A Note on Codimension-1 Foliations

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SUNTO: Sia γ una curva semplice chiusa tangente ad una foliazione $\mathcal{F}$ coorientata di codimensione 1 su una qualsiasi varietà, e sia h l'olonomia di γ. Si esibisce qui una parametrizzazione di un intorno di γ nella quale $\mathcal{F}$ è definita da una 1-forma differenziale i cui coefficienti hanno proprietà globali intimamente legate a quelle di h. Tale parametrizzazione ha inoltre una classe di regolarità superiore a quella che ci si potrebbe attendere in generale.

ABSTRACT: Let γ be a simple closed curve tangent to a cooriented codimension-1 foliation of a manifold, and let h be the holonomy of γ. We provide here a parametrization of a neighbourhood of γ in which $\mathcal{F}$ is defined by a differential 1-form whose coefficients satisfy global properties closely related to the properties satisfied by h. Moreover this parametrization is more regular than one could in general expect it to be.

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The aim of this note is to illustrate a geometric argument which allows, given a germ h of C^k diffeomorphism of $\mathbb{R}$ at 0, to construct a "canonical" model for a C^k codimension-1 foliation near a curve having holonomy h. We regard our model to be canonical because various properties of h have precise counterparts on the coefficients of the form defining the foliation in the model. Recall that in general the holonomy is obtained by "integrating" in a suitable sense the coefficients, so no information on h can lead to a pointwise information on the coefficients of the form.

The motivation for searching for such a model was to understand one of the steps of the proof of a recent result of Eliashberg and Thurston ([1], see also [4]), according to which every foliation on a closed oriented 3-manifold can be approximated by a contact structure. The property of the holonomy which is crucial for [1] is that of being two-sided weakly-contracting, but we think it is of some interest that the same model of foliation reflects several other properties of the holonomy.

Another remarkable feature of our model is the following. Recall that in general, given a C^k foliation and a C^k (or more) chart on a manifold, the foliation is defined in the chart by a differential form which is only C^{k-1}. However in our model the form happens to be C^k.

1 Statement and comments

For all the relevant definitions and basic facts concerning foliations we address the reader to the well-established references [2] and [3]. The technical result proved in this note is the following:

Proposition 1.1. *Let h be a germ at 0 of increasing C^k diffeomorphism of $\mathbb{R}$ with $h(0)=0$. Then we can (constructively) define a germ $\mathcal{F}_h$ of foliation on $S^1_x \times \mathbb{R}_z$ near $S^1_x \times \{0\}$, induced by a form $\mathrm{d}z + a(x,z)\,\mathrm{d}x$, such that a is a C^k-regular function (hence $\mathcal{F}_h$ is a C^k-regular foliation) and $S^1_x \times \{0\}$ is a leaf of $\mathcal{F}_h$ with holonomy h. Moreover:*

1. *If $h'(0)=1$ then $(\partial a/\partial z)(x,0)=0$ for all x;*
2. *If $h'(0)<1$ then $(\partial a/\partial z)(x,0)>0$ for all x;*
3. *If $h(z)<z$ for all $z>0$ then $a(x,z)>0$ for $z>0$ and all x;*
4. *If $h(\zeta)<\zeta$ for some $\zeta>0$ then $a(x,\zeta)>0$ for all x;*
5. *If $h(z')-h(z)<z'-z$ for all $0<z<z'$, then for all x the function $z \mapsto a(x,z)$ is strictly increasing for $z \geq 0$.*

To apply this result, we note that if we have an embedded loop γ contained in a leaf of a C^k cooriented codimension-1 foliation $\mathcal{F}$ on a C^k n-manifold, then we can C^k parametrize a neighbourhood of γ as $S^1_x \times \mathbb{R}^{n-2}_y \times \mathbb{R}_z$ in such a way that $\mathcal{F}$ in these coordinates is invariant under translations in the y-direction. Moreover the leaf $S^1_x \times \{0\}$ in the induced foliation on $S^1_x \times \mathbb{R}_z$ still has holonomy h. Now we recall that the holonomy determines the germ of a

C^k foliation near a compact leaf up to C^k diffeomorphism. As a consequence we get:

Theorem 1.2. *Let $\mathcal{F}$ be a C^k cooriented codimension-1 foliation on an n-manifold and let γ be an embedded closed curve tangent to $\mathcal{F}$. Then a neighbourhood of γ can be C^k-parametrized as $S^1_x \times \mathbb{R}^{n-2}_y \times \mathbb{R}_z$, and in these coordinates $\mathcal{F}$ is the kernel of a form $\mathrm{d}z + a(x,z)\,\mathrm{d}x$ where a is C^k and:*

1. *If γ has trivial linear holonomy then $(\partial a/\partial z)(x,0) = 0$ for all x;*
2. *If γ has contracting linear holonomy then $(\partial a/\partial z)(x,0) > 0$ for all x;*
3. *If γ has contracting holonomy on the positive side then $a(x,z) > 0$ for $z > 0$ and all x;*
4. *If γ has weakly contracting holonomy on the positive side then there exists $\{\zeta_n\}$ such that $\zeta_n > 0$, $\lim_n \zeta_n = 0$ and $a(x,\zeta_n) > 0$ for all x;*
5. *If the holonomy of γ is contracting in a metric sense on the positive side then for all x the map $z \mapsto a(x,z)$ is strictly increasing for $z \geq 0$.*

Remark 1.3. Properties 3-5 could be proved for the negative side, and properties 2-5 have natural analogues with 'expanding' replacing 'contracting'.

Remark 1.4. Recall that in general a C^k foliation is globally the kernel of a 1-form which is only C^{k-1} (but locally the form can be chosen to be constant in foliated coordinates). So under a C^k parametrization of a set which is not a ball one can only expect in general to get a C^{k-1} form. Therefore it is a non-obvious fact that in the C^k parametrization given by the theorem the foliation is defined by a C^k-form. We regard this fact as a manifestation of the canonicity of our model.

2 Geometric construction and proof

Fix the notation of Proposition 1.1. Recall that we want a "preferred" C^k foliation $\mathcal{F}_h$ of $S^1_x \times \mathbb{R}_z$ with holonomy h. The geometric idea to get it goes as follows. We consider the universal cover $\mathbb{R}_x$ of S^1_x acted on by $\mathbb{Z}$, and we foliate $\mathbb{R}_x \times \mathbb{R}_z$ in such a way that each segment joining a point (n,z) with

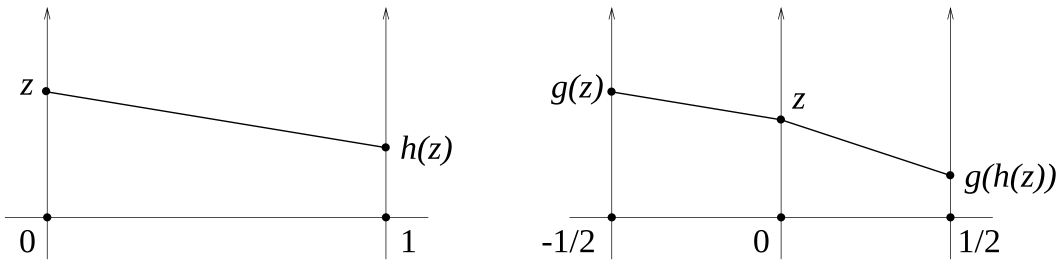

Figure 1: The unregularized foliation in the universal cover

a point $(n+1, h(z))$ is contained in a leaf. This gives rise to a foliation of $S^1_x \times \mathbb{R}_z$ which has holonomy h, but is not even C^1. To obtain the desired foliation then we regularize the leaves of the foliation of $\mathbb{R}_x \times \mathbb{R}_z$ by taking the convolution with a symmetric bell function supported in $[-1/4, 1/4]$. It takes some efforts to prove that this indeed provides a C^k foliation on $S^1_x \times \mathbb{R}_z$, and to establish the desired properties; we will skip some of the calculations, concentrating on the key points.

We first need to formalize the construction of $\mathcal{F}_h$. By simplicity we extend h to a diffeomorphism of $\mathbb{R}$, so that the construction becomes global (but of course the relevant properties only have to be checked in the sense of germs).

We first examine the foliation $\mathcal{G}_h$ before taking the convolution. By definition, in the slice $[0,1]_x \times \mathbb{R}_z$ of the universal cover, the leaves we see are as shown on the left-hand side of Fig. 1. This implies that in the slice $[-1/2, 1/2]_x \times \mathbb{R}_z$ we see what is shown on the right-hand side of the same figure, where g is the diffeomorphism of $\mathbb{R}$ given by $g(z) = (h^{-1}(z) + z)/2$.

Let us define for $w \in \mathbb{R}$ a map $\ell_w : \mathbb{R} \to \mathbb{R}$ as

$$\ell_w(x) = \begin{cases} z + x(z - h^{-1}(z)) & \text{if } x \leq 0 \\ z + x(h(z) - z) & \text{if } x \geq 0 \end{cases}, \qquad \text{where } z = g^{-1}(w).$$

Note that the holonomy of $\mathcal{G}_h$ with basepoint $-1/2$ is given by $\tilde{h}(w) = \ell_w(1/2)$; one easily sees that $\tilde{h} = g \circ h \circ g^{-1}$, which is coherent with the fact that the holonomy of a curve is well-defined only up to conjugation.

Consider now a smooth non-negative even real function $u : \mathbb{R} \to \mathbb{R}$ supported in $[-1/4, 1/4]$, with $u'(t) \leq 0$ for $t \geq 0$ and $\int_{-\infty}^{\infty} u(t)\,\mathrm{d}t = 1$, and define

$$L_w(x) = \int_{-\infty}^{\infty} u(t-x)\ell_w(t)\,\mathrm{d}t.$$

If $w > w'$ we have $\ell_w(t) > \ell_{w'}(t)$ for all t, hence $L_w(x) > L_{w'}(x)$ for all x. This easily implies that by projecting to $S^1_x \times \mathbb{R}_z$ the sets $\{L_w(x) : \ |x| \leq 1/2\}$ we get a C^0-foliation $\mathcal{F}_h$.

Now note that for $|x| \geq 1/4$ the map ℓ_w has the form $t \mapsto at + b$ on $[x - 1/4, x + 1/4]$; using the properties of u one sees that also $L_w(x) = ax + b$. So $L_w(x) = \ell_w(x)$ for $|x| \geq 1/4$. This implies that the holonomy of $S^1_x \times \{0\}$ in $\mathcal{F}_h$ is still $\tilde{h}$ (i.e. h, up to conjugation). Moreover each individual leaf of $\mathcal{F}_h$ is C^∞ regular.

Now we have to deal with the regularity of $\mathcal{F}_h$. To this end we compute the form $\mathrm{d}z + a(x, z)\,\mathrm{d}x$ which defines $\mathcal{F}_h$. Since $a(x, z)$ is the opposite of the slope of the leaf through (x, z) we have for x in a neighbourhood of $[-1/2, 1/2]$ and for all w that

$$a(x, L_w(x)) = -\frac{\mathrm{d}L_w}{\mathrm{d}x}(x) = \int_{-\infty}^{\infty} u'(t - x)\ell_w(t)\,\mathrm{d}t.$$

Now we define $\Psi_x(w) = L_w(x)$ and we claim that:

I. $(x, w) \mapsto \Psi_x(w)$ is C^k regular;

II. $\Psi'_x(0) \neq 0$ for all x.

Assume for a moment these claims to be proved. Then Ψ_x has a local inverse Φ_x near 0 and the map $(x, z) \mapsto \Phi_x(z)$ is C^k-regular because locally

$$\Big((x, z) \mapsto (x, \Phi_x(z))\Big) = \Big((x, w) \mapsto (x, \Psi_x(w))\Big)^{-1}$$

by the implicit function theorem. Moreover with easy computations

$$\begin{aligned} a(x, z) &= \int_{-\infty}^{0} u'(t - x)\Big(g^{-1}(\Phi_x(z)) + 2t(g^{-1}(\Phi_x(z)) - \Phi_x(z))\Big)\,\mathrm{d}t \\ &+ \int_{0}^{\infty} u'(t - x)\Big(g^{-1}(\Phi_x(z)) + 2t(\tilde{h}(\Phi_x(z)) - g^{-1}(\Phi_x(z)))\Big)\,\mathrm{d}t \end{aligned}$$

which immediately implies that a is jointly C^k-regular (at least near $z = 0$, which is sufficient for us). Now we prove the claims. Claim I immediately follows from the formula:

$$\begin{aligned} \Psi_x(w) &= \int_{-\infty}^{0} u(t - x)\Big(g^{-1}(w) + 2t(g^{-1}(w) - w)\Big)\,\mathrm{d}t \\ &+ \int_{0}^{\infty} u(t - x)\Big(g^{-1}(w) + 2t(\tilde{h}(w) - g^{-1}(w))\Big)\,\mathrm{d}t. \end{aligned}$$

To prove claim II we consider the Taylor expansion $h(z) = cz + \mathrm{o}(z)$, which implies that

$$\begin{aligned}
g(z) &= \frac{1+c}{2c} z + \mathrm{o}(z), \qquad g(h(z)) = \frac{1+c}{2} z + \mathrm{o}(z), \\
\Psi_x'(0) &= \lim_{w\to 0} \frac{\Psi_x(w)}{w} = \lim_{z\to 0} \frac{\Psi_x(g(z))}{z} \frac{z}{g(z)} = \frac{2c}{1+c} \lim_{z\to 0} \frac{\Psi_x(g(z))}{z}.
\end{aligned}$$

Moreover:

$$\begin{aligned}
\lim_{z\to 0} \frac{\Psi_x(g(z))}{z} &= \int_{-\infty}^0 u(t-x) \lim_{z\to 0} \frac{z + 2t(z - g(z))}{z} \,\mathrm{d}t \\
&\quad + \int_0^\infty u(t-x) \lim_{z\to 0} \frac{z + 2t(g(h(z)) - z)}{z} \,\mathrm{d}t \\
&= \int_{-\infty}^0 u(t-x) \left(1 + 2t\left(1 - \frac{1+c}{2c}\right)\right) \,\mathrm{d}t + \\
&\quad + \int_0^\infty u(t-x) \left(1 + 2t\left(\frac{1+c}{2} - 1\right)\right) \,\mathrm{d}t \\
&= 1 + \frac{c-1}{c} \int_{-\infty}^0 u(t-x) t \,\mathrm{d}t + (c-1) \int_0^\infty u(t-x) t \,\mathrm{d}t.
\end{aligned}$$

Now note that $\int_{-\infty}^\infty u(t-x) t \,\mathrm{d}t = x$. This easily implies that we can rewrite $\frac{(1+c)}{2c} \Psi_x'(0)$ in the following two ways:

$$\begin{aligned}
&1 + (c-1)x - \frac{(c-1)^2}{c} \int_{-\infty}^0 u(t-x) t \,\mathrm{d}t \\
&1 + \frac{c-1}{c} x + \frac{(c-1)^2}{c} \int_0^\infty u(t-x) t \,\mathrm{d}t.
\end{aligned}$$

In both expressions the third summand is strictly positive; moreover

- if $c \le 1$ then $1 + (c-1)x \ge 0$ for all $x \in [-1/2, 1/2]$ (recall that $c > 0$);
- if $c \ge 1$ then $1 + \frac{c-1}{c} x \ge 0$ for all $x \in [-1/2, 1/2]$.

This immediately implies claim II and hence the regularity of a.

Now we turn to properties 1-5, starting from number 3, which is the easiest one. Using the fact that u is an even function and the expression of a one easily gets

$$a(x, L_w(x)) = \int_0^\infty u'(t) \Big(\ell_w(x+t) - \ell_w(x-t)\Big) \,\mathrm{d}t$$

whence the conclusion because $u'(t) \leq 0$ for $t \geq 0$ and ℓ_w is a strictly decreasing function for $w > 0$ under the assumption that h is contracting.

Next, we prove property 4, which is the most important one in view of the applications to [1]. We define $\zeta_0 = \zeta$ and $\zeta_1 = \tilde{h}^{-1}(\zeta_0)$, so ζ_1 satisfies the equivalent relations $\ell_{\zeta_1}(1/2) = \zeta_0$ and $L_{\zeta_1}(1/2) = \zeta_0$. We claim that:

(i) For all $x \in [-1/2, 1/2]$ there exists $w \in [\zeta_0, \zeta_1]$ such that $L_w(x) = \zeta_0$;

(ii) For all $w \in [\zeta_0, \zeta_1]$ and $x \in \mathbb{R}$, we have $\frac{\mathrm{d}L_w}{\mathrm{d}x}(x) < 0$.

Recalling the expression of a, these claims immediately imply the conclusion.

To show (i) we will check that:

(i-a) $L_{\zeta_0}(x) \leq \zeta_0$ for all $x \in [-1/2, 1/2]$;

(i-b) $L_{\zeta_1}(x) \geq \zeta_0$ for all $x \in [-1/2, 1/2]$;

This is sufficient to establish (i) because we know that the map $w \mapsto L_w(x)$ is continuous and increasing for any fixed x. Since L_{ζ_0} is obtained by taking the average of ℓ_{ζ_0}, to show (i-a) and (i-b) it is sufficient to prove that $\ell_{\zeta_0}(x) \leq \zeta_0$ and $\ell_{\zeta_1}(x) \geq \zeta_0$ for all x. Recall that ℓ_w is the union of the two line segments joining the points

$$(-1/2, w), \qquad (0, g^{-1}(w)), \qquad (1/2, (g \circ h \circ g^{-1})(w)).$$

Therefore to prove (i-a) and (i-b) one has to show the following inequalities:

$$\begin{array}{llll} \text{(i-a-1)} & g^{-1}(\zeta_0) < \zeta_0; & \text{(i-a-2)} & (g \circ h \circ g^{-1})(\zeta_0) < \zeta_0; \\ \text{(i-b-1)} & \zeta_1 > \zeta_0; & \text{(i-b-2)} & g^{-1}(\zeta_1) > \zeta_0. \end{array}$$

Now one easily sees that (i-a-1) is equivalent to $h(\zeta_0) < \zeta_0$ which is true by assumption. For (i-a-2) we note that $g \circ h$ is increasing, so from (i-a-1) we get

$$(g \circ h \circ g^{-1})(\zeta_0) < (g \circ h)(\zeta_0) = (\zeta_0 + h(\zeta_0))/2 < \zeta_0.$$

The inequalities (i-b-1) and (i-b-2) are proved essentially in the same way.

Now we turn to point (ii). Let us recall that the convolution with u of a strictly decreasing function has negative derivative, So it will be sufficient to show that ℓ_w is strictly decreasing in x for $w \in [\zeta_0, \zeta_1]$. Recalling again the definition of ℓ_w one sees that it is sufficient to establish the following inequalities:

$$\begin{array}{rl} \text{(I)} & w > g^{-1}(w) \quad \text{for all } w \in [\zeta_0, \zeta_1], \\ \text{(II)} & g^{-1}(w) > (g \circ h \circ g^{-1})(w) \quad \text{for all } w \in [\zeta_0, \zeta_1]. \end{array}$$

We show (II), the proof of (I) being similar. From above it follows that

$$h(\zeta_0) < g^{-1}(\zeta_0) < \zeta_0 < g^{-1}(\zeta_1) < h^{-1}(\zeta_0).$$

So (II) is implied by the inequality $z > (g \circ h)(z)$ for all $z \in [h(\zeta_0), h^{-1}(\zeta_0)]$; using the formula for g we see that this inequality is equivalent to $z > h(z)$ for all $z \in [h(\zeta_0), h^{-1}(\zeta_0)]$. Now note that h is a continuous monotonic function, so it maps $[h(\zeta_0), \zeta_0]$ to $[h^2(\zeta_0), h(\zeta_0)]$, and $[\zeta_0, h^{-1}(\zeta_0)]$ to $[h(\zeta_0), \zeta_0]$. This implies that $z > h(z)$ respectively for $z \in [h(\zeta_0), \zeta_0]$ and for $z \in [\zeta_0, h^{-1}(\zeta_0)]$, whence the desired inequality. This proves property 4.

Now we turn to properties 1, 2 and 5. In the rest of the proof we will omit most of the computations. We start by remarking that

$$\begin{array}{rcl} a(0, \Psi_0(g(z))) & = & (g^{-1}(z) - g(z))/2 \\ a(-1/2, \Psi_{-1/2}(g(z))) & = & a(-1/2, g(z)) = 2(g(z) - z) = h^{-1}(z) - z \\ a(1/2, \Psi_{1/2}(g(z))) & = & a(1/2, g(h(z))) = 2(z - g(h(z))) = z - h(z) \end{array}$$

(only the first equality requires some work, the other two are immediate). Now, using the definitions and some lengthy computations (in particular the identity $\int_{-\infty}^{+\infty} u'(t-x)t \, \mathrm{d}t = -1$ for all x) we can show that $a(x, \Psi_x(g(z)))$ can be rewritten in the following two ways:

$$\begin{array}{l} (1 - c_+(x)) \cdot a(1/2, \Psi_{1/2}(g(z))) + c_+(x) \cdot a(0, \Psi_0(g(z))), \\ (1 - c_-(x)) \cdot a(-1/2, \Psi_{-1/2}(g(z))) + c_-(x) \cdot a(0, \Psi_0(g(z))) \end{array}$$

with

$$c_+(x) = -2 \int_{-\infty}^{0} u'(t-x)t \, \mathrm{d}t, \qquad c_-(x) = -2 \int_{0}^{+\infty} u'(t-x)t \, \mathrm{d}t.$$

Now $c_+(0) = 1$ and $c_+(1/2) = 0$, and for $x \geq 0$ one sees that $c_+'(x) \leq 0$. So $c_+(x) \in [0, 1]$ for $x \in [0, 1/2]$. In particular for $x \in [0, 1/2]$ the first of the above expressions of $a(x, \Psi_x(g(z)))$ is a convex combination of two functions of z. Similarly for $x \in [-1/2, 0]$ the second expression of $a(x, \Psi_x(g(z)))$ is a convex combination of two functions of z.

Now we can give a unified proof of properties 1, 2, 5. First of all the map $z \mapsto \Psi_x(g(z))$ has positive derivative in 0, so it is sufficient to prove the conclusions for $z \mapsto a(x, \Psi_x(g(z)))$ rather than for $z \mapsto a(x, z)$. Next one shows that the assumption on h allows to prove the desired conclusions for the three functions $z \mapsto a(x, \Psi_x(g(z)))$ for $x = 0$, $x = 1/2$ and $x = -1/2$ (this is rather easy for each of the properties 1, 2 and 5). To conclude it is then sufficient to use the fact that $z \mapsto a(x, \Psi_x(g(z)))$ can always be expressed as a convex combination of two of these three functions, noting that the properties are preserved by convex combinations. This concludes the proof.

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