6,1)\big)$
$-2 : \big(S^2,(3,1),(4,1),(7,1),-1\big)$	$-1 : \big(S^2,(2,1),(5,1),(8,1),-1\big)$
$0 : \big(S^2,(2,1),(3,1),(11,1),-1\big)$	$1 : \big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$
$s118 = M6_{120} = N(3,3)$	
$\infty : L(8,3)$	$-3 : \big(D,(2,1),(4,1)\big)\bigcup_{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)}\big(D,(2,1),(4,1)\big)$
$-2 : \big(S^2,(3,1),(5,1),(5,1),-1\big)$	$-1 : \big(S^2,(2,1),(6,1),(6,1),-1\big)$
$0 : \big(S^2,(2,1),(3,1),(10,1),-1\big)$	$1 : \big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}1 & 2\\ 0 & -1\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$

Table 14: Non-hyperbolic fillings of sporadic $M = N(\alpha,\beta)$'s with $e(M) = 6$. The old SnapPea name of M is also shown when M has complexity more than 4

To begin, we note that the action of f on $E_T(M)$ must preserve the filled manifolds and can have at most 2 fixed points. Using also the fact that $|E_T(M)| \geqslant 5$, we deduce that, among the non-hyperbolic fillings of M, either there is at least one appearing at least three times, or there are at least two appearing at least twice each. If $e(M) > 6$, Tables 7 to 9 show that this happens only for $N(1,2)$ and $N(1,-4)$, and the action of f is given by relations (5) and (6) respectively. If $e(M) = 6$ and M has two cusps, then M is one of the 4 manifolds listed in Table 6, and the action of f is given by one of the relations (2) to (4).

Assume now $e(M) = 6$ and M has one cusp. By Corollary 2.6, M is one of the manifolds listed in Tables 10 to 14. It cannot be one of those in Table 14 because in this table the same filling never appears more than once. The other cases are discussed as follows:

- If M appears in Table 10 or in Table 13, we have $E(M) = \{\infty, a-2, a-1, a, a+1, a+2\}$ for $a = -1$ or $a = -2$ respectively. Since ∞ is the only slope with distance 1 from all other ones in $E(M)$, it is fixed by f. The action is then given by $a + n \mapsto a - n$, but one sees quite easily that $M(a-2)$ is never equal to $M(a+2)$: a contradiction;

- If M appears in Table 11, one sees that f is the unique non-trivial symmetry of the set $E(M) = \{\infty, -3, -5/2, -2, -1, 0\}$, which interchanges ∞ and -2. But $L(3r+2s, r+s)$ never equals $L(4r+11s, r+3s)$: a contradiction;

- If M appears in Table 12, one sees that f permutes non-trivially each set $\{-2, -1, \infty\}$ and $\{-3, -3/2, 0\}$. The three lens spaces $M(\infty), M(-2)$, and $M(-1)$ are distinct except for $r/s \in \{-2/5, -11/3, -13/8\}$, but in each of these cases (which give the cosmetic example found in [4] and mentioned in Subsection 1.2) the manifolds $M(-3), M(-3/2)$, and $M(0)$ are distinct, giving a contradiction.

We are left to discuss the case where $e(M) = 5$, so $E_T(M) = \{\infty, -3, -2, -1, 0\}$, and f is the unique non-trivial symmetry of $E_T(M)$, which fixes ∞ and maps n to $-3-n$. With some effort, one checks from Tables 1 and 2 that $M(0)$ and $M(3)$ are always distinct. □

Proof of 1.10. If $M = N$, then $\alpha, \beta \notin \{\infty, -3, -2, -1, 0\}$ because the fillings listed in Theorem 1.1 are all distinct. Then $N(\alpha) = N(\beta)$ is hyperbolic. If $e(N(\alpha)) = 6$, by Corollary 2.2 we have $\{\alpha, \beta\} = \{-4, -3/2\}$, and the homeomorphism is the orientation-reversing (1). Assume $e(N(\alpha)) = 5$, so $E(N(\alpha)) = E(N(\beta)) = \{\infty, -3, -2, -1, 0\}$, and let $f : N(\alpha) \to N(\beta)$ be the homeomorphism. Note that f acts on $\{\infty, -3, -2, -1, 0\}$. As in the previous proof, ∞ is the only exceptional slope having distance 1 from all other ones, so $f(\infty) = \infty$. Then

$f(n)$ is either n or $-3-n$, so either $N(\alpha,n) = N(\beta,n)$ for $n = -3,-2,-1,0$ or $N(\alpha,n) = N(\beta,-3-n)$ for $n = -3,-2,-1,0$, but Table 1 easily allows to conclude that this is impossible for $\alpha \neq \beta$.

Suppose now $M = N(\gamma)$. If $M(\alpha) = M(\beta)$ is not hyperbolic then α,β are found by examining the repetitions in the fillings in Table 1 and dismissing the pairs related by the automorphisms of Proposition 1.5. The resulting pairs are precisely those of Table 5. Of these pairs, those giving as a filling the solid torus, which is amphichiral, are both truly and reflectively cosmetic. In the other cases the filling is chiral, so the pair cannot be both truly and reflectively cosmetic. For $\gamma \in \{-12/5, -6, -4/3\}$ we now explain how to determine the type of the pair α,β by examining the fillings of $M(\alpha) = M(\beta)$. In fact, using on $\partial M(\alpha) = \partial N(\gamma,\alpha)$ and $\partial M(\beta) = \partial N(\gamma,\beta)$ the homology bases induced by N, we will have $M\big(\alpha, \frac{t}{u}\big) = M\big(\beta, \frac{at+bu}{ct+du}\big)$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$, and the pair α,β is truly or reflectively cosmetic depending on whether $\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is $+1$ or -1. For instance for $\gamma = -4/3$ we have from Table 2

$$\begin{aligned} M(-3,t/u) = N\big(-3,-4/3,t/u\big) &= \big(S^2,(2,1),(5,2),(t+u,u)\big) \\ M(-1,t/u) = N\big(-1,-4/3,t/u\big) &= \big(S^2,(2,1),(5,2),(t+3u,-t-4u)\big) \end{aligned}$$

whence $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \pm\left(\begin{smallmatrix} 4 & 7 \\ -1 & -2 \end{smallmatrix}\right)$, so $-3,-1$ are reflectively cosmetic.

To conclude we are left to show that $M(\alpha) = M(\beta)$ cannot be hyperbolic. Assume first $e(M(\alpha)) > 6$, so $M(\alpha)$ appears in one of the Tables 7 to 9. By construction, the manifolds listed in these tables are distinct and appear with all their possible presentations as fillings of N. Whenever multiple presentations exist one sees they are related by the symmetries of Proposition 1.5, so α and β are equivalent.

Suppose now $e(M(\alpha)) = 6$, so $M(\alpha)$ appears in one of the Tables 10 to 14. Each manifold in Table 14 has a unique presentation as a filling of N, so $M(\alpha)$ cannot appear there. If $M(\alpha)$ appears in Table 10, so $\gamma = 1$, the only filling of $M(\alpha)$ admitting a non-separating torus is $M(\alpha,1)$, and the topological type of $M(\alpha,1)$ determines α, whence $\alpha = \beta$, a contradiction. If $M(\alpha)$ appears in Table 13, the only non-trivial symmetry of $E(M(\alpha))$ is $t/u \mapsto -t/u - 4$, so either $M(\alpha,t/u) = M(\beta,t/u)$ or $M(\alpha,t/u) = M(\beta,-t/u-4)$ for all t/u. Looking at Table 13 itself, one easily sees that the former equality is impossible for $\alpha \neq \beta$. If the latter equality holds, using (3) we get a slope β' equivalent to β such that $M(\alpha,t/u) = M(\beta,-t/u-4) = M(\beta',t/u)$, giving $\alpha = \beta'$ for what just said. Hence α and β are equivalent, a contradiction. A similar argument applies if $M(\alpha)$ appears in Table 11 or 12.

If $e(M(\alpha)) = 5$ then $N(\gamma,\alpha,\delta) = N(\gamma,\beta,h(\delta))$, where h leaves $\{\infty,-3,-2,-1,0\}$ invariant, so either $h(\delta) = \delta$ or $h(\delta) = -3-\delta$. With a little patience one sees from

	S	D	A	T	S^H	T^H
S	**1** ($\geqslant 0$)	**0** ($\geqslant -\infty$)	**2** ($\geqslant -\infty$)	**3** (3)	$-\infty$ **?** ($\geqslant -\infty$)	**1** (1)
D		**1** (1)	**2** (2)	**2** (2)	—	—
A			**5** (5)	**5** (5)	—	—
T				**8** (8)	**2** (2)	**3?** ($\geqslant 3$)
S^H					**0** (0)	**1** (1)
T^H						**1** (1)

Table 15: Known or conjectured values of $\Delta(X_0, X_1)$ shown in boldface, and best lower bounds realized by fillings of N shown between brackets. Every known $\Delta(X_0, X_1)$ with $X_0, X_1 \neq S$ is realized

Table 2 that the latter possibility is actually absurd, while the former implies that $\alpha = \beta$. □

2.2 Distance between exceptional slopes

Following [9], let us say that a manifold is of type D, A, S, or T if it contains respectively an essential disc, annulus, sphere, or torus, and of type S^H or T^H if it contains a Heegaard sphere or torus. Denoting by Δ the distance between slopes on a torus, for $X_0, X_1 \in \{D, A, S, T, S^H, T^H\}$ we define $\Delta(X_0, X_1)$ as the maximum of $\Delta(\alpha_0, \alpha_1)$ over M, α_0, α_1, where M is cusped hyperbolic, α_0, α_1 are slopes on the same cusp of M, and $M(\alpha_i)$ is of type X_i. Most values of $\Delta(X_0, X_1)$ are known, and it turns out that the partial fillings of our chain-link complement N allow to realize many of them, as summarized in Table 15 and explained below. We first discuss the maxima of $\Delta(X_0, X_1)$ not realized, and then arrange the discussion depending on the type of X_0.

Maxima not realized The non-sharp lower bound $\Delta(S, S) \geqslant 0$ is realized by $M = N(n, -4 - n)$ for $n \notin \{-4, -3, -2, -1, 0\}$, because M is hyperbolic and $M(0)$ is reducible. (For later purpose we also note that $M(-2)$ and $M(-1)$ are Seifert manifolds.) No hyperbolic partial filling of N admits a non-closed reducible filling, so no lower bound on $\Delta(S, D) = 0$ or $\Delta(S, A) = 2$ is gained.

Reducible fillings For $X_0 = S$, we are left to discuss $\Delta(X_0, X_1)$ for $X_1 = T$ and $X_1 = T^H$ (no hyperbolic filling of N is a knot complement with reducible fillings, so we do not disprove the conjecture $\Delta(S, S^H) = -\infty$, equivalent to the cabling conjecture). As already noted $M = N(n, -4-n)$ is hyperbolic for $n \notin \{-4, -3, -2, -1, 0\}$ and it has the reducible filling $M(0) = \mathbb{RP}^3 \# L(3,1)$. Moreover $M(-3)$ is toroidal (unless $n = 1$), which shows M realizes $\Delta(S, T) = 3$. In addition, $M(\infty)$ is a lens space, so M realizes $\Delta(S, T^H) = 1$.

Solid torus fillings For $X_0 = D$ we must discuss the cases $X_1 = D, A$, and T. Every $N(\alpha)$ is a knot complement in the solid torus, since $N(\alpha, \infty) = D \times S^1$. The $N(\alpha)$'s having another filling equal to $D \times S^1$ are those of the form $N(-2 + 1/n)$ or $N(-3 + 1/n)$ with $n \in \mathbb{Z}$. These examples realize $\Delta(D, D) = 1$ and $\Delta(D, A) = 2$ Note that by [8] these manifolds are complements of 1-bridge braids in the solid torus. The same examples actually also realize $\Delta(D, T) = 2$, because there is a toroidal filling at distance 2 from the solid torus filling.

Annular fillings There is not much to say here, because the Whitehead sister link exterior $N(1)$ is known to realize $\Delta(A, A) = \Delta(A, T) = 5$.

Toroidal fillings For $X_0 = T$ we must discuss $X_1 \in \{T, S^H, T^H\}$, but $\Delta(T, T) = 8$ is known to be realized by the figure-8 knot complement $N(1, 2)$. Turning to $\Delta(T, S^H) = 2$, we can realize it via

$$M = N\left(-3 + \frac{1}{n}, \frac{\pm(3n+1) + k(6n-1)}{\mp n - k(2n-1)}\right)$$

with $n, k \in \mathbb{Z}$, $n \neq 0$, because M is hyperbolic, it is a knot complement in $M(-1) = S^3$, and $M(-3)$ is toroidal for generic values of (n, k).

Let us now discuss $\Delta(T, T^H)$, for which only the lower bound 3 is known. This bound is realized by $M = N(n, -4 - n + 1/m)$ with $n \in \mathbb{Z} \setminus \{-3, -2, -1, 0\}$, because M is hyperbolic, $M(-3)$ is toroidal, and M has the lens space fillings $M(\infty) = L(n(1-m(4+n))-m, 1-m(4+n))$ and $M(0) = L(6m-1, 2m-1)$. (For later purpose we also note that $M(-2)$ and $M(-1)$ are Seifert manifolds.) These examples probably coincide with those described by Gordon [9], who also noted that $M(0)$ is independent of n, deducing that $L(6m-1, 2m-1)$ contains infinitely many knots with hyperbolic exterior and a toroidal slope at distance 3 from the meridian.

More examples of hyperbolic knot complements in lens spaces with a toroidal slope at distance 3 from the meridian are constructed as $M = N(-1+1/n, -1+1/m)$ with $n, m \in \mathbb{Z} \setminus \{-1, 1\}$, because $M(-3) = L((2n+1)(2m+1)-4, (2n+1)m-2)$ and $M(0)$ is toroidal.

Sphere and lens fillings The celebrated Gordon-Luecke maximum $\Delta(S^H, S^H) = 0$ is realized by many fillings of N, *e.g.* the figure-8 knot complement. Every hyperbolic $N(-5/2, \alpha)$ is a 1-cusped manifold with 3 fillings yielding lens spaces (see Table 12) which realize $\Delta(T^H, T^H) = 1$.

Turning to $\Delta(S^H, T^H) = 1$, we must construct hyperbolic knot complements having lens space fillings. One way is to note that $N(-2 + 1/n)$ has two fillings giving $D \times S^1$, along ∞ and -2, and suitably fill one of its cusps. More precisely, if

$$M = N\left(-2 + \frac{1}{n}, \frac{kn \pm 1}{k - 2(kn \pm 1)}\right)$$

for $k, n \in \mathbb{Z}$, then M is typically hyperbolic, $M(\infty) = S^3$, so M is a knot complement, and $M(-2)$ is a lens space. The same method works starting from $N(-3 + 1/n)$. Other (apparently distinct) examples are given by $M = N(-1+1/n, -1-1/n)$ with $n \in \mathbb{Z}$, $n \neq -1$, because $M(\infty) = S^3$ and $M(-3) = L(4n^2+3, 2n^2+n+2)$. We do not know whether these examples lead to Berge knots, as conjectured in [12].

Seifert manifolds Let us say that M is of type Z if it is an atoroidal closed Seifert manifold. No value $\Delta(Z, X)$ seems to be known. From the previous discussion we deduce that the manifolds $N(\alpha, \beta)$ give the following lower bounds for $\Delta(Z, X)$ for $X \in \{S, T, Z, S^H, T^H\}$:

	S	T	Z	S^H	T^H
Z	2	7	6	1	2

Infinitely many 1-cusped $N(\alpha, \beta)$ realize the following distances:

	S	T	Z	S^H	T^H
Z	2	5	4	1	2

3 Proofs

We give here a unified proof of Theorems 1.1, 1.2, and 1.3. As already mentioned, the ingredients we use are Proposition 1.5, a certain refined version of Theorem 1.1 (proved below as Proposition 3.2), and then an elaborate argument which uses the Gromov-Thurston 2π-theorem and SnapPea for a finite enumeration.

Hyperbolic fillings with symmetries One can see using SnapPea (and check by hand) that the partial Dehn fillings

$$N(-3/2) \qquad N(-4) \qquad N(-5/2) \qquad N(-1/2) \qquad N(1,2) \qquad N(1,-4)$$

are all hyperbolic, that the first two of them are isometric to each other, and that in all cases the symmetry group contains a non-trivial cyclic subgroup that leaves invariant each boundary component. Using the precise form of these non-trivial isometries and their action on the homology bases on the boundary, one can with some patience deduce the identities of Proposition 1.5.

Identities between graph manifolds We slightly relax in this section our notation $\big(F,(p_1,q_1),\ldots,(p_k,q_k),b\big)$ for Seifert manifolds, allowing also $|p_i|\leqslant 1$, which can lead to lens spaces or to connected sums of lens spaces and/or Seifert manifolds. We will always use the conventions of Section 1 on homology bases to encode gluings. The following lemma, which collects well-known [6] or easy properties of graph manifolds, is repeatedly (and often tacitly) used below.

Lemma 3.1. *Suppose F is a surface, (i,j) and (i_n,j_n) are coprime pairs, $k\in\mathbb{Z}$, X is any graph manifold, and $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{GL}_2(\mathbb{Z})$. Let (i',j') be such that $ij'-ji'=\pm 1$ and, when $\partial F\neq\emptyset$, denote by F' the surface obtained from F by capping off one boundary circle. Then the following equalities hold:*

(Seifert manifolds)

$$\big(F,(i_1,j_1),(i_2,j_2),\ldots\big) = \big(F,(i_1,-j_1),(i_2,-j_2),\ldots\big) \tag{7}$$

$$\big(F,(i_1,j_1),(i_1,j_2),\ldots\big) = \big(F,(i_1,j_1{+}ki_1),(i_2,j_2{-}ki_2),\ldots\big) \tag{8}$$

$$\big(F,(i_1,j_1),(i_2,j_2),\ldots\big) = \big(F,(i_1,j_1{+}ki_1),(i_2,j_2),\ldots\big)\quad\text{if } \partial F\neq\emptyset \tag{9}$$

$$\big(S^2,(i,j),(i_1,j_1)\big) = L(i_1j{+}j_1i,\, i_1j'{+}j_1i') \tag{10}$$

$$\big(F,(1,k),(i,j),\ldots\big) = \big(F,(i,j{+}ki),\ldots\big) \tag{11}$$

(Graph manifolds)

$$\big(F,(i,j),\ldots\big)\bigcup\nolimits_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)} X = \big(F,(i,j{+}ki),\ldots\big)\bigcup\nolimits_{\left(\begin{smallmatrix} a+kb & b\\ c+kd & d\end{smallmatrix}\right)} X \tag{12}$$

$$X\bigcup\nolimits_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)} \big(F,(i,j),\ldots\big) = X\bigcup\nolimits_{\left(\begin{smallmatrix} a & b\\ c-ka & d-kb\end{smallmatrix}\right)} \big(F,(i,j{+}ki),\ldots\big) \tag{13}$$

$$\big(D,(i,j)\big)\bigcup\nolimits_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)} \big(F,\ldots\big) = \big(F',(ai{-}bj,ci{-}dj),\ldots\big) \tag{14}$$

(Double fibrations)

$$\big(D,(2,1),(2,1)\big)\bigcup\nolimits_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)} X = (S\,\tilde{\times}\,S^1)\bigcup\nolimits_{\left(\begin{smallmatrix} b & b-a\\ d & d-c\end{smallmatrix}\right)} X \tag{15}$$

$$X\bigcup\nolimits_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)} \big(D,(2,1),(2,1)\big) = X\bigcup\nolimits_{\left(\begin{smallmatrix} a+c & b+d\\ -a & -b\end{smallmatrix}\right)} S\,\tilde{\times}\,S^1 \tag{16}$$

(Reducible manifolds)

$$\big(S^2,(i_1,j_1),(i_2,j_2),(0,1)\big) = L(i_1,j_1)\#L(i_2,j_2) \tag{17}$$

$$\big(D,(0,1),(i,j)\big)\bigcup_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)}\big(F,\ldots\big) = L(i,j)\#\big(F',(b,d),\ldots\big) \tag{18}$$

(Torus bundles)

$$\big(A\times S^1\big)/_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)} = T/_{\left(\begin{smallmatrix} -a & -b\\ c & d\end{smallmatrix}\right)} \tag{19}$$

$$\big(S^2,(2,1),(3,1),(6,1),-1\big) = T/_{\left(\begin{smallmatrix} 1 & 1\\ -1 & 0\end{smallmatrix}\right)} \tag{20}$$

$$\big(S^2,(2,1),(4,1),(4,1),-1\big) = T/_{\left(\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right)} \tag{21}$$

$$\big(S^2,(3,1),(3,1),(3,1),-1\big) = T/_{\left(\begin{smallmatrix} -1 & 1\\ -1 & 0\end{smallmatrix}\right)} \tag{22}$$

$$\big(K,1\big) = T/_{\left(\begin{smallmatrix} -2 & 1\\ -1 & 0\end{smallmatrix}\right)}. \tag{23}$$

The homeomorphisms proving these equalities respect the homology bases on the boundary, except in formulae (7) and (9). Relations (12) and (13) are also valid if $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ is used to glue together two boundary components of the same Seifert block $\big(F,(i,j),\ldots\big)$.

Basic recognition result Recall that we are always denoting by N the complement of the chain-link with 3 components. We establish now the following:

Proposition 3.2. *The following homeomorphisms hold:*

$$N\left(\infty,\frac{r}{s},\frac{t}{u}\right) = \big(S^2,(r,s),(-u,t)\big) \tag{24}$$

$$N\left(-3,\frac{r}{s},\frac{t}{u}\right) = \big(D,(2,1),(r{+}s,s)\big)\bigcup_{\left(\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right)}\big(D,(2,1),(t{+}u,u)\big) \tag{25}$$

$$N\left(-2,\frac{r}{s},\frac{t}{u}\right) = \big(S^2,(3,2),(r{+}2s,-s),(t{+}2u,-u)\big) \tag{26}$$

$$N\left(-1,\frac{r}{s},\frac{t}{u}\right) = \big(S^2,(2,1),(r{+}3s,-s),(t{+}3u,-u)\big) \tag{27}$$

$$N\left(0,\frac{r}{s},\frac{t}{u}\right) = \big(D,(s,r{+}2s),(u,t{+}2u)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\-1&-1\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big) \tag{28}$$

$$N\left(1,1,\frac{t}{u}\right) = \big(A,(u,t{+}u)\big)/_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \tag{29}$$

$$N(1,3,3) = \big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}1&2\\0&-1\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big) \tag{30}$$

$$N(2,2,2) = \big(D,(2,1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}2&3\\-1&-2\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big) \tag{31}$$

Equalities (24) to (29) also hold for partial fillings: if a symbol r/s or t/u is missing on the left-hand side (so the relative cusp is not filled), an open neighborhood of the corresponding fibre on the right-hand side should be removed.

Proof. The first equality is easy: doing ∞-surgery on a component of a link means canceling this component. So $N(\infty)$ is the complement of the Hopf link, which is homeomorphic to $T\times I$, and it is easy to see that the actual homeomorphism leads to relation (24).

The other equalities are proved using spines. Recall that a *spine* of a PL manifold M is a subpolyhedron onto which M collapses. Moreover, if M is cusped, the 2-skeleton of the cellularization dual to an ideal triangulation of M is always a spine P of M. The *singular* (non-surface) points of P give a 4-valent graph $S(P)$ such that $P\setminus S(P)$ consists of discs (called the *faces* of P). A spine of this sort is called *standard*, and it is known to determine M. Moreover, P itself can be reconstructed by a neighbourhood of $S(P)$ in P. Below we will employ also many non-standard spines.

The usual triangulation technique for complements of alternating links shows that our chain-link complement N is obtained by pairing the faces of two triangular prisms. Subdividing the prism and dualizing we see that N has a standard spine with 6 vertices. One can actually show that 6 is the minimal number of vertices for a spine of N, and that there are several spines with 6 vertices, related to the original one by the moves of the Matveev-Piergallini calculus. One such spine, denoted by P and shown in Fig. 3, is particularly suitable for the recognition of fillings. We could give a constructive proof of the fact that P is a spine of N, but we confine ourselves to a quick indirect proof.

Lemma 3.3. *Let P be the standard polyhedron with a neighbourhood of $S(P)$ as in Fig. 3-(1). Then:*

1. *P is a spine of N;*

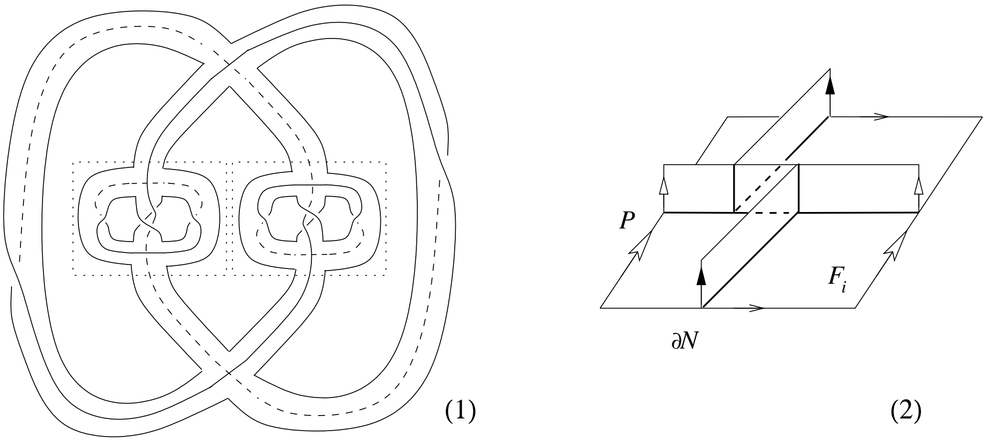


Figure 3: A neighbourhood of $S(P)$ and a neighbourhood of $\overline{F_i}$ (edges with the same label are identified)

2. *P has two faces F_1 and F_2 such that $\overline{F_i}$ is a torus parallel to a component T_i of ∂N, and the rest of P is attached to $\overline{F_i}$ along a θ-shaped graph as shown in Fig. 3-(2); the attachment takes place on the side of $\overline{F_i}$ opposite to the parallelism with T_i;*

3. *There are 4 other faces in P; a hexagon E and a decagon C, each incident to both $\overline{F_i}$'s along an edge, and two squares Q_1 and Q_2, with Q_i incident to $\overline{F_i}$ along an edge.*

Proof. Points (2) and (3) are proved by direct inspection. Let M be the manifold P is a spine of. By point (2) M has at least 3 boundary components, and dual to P there is an ideal triangulation of M involving 6 tetrahedra. We now claim M is (complete, finite-volume) hyperbolic. This is sufficient to prove that $M = N$, because N is the unique hyperbolic manifold with at least three cusps which can be triangulated with at most 6 tetrahedra [5].

We prove the claim recalling that Thurston has associated certain hyperbolicity equations to each triangulated manifold, and that Casson has shown that a solution of the angle part of the consistency equations is already sufficient to ensure hyperbolicity [13]. In terms of spines, we must assign angles to the corners of the faces, and we do this as follows: (1) We put $\pi/3$ on the corners of E, and on the 6 corners opposite to those of E; (2) We put $\pi/2$ on the corners of the Q_i's and on those opposite to them; (3) We put $\pi/6$ on all other corners. One readily sees that angles at opposite corners are always the same and the sum is 2π on all faces, so we have solved the angle equations. □

Let us resume the proof of Proposition 3.2.

Homology bases We now isotope $\overline{F_i}$ so that it coincides with $T_i \subset \partial N$, and note that the complement in T_i of the θ-graph ∂F_i is a disc, so ∂F_i contains three unoriented loops giving three slopes on T_i with mutual distance 1. We can therefore choose a basis of $H_1(T_i)$ so that ∂F_i contains the slopes -1, -2, and ∞. More precisely, we arrange so that -1 (respectively, -2 and ∞) is the slope obtained by erasing from ∂F_i the edge which belongs to Q_i (respectively, E and C). We will also say that -1 (respectively, -2 and ∞) is the slope associated to Q_i (respectively, E and C).

We will now proceed using these homology bases, recognize many fillings of N, and only later show that these bases are actually (up to sign) the natural ones. For this reason we recognize again $N(\infty)$, because it is not obvious at this stage that ∞ represents the same slope it did with respect to the meridian-longitude basis.

General recognition strategy Our recognition of $N(\alpha)$ for $\alpha \in \{\infty, -3, -2, -1, 0\}$ is based on the construction of a spine $P(\alpha)$. We first illustrate here how to build $P(\alpha)$ for arbitrary α. The same method also gives a spine of $N(\alpha, \beta)$, that we will use to recognize $N(1,1)$, but it does not apply to the closed fillings $N(\alpha, \beta, \gamma)$. To recognize $N(1,3,3)$ and $N(2,2,2)$ we have constructed spines of the punctured manifolds via a slightly different technique, sketched below.

The spine $P(\alpha)$ of $N(\alpha)$ is constructed from P by removing the (open) hexagon F_1, thus getting a θ-shaped "boundary", and attaching a polyhedron $Q(\alpha)$ having θ-shaped boundary too, with a map matching the two θ's. More precisely: consider $N(\alpha) = N \cup V$ where V is an open solid torus, glued to N along $T_1 \subset \partial N$. Since $\overline{F_1} = T_1$, then $N(\alpha) \setminus P$ is the union of V and two copies of $T \times (0,1]$'s. Suppose we have a polyhedron $Q(\alpha) \subset \overline{V}$ so that (i) $Q(\alpha) \cap \partial V$ is contained in the θ-shaped graph ∂F_1, (ii) $V \setminus Q(\alpha)$ is an open ball. By (i), the complement of $P \cup Q(\alpha)$ inside $N(\alpha)$ is the union of two $T \times (0,1]$'s and one open ball. By (ii), F_1 is again an (open) face of $P \cup Q(\alpha)$, which separates the ball from one $T \times (0,1]$. Therefore $(P \cup Q(\alpha)) \setminus F_1$ is a spine of $N(\alpha)$, which we call $P(\alpha)$.

We explain now how to construct $Q(\alpha)$ for all α (even if we will only be using the construction for a few α's). If $\alpha \in \{-2, -1, \infty\}$ then the meridian of V is isotopic to one of the three loops in ∂F_1, and we are in a "degenerate" case, because $Q(\alpha)$ can be taken to be just a meridinal disc of V, as in Fig. 4-(1), so $P(\alpha)$ is not standard. If $\alpha \in \{-3, -3/2, 0\}$, the θ-shaped graph ∂F_1 does not contain the meridian of V, but it intersects the meridian in two points. We can then take as $Q(\alpha)$ the Möbius triplet shown in Fig. 4-(2). For all remaining α's the θ-shaped graph intersects the meridian in more than two points. Therefore we change the θ-shaped boundary of

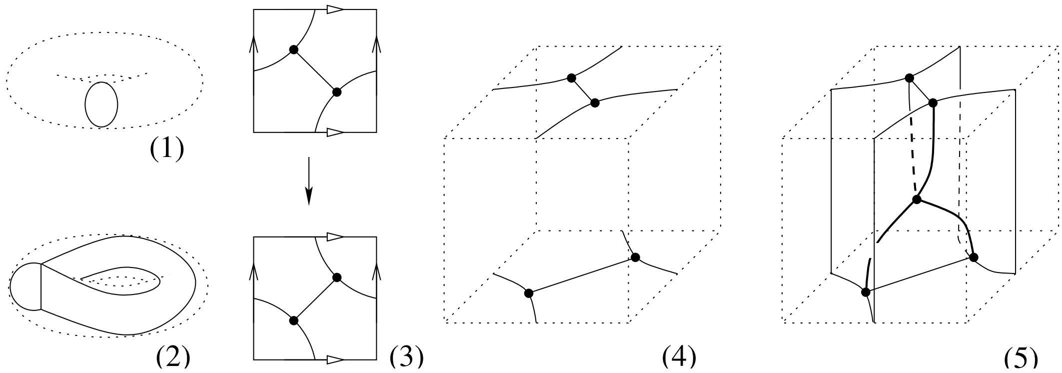


Figure 4: Two polyhedra in the solid torus (1,2) and a flip (3,4) realized by adding a vertex (5)

the Möbius triplet via some *flips* (see Fig. 4-(3,4)) in order to get a θ-shaped graph which matches ∂F_1, each flip adding a vertex to $Q(\alpha)$ as shown in Fig. 4-(5). (This $Q(\alpha)$ is dual to a layered triangulation.)

The same technique, applied to both F_1 and F_2, allows to get a spine of any $N(\alpha, \beta)$, as we will do below for $N(1,1)$, while for $N(\alpha, \beta, \gamma)$ we will use a slightly different method.

Recognition of degenerate fillings As we said, if $\alpha \in \{-2, -1, \infty\}$ we are in a special case, because $Q(\alpha)$ does not contain the whole θ-graph ∂F_1, but only a circle in it. The edge in ∂F_1 not contained in $Q(\alpha)$ is therefore adjacent to a single face Σ in $P(\alpha)$, namely the face (Q_1, E or C) of P associated to α. We can therefore collapse Σ and then further collapse as much as possible, getting a polyhedron $P'(\alpha)$ which is still a spine of $N(\alpha)$. Along the collapse we kill some vertices of P, so $P'(\alpha)$ has less than 6 vertices and can be easily analyzed.

To construct $P'(\alpha)$ we must first replace by three parallel strands the portion of Fig. 3-(1) within a dotted rectangle and then remove the attaching circle of the face Σ. If Σ is doubly incident to some edge e then, after collapsing Σ, e is again incident to one face Σ' only, so we can collapse Σ' too, and so on. The spine $P'(\alpha)$ we get here at the end of the process has a (maybe empty) singular set $S(P'(\alpha))$ which again is 4-valent, but $P'(\alpha) \setminus S(P'(\alpha))$ may not consist of discs only, because some of the original discs of P have been glued together, maybe producing annuli, Möbius strips, or even more complicated surfaces.

For $\alpha = \infty$, everything collapses to the torus $T_2 = \overline{F_2}$, as shown in Fig. 5, so $N(\infty) = T_2 \times I$. For $\alpha = -1$, it is shown in Fig. 6-(1, 2) how $P'(-1)$, which has two vertices, can be transformed via an inverse "lune move" into a spine without

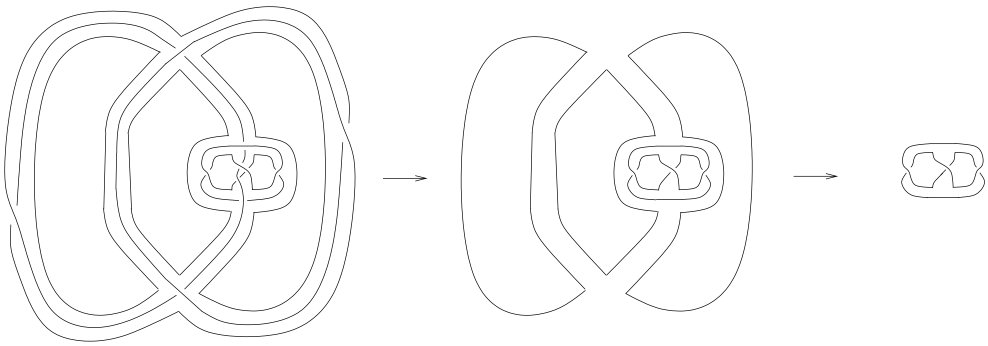


Figure 5: The spine $P'(\infty)$ is a torus

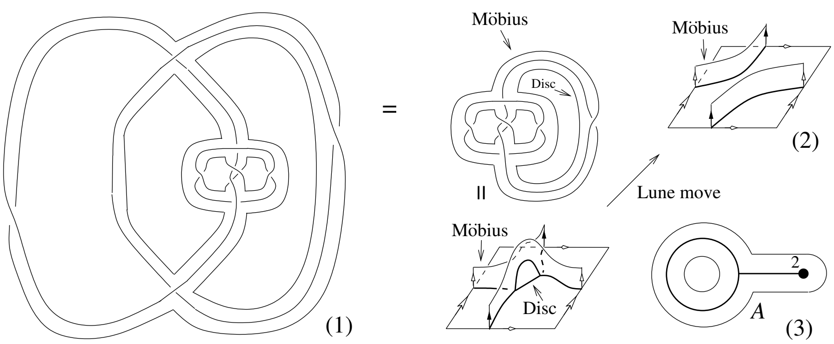


Figure 6: The spine $P'(-1)$, its modification, and the manifold it gives

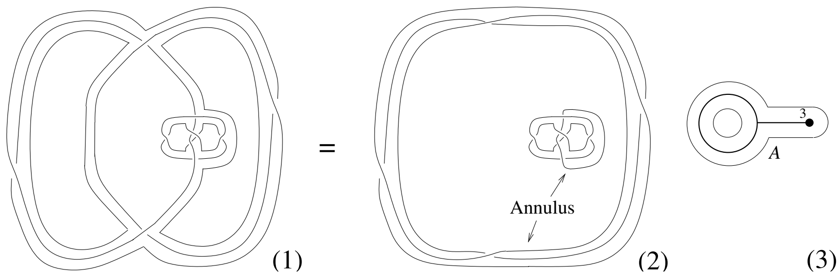


Figure 7: The spine $P'(-2)$, and the manifold it gives

vertices of the same manifold. This spine consists of a torus with a Möbius strip attached, and it is a spine of $\big(A, (2,1)\big)$, since it is the inverse image in the Seifert fibration of $\big(A, (2,1)\big)$ of the graph contained in the base annulus, as shown in Fig. 6-(3). Note that the endpoint of corresponds to the singular fibre.

The spine $P'(-2)$ is shown in Fig. 7-(1,2). It is made of one circle, one torus, and one annulus having one boundary component on the torus, and the other one turning three times along the circle. This is a spine of $\big(A, (3,1)\big)$ since it is the inverse image of the graph in the base annulus of the fibration as in Fig. 7-(3).

Next-to-degenerate fillings If $\alpha \in \{-3, -3/2, 0\}$, the polyhedron $P'(\alpha)$ is constructed by attaching the Möbius triplet to $P \setminus F_1$ along their θ-shaped boundaries. Drawing $P'(\alpha)$ is easy: just replace the portion of Fig. 3-(1) within a dotted rectangle by three "monotonic" strands. Two of these strands must cross each other, while the other one must be straight vertical: the straight one is that representing the face associated to the slope in $\{-2, -1, \infty\}$ which is opposite to α (*i.e.* the only slope having distance 2 from α).

The polyhedron $P'(-3)$ is as shown in Fig. 8-(1), because -3 is opposite to -1, which corresponds to Q_1. Since $P'(-3)$ has one face with two vertices, we can apply to $P'(-3)$ a negative lune move, getting a polyhedron which consists of a torus, a disc, and a Möbius strip, as shown in Fig. 8-(2). Actually, if we remove the Möbius strip we get a torus and another Möbius strip, as for $\alpha = -1$, whence a spine of $\big(A, (2,1)\big)$. The boundary loop of the removed Möbius strip intersects the fibre of $\big(A, (2,1)\big)$ in one point, so it contributes with a $\big(D, (2,1)\big)$, with a vertical strip of $\big(D, (2,1)\big)$ glued to a horizontal strip of $\big(A, (2,1)\big)$, as suggested by Fig. 8-(3). The result is $N(-3) = \big(A, (2,1)\big) \bigcup_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} \big(A, (2,1)\big)$.

The polyhedron $P'(0)$ is shown in Fig. 9-(1) and also in the top of part (2) of the same figure, where each boundary arc should be glued to some other one. We

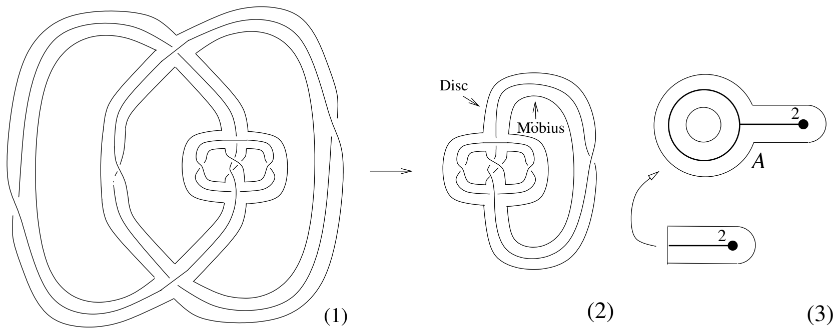


Figure 8: The spine $P'(-3)$, its modification, and the manifold it gives

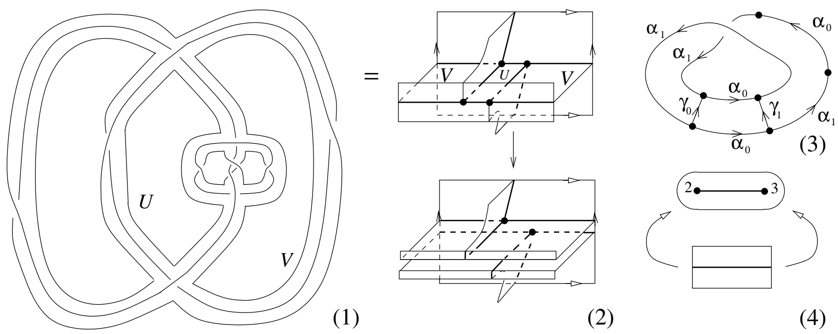


Figure 9: The spine $P'(0)$, its modification, and the manifold it gives

do not show all the gluings, but we note that the left $\vdash\!\dashv$ is identified "straight across" with the $\vdash\!\dashv$ on the right. The union of the two squares U and V gives an annulus in $P'(0)$. We can now modify $P'(0)$ as suggested in Fig. 9-(2), *i.e.* we imagine $\overline{U \cup V}$ to be made of two annular films glued together, and we blow air between the films from the front of the figure. The effect of this move on left and right boundary graphs is given by $\vdash\!\dashv \;\to\; \vDash\!\Dashv$. Using the exact identifications one sees that the new spine R, which has two vertices, consists of one annulus and two discs. By removing the annulus, we get a polyhedron Q without vertices, with $S(Q) = S^1$, and $Q \setminus S(Q)$ consisting of one Möbius strip whose boundary turns 3 times along the circle $S(Q)$. This is a spine of $\big(D, (2,1), (3,1)\big)$, since it is the inverse image of a segment connecting the two singular points in the base disc, as suggested by Fig. 9-(3). The annulus $R \setminus Q$ we have removed will contribute with a $D \times S^1$ with two "vertical" (*i.e.* fibrewise) strips on $\partial(D \times S^1)$ glued to $\partial\big(D, (2,1), (3,1)\big)$. To analyze the gluing we must find which curves on $\partial\big(D, (2,1), (3,1)\big)$ project in Q to the boundary curves γ_0, γ_1 of the annulus $R \setminus Q$. Now γ_0 and γ_1 appear in the unfolded version of Q as in Fig. 9-(3), and it is easy to see that a curve on $\partial\big(D, (2,1), (3,1)\big)$ projecting in Q to γ_i must intersect the fibre once, so it has slope $1/k$ for some $k \in \mathbb{Z}$ with respect to the natural homology basis. We deduce that

$$N(0) = \big(P \times S^1\big) \bigcup\nolimits_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & k \end{smallmatrix}\right)} \big(D, (2,1), (3,1)\big).$$

Note that we cannot modify k by changing bases, because there is no free boundary component on the right block. A more detailed analysis of R would allow to compute the exact value of k, but we will compute it later by an easier indirect argument.

Double filling Note that so far we have not used the flip of Fig. 4-(3,4,5), but we do use it now to get a spine $P(1,1)$ for $N(1,1)$. A neighbourhood of the singular set of $P(1,1)$ is obtained from Fig. 3-(1) by removing both the portions within the dotted boxes and replacing them by a "curl," as shown in Fig. 10-(1). The curl arises because the flip adds a vertex to the spine. Parts (2) and (3) of Fig. 10 show how to transform $P(1,1)$ into another spine Q of $N(1,1)$ having only one vertex and two faces (a disc and an annulus). The closure of the disc is a surface, which is easily seen to be a torus. Therefore Q is a torus with an annulus glued to it, and the two boundary circles of the annulus intersect in one point on the torus (giving the single vertex of Q). This implies that $N(1,1) = \big(P \times S^1\big)/_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)}$.

Recognition of $N(1,3,3)$ and $N(2,2,2)$ To get a spine for $N(\alpha, \beta, \gamma)$, the above method based on the spine P of N does not work. We have used instead a polyhedron

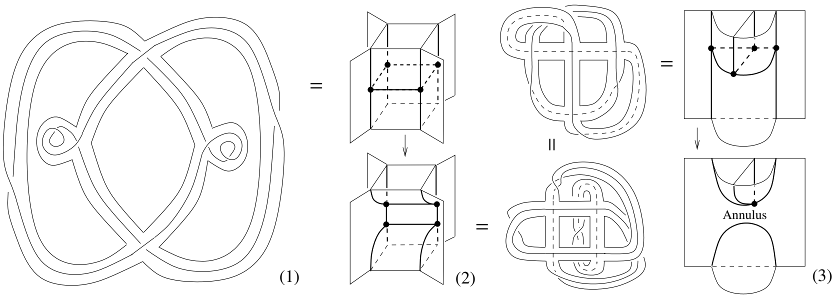


Figure 10: The spine $P(1,1)$ and its modification

$\widetilde{P} \subset N$ which is a spine of N minus a ball (not of N), and has 3 hexagonal faces $\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3$ appearing in $\widetilde{P}$ as F_1 and F_2 appear in P. Now the construction leading from P to $P(\alpha)$ can be repeated for all the $\widetilde{F}_i$'s and gives a spine $\widetilde{P}(\alpha,\beta,\gamma)$ of $N(\alpha,\beta,\gamma)$. The polyhedron $\widetilde{P}$ is more complicated than P, and it is shown in [14] as a "skeleton" of the "brick" B_9. Since $\widetilde{P}$ has 9 vertices, $\widetilde{P}(1,3,3)$ has $9+1+3+3=16$, while $\widetilde{P}(2,2,2)$ has $9+2+2+2=13$. Recognizing $N(1,3,3)$ and $N(2,2,2)$ has therefore required several steps, that we have decided to omit here.

Naturality of bases To show that the homology bases used so far in the proof are (up to sign) the meridian-longitude bases, we state the following:

Lemma 3.4. *With respect to the natural homology bases on ∂N, the slopes α such that $N(\alpha)$ is non-hyperbolic are $\infty, -3, -2, -1, 0$.*

This result can (and will) be proved very easily, using SnapPea for a finite enumeration and the Gromov-Thurston 2π-theorem. Since these tools are part of a more elaborate argument that we use to conclude the proof of Theorems 1.1 to 1.3, we defer the proof of Lemma 3.4 to that argument, presented at the end of the paper.

Back to naturality, let $h : \mathbb{Q}\cup\{\infty\} \to \mathbb{Q}\cup\{\infty\}$ be the map which describes how the slopes change under the change of basis. Lemma 3.4 implies that h sends the set $\{\infty,-3,-2,-1,0\}$ to itself. Therefore it fixes ∞, and it maps t/u either to t/u or to $-3-t/u$. The above recognition of $N(0)$ and $N(-3)$, with coefficients with respect to the basis coming from the spine P, easily implies that $N(0)$ contains an incompressible annulus connecting two distinct boundary components, while $N(-3)$ does not. It is now easy to see that, with respect to the meridian-longitude basis, $N(0)$ does contain such an annulus: a spanning disc of a component C of the chain-link of Fig. 1 gives a pair of pants P in N and, if we 0-fill C, the union of P and a filling disc gives the desired annulus. Therefore $h(0)=0$, and h is the identity, as required.

Conclusion for the slopes -1 and -2 We will now deduce relations (25) to (29) from (24) and the topological informations on the fillings of N just discovered. We start by proving (27). Since $N(-1)=\big(A,(2,1)\big)$, using the fact that the boundary components of $N(-1)$ are symmetric, we deduce that there exists $\left(\begin{smallmatrix} m & n \\ p & q\end{smallmatrix}\right)\in \mathrm{GL}_2(\mathbb{Z})$ such that

$$N\left(-1,\frac{r}{s},\frac{t}{u}\right)=\big(S^2,(2,1),(mr+ns,pr+qs),(mt+nu,pt+qu)\big).$$

Without loss of generality we can also assume $m\geqslant 0$. Since $N(-1,\infty,\infty)$ equals $\big(S^2,(1,0),(1,1)\big)=S^3$, we have $\big(S^2,(2,1),(m,p),(m,p)\big)=S^3$, which implies that $m=1$ and $p=0$.

Now recall that, among Seifert fibrations over S^2 with at most 3 singular fibres, only 3 contain non-separating tori, namely those described in (20), (21), and (22). We have shown that $N(1,1)$ contains a non-separating torus, so $N(-1,1,1)=\big(S^2,(2,1),(1+n,q),(1+n,q)\big)$ also does, therefore it must be (up to orientation) one of the 3 manifolds just mentioned. We easily deduce that either $1+n=4$ and $q=-1$, whence relation (27), or $1+n=-4$ and $q=1$. The latter possibility is however absurd, because it would imply that $N(-1,-2)$ is $\big(D,(2,1),(7,-1)\big)$, but we know that $N(-2)=\big(A,(3,1)\big)$.

Relation (26) is proved in the same way. We first note that

$$N\left(-2,\frac{r}{s},\frac{t}{u}\right)=\big(S^2,(3,2),(mr+ns,pr+qs),(mt+nu,pt+qu)\big).$$

Using the identity $N(-2,\infty,\infty)=S^3$ we then get $m=1$ and $p=0$. Moreover $N(-2,1,1)$ contains a non-separating torus, so $1+n=3$ and $q=-1$.

Conclusion for the slope -3 Concerning (25), we have

$$N\left(-3,\frac{r}{s},\frac{t}{u}\right)=\big(D,(2,1),(mr+ns,pr+qs)\big)\bigcup_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)}\big(D,(2,1),(mt+nu,pt+qu)\big)$$

for some $\left(\begin{smallmatrix} m & n \\ p & q\end{smallmatrix}\right)\in \mathrm{GL}_2(\mathbb{Z})$ with $m\geqslant 0$. We first note that by reversing orientations we get two blocks of type

$$\big(D,(2,-1),(mr+ns,-pr-qs)\big)=\big(D,(2,1),(mr+ns,-(m+p)r-(n+q)s)\big),$$

glued along the same $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Since $\det\left(\begin{smallmatrix}m&n\\-(m+p)&-(n+q)\end{smallmatrix}\right)=-\det\left(\begin{smallmatrix}m&n\\p&q\end{smallmatrix}\right)$, we can suppose $\det\left(\begin{smallmatrix}m&n\\p&q\end{smallmatrix}\right)=1$. Now, we know that

$$L(3,1)=N(-3,\infty,\infty)=\big(D,(2,1),(m,p)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(2,1),(m,p)\big).$$

An obviously necessary condition for this to hold is that $m=1$. In addition $\big(D,(2,1{+}2p)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(2,1{+}2p)\big)=\big(S^2,(2,1{+}2p),(-1{-}2p,2)\big)=L\big(4{-}(1{+}2p)^2,p(1{+}2p){-}2\big)$ must be equal to $L(3,1)$, whence $4-(1+2p)^2=\pm3$. This implies that $p\in\{0,-1\}$.

Now we use the fact that $N(-3,-2)$ and $N(-3,-1)$, being single fillings of $\big(A,(2,1)\big)$ and $\big(A,(3,1)\big)$ respectively, are either reducible or atoroidal. However

$$\begin{aligned} N(-3,-2) &= \big(D,(2,1),(2{-}n,2p{-}q)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(A,(2,1)\big)\\ N(-3,-1) &= \big(D,(2,1),(1{-}n,p{-}q)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(A,(2,1)\big)\end{aligned}$$

so in both these expressions the block on the right-hand side must be either reducible or a solid torus. This implies that $|2-n|\leqslant1$ and $|1-n|\leqslant1$, whence $n\in\{1,2\}$.

To conclude we examine $N(-3,1,1)$ and note that it can be expressed both as $\big(A,(a,b)\big)/_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}$, for suitable a and b, and as

$$\big(D,(2,1),(1{+}n,p{+}q)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(2,1),(1{+}n,p{+}q)\big).$$

Uniqueness of the JSJ decomposition implies that one block of this expression, and hence also the other one, must have a non-unique fibration. So $n=1$, in which case $\big(D,(2,1),(2,1)\big)$ can be fibred also as $S\,\tilde{\times}\,S^1$. In addition, the alternative fibration of one block must be matched to either the original or the alternative fibration of the other block. Using relations (12), (15), and (16) one sees that this can only happen for $(p,q)=(0,1)$ and $(p,q)=(-1,-2)$, but $\det\left(\begin{smallmatrix}1&1\\p&q\end{smallmatrix}\right)=1$, so $(p,q)=(0,1)$, as required. Even if not necessary, note that we have shown the equality $N(-3,1,1)=(K,1)$, which is coherent with (29) and (23).

Conclusion for the slope 0 Concerning (28), we have

$$N\left(0,\frac{r}{s},\frac{t}{u}\right)=\big(D,(mr{+}ns,pr{+}qs),(mt{+}nu,pt{+}qu)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\-1&k\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$$

for some $\left(\begin{smallmatrix} m & n \\ p & q \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ with $p \geqslant 0$ and some $k \in \mathbb{Z}$. Since $S^2 \times S^1 = N(0,\infty,\infty) = \big(D, (m,p), (m,p)\big) \bigcup_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & k \end{smallmatrix}\right)} \big(D, (2,1), (3,1)\big)$, we have $m = 0$, $p = 1$, and

$$S^2 \times S^1 = (S^2 \times S^1)\#\big(S^2, (2,1), (3,1), (1,k)\big).$$

Therefore $\big(S^2, (2,1{+}2k), (3,1)\big) = S^3$, which implies that $5 + 6k = \pm 1$, hence $k = -1$.

Having shown that $p = 1$, we have $n = \pm 1$. We prove by contradiction that $n = 1$. Assuming $n = -1$ we get

$$\begin{aligned} N(0,-2,t/u) &= \big(D, (1, 2{-}q), (-u, t{+}qu)\big) \bigcup_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)} \big(D, (2,1), (3,1)\big) \\ &= \big(S^2, (2,-1), (3,1), (t{+}2(q{-}1)u, -u)\big). \end{aligned}$$

Equation (26), now proved, shows that for all t/u this manifold must be equal to $\big(S^2, (2,-1), (3,2), (t{+}2u, -u)\big)$, which is clearly impossible whatever q.

Having shown that $n = 1$, we determine q noting that

$$\begin{aligned} N(0,1,1) &= \big(D, (1, 1{+}q), (1, 1{+}q)\big) \bigcup_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)} \big(D, (2,1), (3,1)\big) \\ &= \big(S^2, (2,1), (3,1), (2(1{+}q), 1), -1\big) \end{aligned}$$

contains a non-separating torus, hence $q = 2$, as required.

Conclusion for the slopes (1,1) Finally, we prove (29). We have

$$N\left(1,1,\frac{t}{u}\right) = \big(A, (mt{+}nu, pt{+}qu)\big) / \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$$

for some $\left(\begin{smallmatrix} m & n \\ p & q \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ with $p \geqslant 0$. First, $S^2 \times S^1 = N(1,1,\infty) = \big(A, (m,p)\big) / \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ gives $m = 0$ and $p = 1$, hence $n = \pm 1$. We also deduce from relations (25) to (28) and (20) to (23) that $N(1,1,\alpha) = T_{\left(\begin{smallmatrix} \alpha+1 & 1 \\ -1 & 0 \end{smallmatrix}\right)}$ for $\alpha \in \{-3,-2,-1,0\}$. Using (19) we easily get $q = 1$ and $n = 1$, as required. $\square$

We can now prove our main results.

Proof of 1.1, 1.2, and 1.3 The proof is divided in 4 steps:

1. we use Proposition 3.2 and Lemma 3.1 to recognize the JSJ decomposition of many non-hyperbolic fillings;
2. we apply the symmetries listed in Proposition 1.5 finding more non-hyperbolic fillings;

3. we use the Gromov-Thurston 2π-theorem to prove that M is negatively curved except in the cases listed (up to permutation) in Steps 1 and 2 and in a finite number of other cases. As already mentioned, the orbifold theorem shows that M is hyperbolic if it is negatively curved;

4. using SnapPea, we examine the finite number of cases not covered by the previous steps, and we check that the corresponding M is always hyperbolic.

Step 1: Recognition of non-hyperbolic fillings The equalities listed in Theorem 1.1 are deduced from Proposition 3.2. We now prove most of the equalities listed in Tables 1, 2, and 3, by studying the "degenerate" cases in which the expressions in Proposition 3.2 do not describe the JSJ decomposition of the manifold. We say that $\big(F, {\scriptstyle(i_1,j_1),\ldots,(i_k,j_k)}\big)$ is *genuine* if it is irreducible and not a solid torus. Degenerate cases occur mostly in presence of non-genuine blocks.

1-cusped fillings We first prove all the equalities listed in Table 1, except the last two. For $(p/q, r/s) = (1, 1)$ the equality follows from Proposition 3.2 directly. For $p/q \in \{-3, -2, -1, 0\}$ the argument is quite simple: we only spell it out here for $p/q = -3$, the other cases being even easier.

So, assume $p/q = -3$ and $r/s \in \mathbb{Q}$. By (25) we have

$$M = \big(D, {\scriptstyle(2,1)}, {\scriptstyle(r+s,\,s)}\big) \bigcup\nolimits_{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)} \big(A, {\scriptstyle(2,1)}\big).$$

this formula describes the JSJ decomposition of M except in the following cases:

(A) the block on the left-hand side is non-genuine;

(B) the block on the left-hand side admits an alternative fibration which is matched to the fibration of the block on the right-hand side.

Case (A) occurs for $|r + s| \leqslant 1$. When $|r + s| = 0$, *i.e.* $r/s = -1$, we have $M = \mathbb{RP}^3\#\big(D, {\scriptstyle(2,1)}\big) = \mathbb{RP}^3\#(D \times S^1)$ by (18). If $|r + s| = 1$, *i.e.* $r/s = -1 + \frac{1}{n}$ for some $n \in \mathbb{Z} \setminus \{0\}$, we have

$$M = \big(D, {\scriptstyle(1,\,1+2n)}\big) \bigcup\nolimits_{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)} \big(A, {\scriptstyle(2,1)}\big) = \big(D, {\scriptstyle(1+2n,\,-2)}, {\scriptstyle(2,1)}\big) = \big(D, {\scriptstyle(1+2n,\,2)}, {\scriptstyle(2,1)}\big)$$

by (14), (7), and (9), which is genuine except for $|1 + 2n| \leqslant 1$, which only occurs for $n = -1$, giving $D \times S^1$.

Using Lemma 3.1 it is not hard to see that case (B) occurs only for $r/s = -3$, in which case $M = \big(S, {\scriptstyle(2,1)}\big)$.

We now prove all the equalities in Table 2, and the equalities in Table 3 corresponding to the case $(p/q, r/s) = (1, 1)$

The slope -3 We start by considering the case where $p/q=-3$. Relation (25) then shows that M is homeomorphic to

$$\big(D,(2,1),(r{+}s,s)\big)\bigcup\nolimits_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(2,1),(t{+}u,u)\big) \tag{32}$$

The first case where the block on the left-hand side is non-genuine occurs when $r/s=-1$, in which case the block becomes $\big(D,(2,1),(0,1)\big)$ and we deduce from (18) and (11) that

$$M=\mathbb{RP}^3\#\big(S^2,(2,1),(t{+}u,u)\big).$$

Now we employ (10) to see that the second summand is $L(t{+}3u,u)$. Summing up, we have $M=\mathbb{RP}^3\#L(t{+}3u,u)$ when $r/s=-1$ (and $p/q=-3$).

Another case where the block on left-hand side of (32) is non-genuine occurs when $|r+s|=1$, namely when $r/s=-1+1/n$ for some $n\in\mathbb{Z}$, $n\neq 0$. In this case the block is $\big(D,(2,1),(1,n)\big)=\big(D,(2,1{+}2n)\big)$. We then use (14) to deduce that

$$M=\big(S^2,(2n{+}1,-2),(2,1),(t{+}u,u)\big)$$

in this case. Now this manifold is Seifert with 3 exceptional fibres unless $|2n+1|\leqslant 1$ or $|t+u|\leqslant 1$. The former inequality is only possible for $n=-1$ (recall that $n\neq 0$), *i.e.* for $r/s=-2$, and $M=\big(S^2,(2,1),(t{+}u,u),(1,2)\big)=\big(S^2,(2,5),(t{+}u,u)\big)$ in this case. Now we apply (10) to see that

$$M=L\big(2u{+}5(t{+}u),u{+}2(t{+}u)\big)=L(5t{+}7u,2t{+}3u).$$

Let us assume instead that $|t+u|\leqslant 1$. We dismiss the case $t/u=-1$ which was already considered up to permutation, so we have $t/u=-1+1/m$ for $m\in\mathbb{Z}$, $m\neq 0$, and $M=\big(S^2,(2,1),(1,m),(2m{+}1,-2)\big)=\big(S^2,(2,2m{+}1),(2m{+}1,-2)\big)$. We now apply (10), eventually getting

$$M=L\big((2n{+}1)(2m{+}1){-}4,(2n{+}1)m{-}2\big).$$

We also note that the lack of symmetry in n and m is only apparent, because

$$(2nm+m-2)(2nm+n-2)\equiv 1 \pmod{(2n+1)(2m+1)-4}.$$

We have now recognized M whenever r/s equals -1 or $-1+1/n$ (and $p/q=-3$). So we assume neither r/s nor t/u have this form, *i.e.* $|r+s|>1$ and $|t+u|>1$. So (32) describes the gluing of two genuine Seifert blocks, and the gluing matrix does not match the fibres of the fibrations implicit in the expressions of the blocks. Therefore (32) gives the JSJ decomposition of M except in the following cases (up to permutation):

(A) the block on left-hand side has an alternative fibration which is matched to the original fibration of the block on the right-hand side;

(B) both the blocks admit alternative fibrations, and these fibrations are matched.

Now the left-hand side has an alternative fibration precisely when $|r+s|=2$, namely when $r/s=-1+\frac{2}{2n+1}$ for some $n\in\mathbb{Z}$, and in this case we deduce from (12) and (15) that

$$\begin{aligned} M &= \big(D,(2,1),(2,2n{+}1)\big)\bigcup\nolimits_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(2,1),(t{+}u,u)\big)\\ &= \big(D,(2,1),(2,1)\big)\bigcup\nolimits_{\left(\begin{smallmatrix}-n&1\\1&0\end{smallmatrix}\right)}\big(D,(2,1),(t{+}u,u)\big)\\ &= \big(S\mathbin{\tilde{\times}}S^1\big)\bigcup\nolimits_{\left(\begin{smallmatrix}1&n+1\\0&-1\end{smallmatrix}\right)}\big(D,(2,1),(t{+}u,u)\big). \end{aligned}$$

Now case (A) occurs precisely when $n+1=0$, whence $r/s=-3$, and

$$M=\big(S\mathbin{\tilde{\times}}S^1\big)\bigcup\nolimits_{\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)}\big(D,(2,1),(t{+}u,u)\big).$$

Up to changing sign the gluing matrix is now properly matching both the sections and the fibres, so M is Seifert fibred over the gluing of S and D, which gives

$$M=\big(\mathbb{RP}^2,(2,1),(t{+}u,u)\big)$$

for $r/s=-3$ (and $p/q=-3$).

Assume case (B) occurs. Then t/u also has the form $-1+\frac{2}{2m+1}$, and by (13) and (16) we have

$$\begin{aligned} M &= \big(S\mathbin{\tilde{\times}}S^1\big)\bigcup\nolimits_{\left(\begin{smallmatrix}1&n+1\\0&-1\end{smallmatrix}\right)}\big(D,(2,1),(2,2m{+}1)\big)\\ &= \big(S\mathbin{\tilde{\times}}S^1\big)\bigcup\nolimits_{\left(\begin{smallmatrix}1&n+1\\m&mn+m-1\end{smallmatrix}\right)}\big(D,(2,1),(2,1)\big)\\ &= \big(S\mathbin{\tilde{\times}}S^1\big)\bigcup\nolimits_{\left(\begin{smallmatrix}m+1&m+n+mn\\-1&-n-1\end{smallmatrix}\right)}\big(S\mathbin{\tilde{\times}}S^1\big). \end{aligned}$$

Moreover the fibres should match, so $m+n+mn=0$. Therefore we have either $n=m=0$ or $n=m=-2$, namely either $r/s=t/u=1$ or $r/s=t/u=-5/3$. The two manifolds we get are

$$M=\big(S\mathbin{\tilde{\times}}S^1\big)\bigcup\nolimits_{\left(\begin{smallmatrix}-1&0\\\pm1&1\end{smallmatrix}\right)}\big(S\mathbin{\tilde{\times}}S^1\big)$$

which are easily seen to be $(K,\pm1)$, where K is the Klein bottle. Finally, we have $(K,+1)=(K,-1)$.

The slope -2 Let us consider now the case $p/q = -2$, where equation (26) shows that M is

$$\big(S^2, (3,2), (r+2s, -s), (t+2u, -u)\big).$$

There are three exceptional fibres unless $|r+2s| \leqslant 1$ (up to permutation). If $r/s = -2$, we have $M = \big(S^2, (3,2), (0,1), (t+2u, -u)\big)$, which is equal to $L(3,1)\#L(t+2u, u)$ by (17). If $|r+2s| = 1$, *i.e.* $r/s = -2+1/n$, then $M = \big(S^2, (3,2), (1,-n), (t+2u, -u)\big) = \big(S^2, (3, 2-3n), (t+2u, -u)\big)$. We apply (10), getting

$$M = L\big(3u-(2-3n)(t+2u), u-(1-n)(t+2u)\big) = L\big(3n(t+2u)-2t-u, n(t+2u)-t-u\big).$$

The slope -1 The case $p/q = -1$ is similar to the previous one. Relation (27) proves that M is

$$\big(S^2, (2,1), (r+3s, -s), (t+3u, -u)\big).$$

This manifold is Seifert fibred over S^2, and there are three exceptional fibres unless $|r+3s| \leqslant 1$ up to permutation. If $r/s = -3$ then $M = \big(S^2, (2,1), (1,0), (t+3u, -u)\big)$, which is equal to $\mathbb{RP}^3\#L(t+3u, u)$ by (17). If $|r+3s| = 1$, *i.e.* $r/s = -3+1/n$, we have $M = \big(S^2, (2,1), (1,-n), (t+3u, -u)\big) = \big(S^2, (2, 1-2n), (t+3u, -u)\big)$. Now we use (10), deducing that

$$M = L\big(-2u+(1-2n)(t+3u), u+n(t+3u)\big) = L\big(2n(t+3u)-t-u, n(t+3u)+u\big).$$

The slope 0 Let us turn now to the case $p/q = 0$, where (28) shows that M is

$$\big(D, (s, r+2s), (u, t+2u)\big) \bigcup_{\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}} \big(D, (2,1), (3,1)\big) \qquad (33)$$

The right-hand side block is now always genuine, but the left-hand side block is not if $|s| = 1$ (recall that $s \neq 0$), namely when $r/s = n \in \mathbb{Z}$. In this case the block becomes $\big(D, (1, n+2), (u, t+2u)\big) = \big(D, (u, t+(n+4)u)\big)$. Now we use (14) to see that

$$M = \big(S^2, (2,1), (3,1), (t+(4+n)u, -t-(3+n)u)\big) = \big(S^2, (2,-1), (3,1), (t+(4+n)u, u)\big).$$

So M is Seifert with three exceptional fibres unless $|t+(4+n)u| \leqslant 1$. If $t+(4+n)u = 0$, namely $t/u = -4-n$, we use (17) and see that $M = \mathbb{RP}^3\#L(3,1)$. If $|t+(4+n)u| = 1$, *i.e.* $t/u = -4-n+1/m$, we have $M = \big(S^2, (2,1), (3,1), (1, m-1)\big) = \big(S^2, (3,1), (2, 2m-1)\big)$. We eventually use (10) and see that $M = L(6m-1, 2m-1)$.

We consider now the case where $r/s, t/u \notin \mathbb{Z}$ in (33), so both the blocks involved are genuine Seifert, and the gluing matrix does not match the fibrations. In addition, the right-hand side block has a unique fibration, so the JSJ decomposition of M is given by

$$\big(D, (s, r+2s), (u, t+2u)\big) \bigcup_{\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}} \big(D, (2,1), (3,1)\big) \qquad (34)$$

unless the left-hand side block has an alternative fibration whose fibre is matched to the fibre of $(D,(2,1),(3,1))$. Assuming this happens, we must have $|s|=|u|=2$, so $r/s=n+1/2$ and $t/u=m+1/2$. By (12) and (15) we then have

$$\begin{aligned} M &= \big(D,(2,2n{+}5),(2,2m{+}5)\big)\bigcup\nolimits_{\begin{pmatrix}0&-1\\1&1\end{pmatrix}}\big(D,(2,1),(3,1)\big)\\ &= \big(D,(2,1),(2,1)\big)\bigcup\nolimits_{\begin{pmatrix}m+n+4&-1\\-m-n-3&1\end{pmatrix}}\big(D,(2,1),(3,1)\big)\\ &= \big(S\,\tilde{\times}\,S^1\big)\bigcup\nolimits_{\begin{pmatrix}-1&-m-n-5\\1&m+n+4\end{pmatrix}}\big(D,(2,1),(3,1)\big). \end{aligned}$$

Fibres match for $n+m+5=0$, *i.e.* $m=-n-5$. Summing up, we have shown that for $r/s,t/u\notin\mathbb{Z}$ the JSJ decomposition of M is given by (34) unless $r/s=n+1/2$ and $t/u=-n-9/2$. In this case we have, using (13) with $l=-1$ and switching the orientation of $S\,\tilde{\times}\,S^1$:

$$\begin{aligned} M &= \big(S\,\tilde{\times}\,S^1\big)\bigcup\nolimits_{\begin{pmatrix}-1&0\\1&-1\end{pmatrix}}\big(D,(2,1),(3,1)\big)\\ &= \big(S\,\tilde{\times}\,S^1\big)\bigcup\nolimits_{\begin{pmatrix}-1&0\\0&-1\end{pmatrix}}\big(D,(2,-1),(3,1)\big)\\ &= \big(S\,\tilde{\times}\,S^1\big)\bigcup\nolimits_{\begin{pmatrix}-1&0\\0&1\end{pmatrix}}\big(D,(2,-1),(3,1)\big)\\ &= \big(\mathbb{RP}^2,(2,1),(3,1),-1\big). \end{aligned}$$

The slopes (1,1) We now study the case $p/q=r/s=1$, where equation (29) shows that

$$M=\big(A,(u,t{+}u)\big)\Big/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}.$$

The block which is being glued to itself is non-genuine if $t/u=n\in\mathbb{Z}$. If this is the case the block becomes $\big(A,(1,n{+}1)\big)$ and we deduce from (12) and (19) that

$$M=\big(A\times S^1\big)\Big/_{\begin{pmatrix}-1-n&1\\1&0\end{pmatrix}}=T_{\begin{pmatrix}n+1&1\\-1&0\end{pmatrix}}$$

If $t/u\notin\mathbb{Z}$ then $\big(A,(u,t{+}u)\big)$ is genuine Seifert with a unique fibration and the self-gluing does not match the fibres, so the JSJ decomposition of M is as required.

Step 2: Recognition of more non-hyperbolic fillings As announced, we use now relations (1) to (6) to find triples which were not listed in Step 1 but give the same fillings as triples listed in Step 1. This will complete the proof of all

Original triple	Move	New triple
$-\frac{3}{2},-\frac{5}{2},\frac{t}{u}$	(3)	$-\frac{3}{2},0,-\frac{u}{t+2u}-2$
$-4,-\frac{1}{2},\frac{t}{u}$	inverse of (1)	$-\frac{3}{2},0,-\frac{t}{u}-3$
$-\frac{3}{2},-\frac{7}{3},-\frac{7}{3}$	(3)	$-\frac{3}{2},1,1$
$-4,-4,-\frac{2}{3}$	inverse of (1)	$-\frac{3}{2},1,1$
$-\frac{3}{2},-\frac{3}{2},-4$	(1)	$-4,1,1$
$-\frac{5}{2},-\frac{5}{2},-\frac{4}{3}$	(2)	$-\frac{5}{2},1,1$
$-\frac{1}{2},-5,-5$	(4)	$-\frac{1}{2},1,1$
$1,2,n$ $n=2,3,4,5$	(5)	$1,2,2-n$
$1,-4,-\frac{1}{3}$	(6)	$1,-4,-3$
$-\frac{3}{2},-\frac{3}{2},-\frac{8}{3}$	(1)	$-4,-\frac{1}{3},1$
$-\frac{5}{2},-\frac{5}{3},-\frac{5}{3}$	(2)	$-\frac{5}{2},-4,-\frac{1}{2}$

Table 16: Sporadic triples giving non-hyperbolic fillings

the equalities listed in Tables 1 to 4. We can immediately conclude with Table 1: Proposition 1.5 gives $N(-3/2,-5/2)=N(-3/2,0)=N(-4,-1/2)$, therefore the last two lines of Table 1 are deduced from the case $(p/q,r/s)=(0,-3/2)$ in the same table.

All equalities of table 2 have already been proved. We therefore turn to Tables 3 and 4. For the sake of conciseness and clarity, we have organized our constructions in Table 16. Let us first comment on the first nine lines of this table. Here we find on the first column a triple not found in Step 1 (except for $t/u\in\{-3,-2,-1,0\}$ in the first two lines). In the second column there is a reference to one of the symmetry relations listed in Proposition 1.5. We view this relation as a move acting on the triple to its left, and in the third column we give the triple resulting from this action; this triple is readily seen to be one of those of Step 1. The last two lines are similar, except that in the last column the triple is *not* one of those of Step 1. Instead, it appears in the first column of the ninth and second line respectively.

Let us now recognize M in all the lines listed in the first column of Table 16. Of course this is trivial for all the lines except the first two, because for the other cases we only have to use Step 1 for the triple on the last column (but in Table 4 we have often simplified the expression using formulae (11), (12), and (13), and changes of orientation).

Then, let us start from the case $-\frac{3}{2}, -\frac{5}{2}, \frac{t}{u}$, so M is the same as the manifold obtained with the filling coefficients $0, -\frac{3}{2}, -\frac{2t+5u}{t+2u}$. First, if $t+2u = 0$, *i.e.* $t/u = -2$, we easily get from the case $p/q = \infty$ that $M = L(2,1) = \mathbb{RP}^3$. Otherwise we have to refer to the case $p/q = 0$. Now $-\frac{2t+5u}{t+2u}$ is an integer when $|t+2u| = 1$, *i.e.* $t/u = -2 + 1/n$, and in this case $-\frac{2t+5u}{t+2u} = -n-2$. We further need to make distinctions depending on whether

$$-3 + (4 - (n+2))2 = -2n + 1$$

has value 0 or ± 1. Of course 0 cannot be attained, and $+1$ also cannot (because $n \neq 0$), so the only special case occurs for $n = 1$, namely for $t/u = -1$, and $M = L(13,5)$, as we knew already from the discussion of slope -1. We then deduce that for $t/u = -2 + 1/n$ but $n \neq 1$ we have

$$M = \big(S^2, (2,-1), (3,1), (3-(4-n-2)2, -2)\big) = \big(S^2, (2,-1), (3,1), (2n-1, -2)\big).$$

When $-\frac{2t+5u}{t+2u}$ is neither an integer nor infinity, *i.e.* when t/u is not -2 or $-2+1/n$, we will have, after easy calculations,

$$M = \big(D, (2,1), (3,1)\big) \bigcup_{\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}} \big(D, (2,1), (t+2u, u)\big)$$

except in the special case where $-\frac{2t+5u}{t+2u} = -(-\frac{3}{2}) - 4$, which occurs for $t/u = 0$, and

$$M = \big(\mathbb{RP}^2, (2,1), (3,1), -1\big)$$

in this case, as we knew already.

The discussion of the cases $-4, -1/2, t/u$ is easier, because it reduces to that of $0, -3/2, -t/u - 3$. Namely, for $t/u = n$ we have to consider the special cases $n = 0, -1$, leading to $L(13,5)$ and $L(11,3)$ respectively (as we knew already), otherwise

$$M = \big(S^2, (2,-1), (3,1), (2n+1, -2)\big).$$

Moreover we have the exceptional case where $-t/u - 3 = -(-3/2) - 4$, which occurs for $t/u = -1/2$ and leads to

$$M = \big(\mathbb{RP}^2, (2,1), (3,1), -1\big).$$

In all other cases we have

$$M = \big(D, (2,1), (3,1)\big) \bigcup_{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}} \big(D, (2,1), (u,t)\big).$$

Now that we have recognized the M's arising from the triples on the first column of Table 16, we could show, with a little patience, that even a repeated application of the symmetries (1) to (6) to any of the triples found so far does not lead to new triples. This will actually be deduced from the rest of our argument, because we will see that all the triples, except those listed so far (up to permutation), give a hyperbolic M.

Step 3: Application of the 2π-theorem As announced, we will now use the Gromov-Thurston 2π-theorem [2] to show that a Dehn filling of N is negatively curved for all fillings except those listed in Steps 1 and 2 and a finite number of other ones. More precisely, let us define $\mathcal{G}$ as the set of all slopes, pairs of slopes, and triples of slopes that were proved in Steps 1 and 2 to give a non hyperbolic filling of N. The aim of Step 3 is to construct three finite sets of slopes S_1, S_2, S_3 such that *if α, β, γ are slopes on ∂N and $N(\alpha, \beta, \gamma)$ is not negatively curved then either $(\alpha, \beta, \gamma) \in \mathcal{G}$ or, up to permutation, $(\alpha, \beta, \gamma) \in S_1 \times S_2 \times S_3$.*

We begin by recalling the statement of the 2π-theorem, so we consider a cusped hyperbolic manifold M and a horospherical cusp section H of M, that is an embedded surface parallel to ∂M such that the metric of M restricts to a Euclidean metric on H. Then $H = \sqcup_{i=1}^k T_i'$, where T_i' is a Euclidean torus isotopic to the i-th component T_i of ∂M. If α_i is a slope on T_i we now define the length of α_i with respect to H (or H-length for short) as the shortest length of a curve in T_i' parallel to α_i. The Gromov-Thurston theorem now asserts that $M(\alpha_{i_1}, \ldots, \alpha_{i_h})$ is negatively curved whenever all filling slopes α_{i_j} have H-length larger than 2π.

Before proceeding we note that if a homology basis (a, b) is fixed on a torus T, the Euclidean structures on T up to isotopy are determined by two parameters $x + iy \in \mathbb{C}$ with $y > 0$ and $A \in \mathbb{R}$ with $A > 0$. Here $x + iy$ gives the *shape* of T (*i.e.* the Euclidean structure up to scaling) while A gives the area. The relation with the fixed basis (a, b) comes from the fact that there should exist a locally isometric covering $\mathbb{C} \to T$ on which a and b act as the translations $z \mapsto z + \sqrt{A/y}$ and $z \mapsto z + (x + iy)\sqrt{A/y}$ respectively. It is now an easy exercise to show that the minimal length $\ell(p/q)$ of a slope $pa + qb$ on T satisfies

$$\ell(p/q)^2 = \frac{A}{y}\big((p + xq)^2 + (yq)^2\big). \tag{35}$$

We now specialize to our chain-link exterior N and note that it has symmetries realizing any permutation of the boundary components, so the shapes of its three cusps are actually the same. We describe this common shape using the homology bases (a_i, b_i) chosen by SnapPea, where $a_i = \mu_i$ and $b_i = 2\mu_i - \lambda_i$ for $i = 1, 2, 3$, with (μ_i, λ_i) being the meridian-longitude basis used above. The shape is then described by the parameter $1/2+i\sqrt{7}/2$. Moreover, if we choose a maximal horospherical cusp section H consisting of three tori having the same area, this area turns out to be $\sqrt{7}/2$. Using this information and (35) it is now easy to prove that the slopes on any component of ∂N having H-length less that 2π are those in the following set:

$$\begin{aligned}\tilde{S}_1 &= \Big\{\infty,-6,-5,-4,-3,-\frac{5}{2},-2,-\frac{5}{3},-\frac{3}{2},-\frac{4}{3},-\frac{5}{4},-1,\\ &\quad -\frac{3}{4},-\frac{2}{3},-\frac{1}{2},-\frac{1}{3},-\frac{1}{4},0,\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},1,\frac{3}{2},2,3,4,5\Big\}.\end{aligned}$$

Now we remark that $\tilde{S}_1 \cap \mathcal{G} = \{\infty,-2,-1,0,1\}$ (recall that the homology bases are changed: a slope previously corresponding to p/q is now given by $-2-p/q$, and conversely). And we set $S_1 = \tilde{S}_1 \setminus \mathcal{G}$, noting that S_1 has 23 elements. Using SnapPea it is now easy to check that $N(\alpha)$ is actually hyperbolic for all $\alpha \in S_1$. Moreover, since there is a symmetry of N fixing any boundary component and exchanging the other two, the two cusps of $N(\alpha)$ have the same shape given by a certain $z_\alpha \in \mathbb{C}$. Let us consider as above the maximal cusp section H_α in $N(\alpha)$ consisting of two tori of the same area, and let us denote by A_α this area. Again using SnapPea one can determine z_α and A_α for all $\alpha \in S_1$, and correspondingly compute, via (35), the set $\tilde{S}_2(\alpha)$ of slopes on (any component of) $\partial N(\alpha)$ having H_α-length less than 2π. We now define

$$S_2(\alpha) = \big\{\beta \in \tilde{S}_2(\alpha) : \ (\alpha,\beta) \notin \mathcal{G}\big\}, \qquad S_2 = \bigcup\nolimits_{\alpha\in S_1} S_2(\alpha).$$

This set S_2 consists of 55 elements. Again using SnapPea one can now check that $N(\alpha,\beta)$ is actually hyperbolic for all $(\alpha,\beta) \in (S_1 \times S_2) \setminus \mathcal{G}$, and compute the shape $z_{\alpha,\beta} = x_{\alpha,\beta} + iy_{\alpha,\beta}$ and the area $A_{\alpha,\beta}$ of the maximal cusp $H_{\alpha,\beta}$ of $N(\alpha,\beta)$. With the list of actual values at hand one can check that

$$-1 \leqslant x_{\alpha,\beta} \leqslant 4, \quad 0.5 \leqslant y_{\alpha,\beta} \leqslant 4, \quad A_{\alpha,\beta} \geqslant 1 \qquad \forall(\alpha,\beta) \in (S_1 \times S_2) \setminus \mathcal{G}$$

(the last inequality is actually known from the theory, but one can also check it experimentally). Now let p/q be a slope on $\partial N(\alpha,\beta)$. Up to switching signs we can assume $q \geqslant 0$, whence, by (35) and the previous inequalities,

$$\ell(p/q)^2 = \frac{A_{\alpha,\beta}}{y_{\alpha,\beta}}\big((p+q\cdot x_{\alpha,\beta})^2 + (q\cdot y_{\alpha,\beta})^2\big) \geqslant \frac{1}{4}\big(f(p,q)^2 + q^2\cdot(0.5)^2\big) =: g(p,q)$$

where $f(p,q)$ is 0 if $0 \in (p-q, p+4q)$, and $\min\{|p-q|, |p+4q|\}$ otherwise. We then define $S_3 = \{p/q : \ g(p,q) \leqslant 4\pi^2\}$, and note that S_3 has 200 elements.

Having constructed S_1, S_2, S_3, let us check that they work. If $N(\alpha,\beta,\gamma)$ is not negatively curved then, up to permutation, we have $\alpha \in \tilde{S}_1$ by the 2π-theorem. So either $\alpha \in S_1$ or $\alpha \in \mathcal{G}$. And $(\alpha, \beta, \gamma) \in \mathcal{G}$ in the latter case. Assuming $\alpha \in S_1$, we know $N(\alpha)$ is hyperbolic, so $\beta \in \tilde{S}_2(\alpha)$ up to permutation by the 2π-theorem. So either $\beta \in S_2$, or $(\alpha, \beta) \in \mathcal{G}$, or both. And $(\alpha, \beta, \gamma) \in \mathcal{G}$ if $(\alpha, \beta) \in \mathcal{G}$. Assuming $(\alpha, \beta) \notin \mathcal{G}$ we know that $N(\alpha, \beta)$ is hyperbolic, so $\gamma \in S_3$ by the 2π-theorem.

Step 4: Finite enumeration To conclude we are left to investigate all the $23 \cdot 55 \cdot 200 = 253000$ triples of slopes in $S_1 \times S_2 \times S_3$, and check that those not lying in $\mathcal{G}$ define hyperbolic fillings. We have done this using the scriptable Python version of SnapPea. Many of the triples in $(S_1 \times S_2 \times S_3) \setminus \mathcal{G}$ turned out to give solutions of the hyperbolic Dehn filling equations with some inverted tetrahedra. For these triples we have tried with several alternative triangulations of N and of some partial fillings of N, getting a genuine solution after some attempts. The only triple resisting to this method was $\{1, -5, -1/2\}$, which is known [11] to give the closed hyperbolic manifold with volume 1.0149416... (and complexity 9). $\square$

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