From Enumerative Geometry to Representation Theory, and back via Cohomological Hall algebras

(Slides available on https://web.dm.unipi.it/sala/talks_trips/)

1. Overview

There are several instances of the interplay between

Algebraic Geometry	Representation Theory
(moduli spaces, stacks, enumerative invariants, ...)	(Quantum groups, vertex algebras, ...)

First part of the talk:

Alday-Gaiotto-Tachikawa conjectures concerning 'geometric' representations of vertex algebras (Schiffmann-Vasserot, Maulik-Okounkov)

Geometry $\longleftrightarrow$ Repr. Theory

$r=1$ $\quad M(n) = \mathrm{Hilb}^n(\mathbb{C}^2) \longleftrightarrow \mathrm{Heis}$

(Hilbert scheme) $\qquad$ (Heisenberg algebra)

arbitrary r $\quad M(r,n) \longleftrightarrow \mathcal{W}_k(\mathfrak{gl}(r))$

(affine W-algebra)

$\Uparrow$

Maulik-Okounkov affine Yangian $Y^{MO}_{1\text{-loop}}$ of $\mathfrak{gl}(1)$

The second part focuses on :

$$\boxed{\mathbb{C}^2 \rightsquigarrow \mathbb{C}^2/\mathbb{Z}_2 \longleftarrow T^*\mathbb{P}^1}$$

$$\mathbb{Z}_2 \hookrightarrow SL(2,\mathbb{C}) \,, \quad e^{\pi i} \longmapsto \begin{pmatrix} e^{\pi i} & 0 \\ 0 & e^{-\pi i} \end{pmatrix}$$

$$\Longrightarrow \mathbb{Z}_2 \hookrightarrow SL(2,\mathbb{C}) \circlearrowright \mathbb{C}^2$$

$\mathbb{C}^2/\mathbb{Z}_2$ complex algebraic surface, singular at 0

$$\Longrightarrow \mathbb{C}^2/\mathbb{Z}_2 \xleftarrow[\text{resolution}]{\pi} T^*\mathbb{P}^1 \supset \pi^{-1}(0) = \mathbb{P}^1$$

(= zero section)

Geometry $\longleftrightarrow$ Repr. Theory

$r=1$ $\quad \mathrm{Hilb}^n([\mathbb{C}^2/\mathbb{Z}_2]), \mathrm{Hilb}^n(T^*\mathbb{P}^1) \longleftrightarrow \widehat{sl}(2) \overset{\text{v.s.}}{=} sl(2)[t^{\pm 1}] \oplus \mathbb{C}c$

(Hilbert scheme) $\qquad$ (affine Lie algebra)

arbitrary r $\quad \mathcal{M}_{\mathbb{C}^2/\mathbb{Z}_2}(r), \mathcal{M}_{T^*\mathbb{P}^1}(r) \overset{??}{\longleftrightarrow}$??

$\Uparrow ?$

Yangian ??

Families of Yangians over Bridgeland's space of stability conditions

(Diaconescu-Porta-S-Schiffmann-Vasserot,
S-Schiffmann-Shimpi)

2. Alday-Gaiotto-Tachikawa conjecture for $r=1$

2.1. Algebraic Prelude: Heisenberg Lie algebra

Fix a field $F \supset \mathbb{C}$.
For any $R, N \in \mathbb{N}_{>0}$, $R > N$, consider

$$\begin{array}{ccc} F[z_1, z_2, \dots, z_N] & \hookrightarrow & F[z_1, z_2, \dots, z_N, z_{N+1}, \dots, z_R] \\ z_1, \dots, z_N & \longmapsto & z_1, \dots, z_N \end{array}$$

$$\Longrightarrow \operatorname{colim} F[z_1, z_2, \dots, z_N] = \bigcup_{N \geq 1} F[z_1, z_2, \dots, z_N]$$

$=: F[z_1, z_2, \dots] = M =$ ring of polynomials in infinitely many variables

Fix $0 \neq \varkappa \in F$. Define endomorphisms of M:

- $c := \varkappa \, \mathrm{id}_M$
- For $n \in \mathbb{N} \setminus \{0\}$:

$$\begin{cases} b_n := \varkappa\, n \dfrac{\partial}{\partial z_n} & \text{– annihilation operators} \\ b_{-n} := \text{multiplication by } z_n & \text{– creation operators} \end{cases}$$

Then, the operators b_n, $n \in \mathbb{Z} \setminus \{0\}$, and c satisfy

$$[b_n, c] = 0 \quad \text{and} \quad [b_n, b_m] = n\,\delta_{n+m,0}\, c$$

Definition

The (infinite-dimensional) Heisenberg Lie algebra is the Lie algebra Heis generated by c and b_n $\forall n \neq 0$.

$M = F[z_1, z_2, \dots]$ is called the Fock space of Heis.

Note that w.r.t. $\text{Heis} \curvearrowright M = F[z_1, z_2, \dots]$, we get

- $b_n(1) = 0$ $\forall n \in \mathbb{N} \setminus \{0\}$, and
- $c(1) = \varkappa$, and
- $\{ b_{-n_1} \cdots b_{-n_t}(1) : t \geq 1, n_j \in \mathbb{N} \setminus \{0\} \}$ is a basis of M/F

i.e., M is a highest weight module with highest vector 1

We now introduce an "enumerative" quantity associated to $\mathcal{H}eis \curvearrowright M$.

Define the "degree" operator: $L_0 := \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} m z_m \frac{\partial}{\partial z_m}$

We obtain

$$[L_0, b_n] = -n b_n \quad \forall n \neq 0$$

For $K \in \mathbb{N}$, set $M_K := \{ v \in M : L_0(v) = Kv \}$. Then

$$M = \bigoplus_K M_K$$

Therefore we get the character formula:

$$\boxed{\mathrm{Tr}_M(q^{L_0}) := \sum_{K\geq 0} q^K \dim_F M_K = \prod_{i\geq 1} \frac{1}{(1-q^i)} = \frac{q^{-1/24}}{\eta(q)}}$$

where $\eta(q)$ = Dedekind eta function.

The question I would like to address now is:

Is there a geometric interpretation of the "enumerative" quantity $\mathrm{Tr}_{M}(q^{L_0})$?

Let's introduce the geometric framework that will allow us to answer this question.

2.2. Geometric Prelude: Hilbert schemes of points

Fix $n \in \mathbb{N}$. Define

$$\mathcal{M}(n) := \left\{ (A,B,i) : \begin{array}{l} 1)\ [A,B] = 0 \\ 2)\ \mathbb{C}\langle A,B \rangle \, \mathrm{Im}(i) = \mathbb{C}^n \end{array} \right\} \Big/ GL_n(\mathbb{C})$$

$$\begin{tikzcd} \mathbb{C}^n \arrow[loop left, "A"] \arrow[loop right, "B"] \\ \mathbb{C} \arrow[u, "i"] \end{tikzcd}$$

linear maps

via conjugation (gAg^{-1}, gBg^{-1}, gi) for $g \in GL_n(\mathbb{C})$

$\mathcal{M}(n)$ is a smooth algebraic variety$/_{\mathbb{C}}$ of dim. $2n$

For $(A,B,i) \in \mathcal{M}(n)$:

$$\begin{aligned} \mathbb{C}[x,y] &\xrightarrow{\ \phi\ } \mathbb{C}^n \\ f(x,y) &\longmapsto f(A,B)i(1) \end{aligned}$$

$\Longrightarrow$ $I = \mathrm{Ker}(\phi) \subset \mathbb{C}[x,y]$ ideal

$\Longrightarrow$ $\mathcal{M}(n) \simeq$ Hilbert scheme of n pts on $\mathbb{C}^2$

$=$ moduli space of ideals $I \subset \mathbb{C}[x,y]$ such that $\dim_{\mathbb{C}} \mathbb{C}[x,y]/_I = n$

$=$ moduli space of 0-dimensional subschemes $Z \subset \mathbb{C}^2$ such that $\dim_{\mathbb{C}} H^0(Z, \mathcal{O}_Z) = n$

Now, we are going to interpret

$\mathrm{Tr}_{\mathcal{M}}(q^{L_0}) \longleftrightarrow$ equivariant cohomology of $\mathcal{M}(n)$

The action of the torus $T := (\mathbb{C}^*)^{\times 2}$ on $\mathbb{C}^2$ lifts to an action on $\mathcal{M}(n)$:

$$(t_1, t_2) \cdot (A, B, i) = (t_1 A, t_2 B, i)$$

Recall: $H_T^*(pt) = \mathbb{C}[\varepsilon_1, \varepsilon_2] \Longrightarrow$ Set $F := \mathbb{C}(\varepsilon_1, \varepsilon_2)$

The F-vector space to consider on the geometric side:

$\mathbb{L}_{loc\,n} :=$ localized T-equivariant cohomology of $\mathcal{M}(n)$ $= H_T^*(\mathcal{M}(n)) \otimes_{\mathbb{C}[\varepsilon_1, \varepsilon_2]} \mathbb{C}(\varepsilon_1, \varepsilon_2)$

$$\mathbb{L}_{loc} := \bigoplus_{n \geq 0} \mathbb{L}_{loc,n}$$

Göttsche-Soegel '93 proved that:

Generating function of the Betti numbers of the $\mathcal{M}(n)$'s

$$:= \sum_{n\geq 0} q^n \, \mathrm{gr.dim}_F \, \mathbb{L}_{loc,n} = \frac{q^{-1/24}}{\eta(q)} = \mathrm{Tr}\left(q^{L_0}\right)$$

(grading w.r.t. coh. degree)

The following theorem lifts these equalities from the level of enumerative invariants to that of vector spaces:

Theorem (Grojnowski '96, Nakajima '97, '99)

Set $\varkappa := -\frac{\varepsilon_1}{\varepsilon_2}$. $\exists$ an action of Heis on $\mathbb{L}_{loc}$ such that

$$\mathbb{L}_{loc} \simeq M = F[z_1, z_2, \ldots]$$

$$[\mathcal{M}(0)]_T \longmapsto 1$$

(equivariant fundamental class)

Attention ⚠:

$\left\{ b_{-n_1} \cdots b_{-n_t} [\mathcal{M}(0)]_T \right\}$ is a F-basis of $\mathbb{L}_{loc}$

2.3. Alday-Gaiotto-Tachikawa conjecture for arbitrary r

Now, we introduce the generalization:

- $M(n) \rightsquigarrow M(r,n) = $ ADHM moduli space:

$$M(r,n) := \left\{ (A,B,i,J) : \begin{array}{l} 1)\ [A,B] + iJ = 0 \\ 2)\ \mathbb{C}\langle A,B\rangle \operatorname{Im}(i) = \mathbb{C}^n \end{array} \right\} \Big/ GL_n(\mathbb{C})$$

linear maps

$$\begin{tikzcd} \mathbb{C}^n \arrow[loop left, "A"] \arrow[loop right, "B"] \arrow[d, bend right, "J"'] \\ \mathbb{C}^r \arrow[u, bend right, "i"'] \end{tikzcd}$$

via conjugation
$(gAg^{-1}, gBg^{-1}, gi, Jg^{-1})$
for $g \in GL_n(\mathbb{C})$

- $T := (\mathbb{C}^*)^2 \circlearrowright M(n) \rightsquigarrow T_r := (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^r \circlearrowright M(r,n)$:

$$(t_1,t_2,D)\cdot(A,B,i,J) = (t_1A, t_2B, iD^{-1}, t_1t_2DJ)$$

(Here, D = diagonal matrix $\in GL_r(\mathbb{C})$.)

$$H^*_{T_r}(pt) = \mathbb{C}[\varepsilon_1,\varepsilon_2,a_1,\dots,a_r] \Longrightarrow F = \mathbb{C}(\varepsilon_1,\varepsilon_2,a_1,\dots,a_r)$$

- $\mathbb{L}_{loc} \rightsquigarrow \mathbb{L}^{(r)}_{loc} := \bigoplus_{n\geq 0} H^*_{T_r}(\mathcal{M}(r,n))_{loc}$

Alday-Gaiotto-Tachikawa '09 conjectured that:

- $\mathcal{H}eis \rightsquigarrow \mathcal{W}_k(gl(r)) := \mathcal{H}eis \otimes \mathcal{W}_k(sl(r))$

where $\mathcal{W}_k(sl(r))$ = principal affine W-algebra of $sl(r)$ at level $k := -\varepsilon_2/\varepsilon_1 - r$

- $M = F[z_1, z_2, \ldots] \rightsquigarrow M_\beta$ = Verma module, with $\beta \in F^{\oplus r}$

Attention ⚠ : for $r \geq 4$, $\mathcal{W}_k(sl(r))$ does not have an explicit presentation by gen.s and rel.s

$\Longrightarrow$ Hard to construct representations

Theorem (Schiffmann-Vasserot '12; Maulik-Okounkov '12)
$\exists$ an action of $\mathcal{W}_k(gl(r))$ on $\mathbb{L}^{(r)}_{loc}$ such that

$$\mathbb{L}^{(r)}_{loc} \simeq M_\beta \quad \text{for a well-defined } \beta \in F^{\oplus r}$$

Attention ⚠: $\mathbb{L}^{(r)}_{loc}$ has a "combinatorial" description

$\Longrightarrow$ $\mathcal{W}_k(\mathfrak{gl}(r))$-action on $\mathbb{L}^{(r)}_{loc}$ deeply studied in combinatorics and theoretical physics

Attention ⚠: How have these Theorems been proved?

- $r=1$: Grojnowski and Nakajima defined geometrically the b_n's and checked the rels
- On the other hand, $\mathcal{W}_k(\mathfrak{gl}(r))$ does not have an explicit presentation by gen.s and rel.s for $r \geq 4$.

  SV and MO did not construct "directly" the $\mathcal{W}_k(\mathfrak{gl}(r))$-action, rather they proved:

Theorem (Schiffmann-Vasserot '12; Maulik-Okounkov; '12)
$\exists$ a Hopf algebra
(i.e. $(A, m: A\otimes A \longrightarrow A, \text{unit}, \Delta: A \longrightarrow A\otimes A, \text{counit}, S \text{ antipode})$)

$$\mathbb{Y}^{MO}_{1\text{-loop}} = \text{MO's affine Yangian of } \mathfrak{gl}(1)$$
$$= \text{SV's degenerate spherical DAHA of } GL_\infty$$

such that

certain loc.

1. $\exists$ a faithful representation of $\left(\mathbb{Y}^{MO}_{1\text{-loop}}\right)_{loc(r)}$ on $\mathbb{L}^{(r)}_{loc}$, generated by $[M(r,0)]_T$.
2. $\left(\mathbb{Y}^{MO}_{1\text{-loop}}\right)_{loc(r)} \circlearrowright \mathbb{L}^{(r)}_{loc}$ induces $\mathcal{W}_k(\mathfrak{gl}(r)) \circlearrowright \mathbb{L}^{(r)}_{loc}$.

This result is an instance of Brundan-Kleshev's paradigma:

$$\boxed{\text{Yangians are 'universal' algebraic structures to construct representations of } \mathcal{W}\text{-algebras}}$$

Remark

$\mathbb{Y}^{MO}_{1\text{-loop}}$ is an example of Maulik-Okounkov Yangians:

Q = quiver (i.e., oriented graph) $\rightsquigarrow$ there exist

- $\mathfrak{g}_Q^{MO}$ = Maulik-Okounkov Lie algebra
- Y_Q^{MO} = "deformation" of $U(\underbrace{\mathfrak{g}_Q^{MO} \otimes \mathbb{C}[u]}_{\text{current algebra}})$

More precisely, i.e.,

- Y_Q^{MO} = Hopf algebra $/ H_T^*(pt)$ (T = torus associated to Q)
- $Y_Q^{MO} = \mathrm{Sym}(\mathfrak{g}_Q^{MO} \otimes \mathbb{C}[u] \otimes H_T^*(pt))$ as vector spaces

3. Summary

	Geometry	$\longleftrightarrow$	Repr. Theory
$r=1$	$\mathcal{M}(n)=\mathrm{Hilb}^n(\mathbb{C}^2)$ (Hilbert scheme)	$\longleftrightarrow$	Heis (Heisenberg algebra)
arbitrary r	$\mathcal{M}(r,n)$	$\longleftrightarrow$	$\mathcal{W}_k(\mathfrak{gl}(r))$ (affine W-algebra)

$$\Uparrow$$

Maulik-Okounkov affine Yangian $Y^{MO}_{1\text{-loop}}$ of $\mathfrak{gl}(1)$

Now, we pass from $\mathbb{C}^2$ to:

$$\mathbb{C}^2/\mathbb{Z}_2 \xleftarrow[\text{resolution}]{\pi} T^*\mathbb{P}^1 \supset \pi^{-1}(0)=\mathbb{P}^1$$

	Geometry	$\longleftrightarrow$	Repr. Theory
$r=1$	$\mathrm{Hilb}^n([\mathbb{C}^2/\mathbb{Z}_2]),\ \mathrm{Hilb}^n(T^*\mathbb{P}^1)$ (Hilbert scheme)	$\longleftrightarrow$	$\widehat{\mathfrak{sl}}(2)\overset{\text{v.s.}}{=}\mathfrak{sl}(2)[t^{\pm1}]\oplus\mathbb{C}c$ (affine Lie algebra)

arbitrary r: $\mathcal{M}_{\mathbb{C}^2/\mathbb{Z}_2}(r)$, $\mathcal{M}_{T^*\mathbb{P}^1}(r)$ $\overset{??}{\longleftarrow\,-\,-\,-\,-\longrightarrow}$??

$\Uparrow ?$

Yangian ??

The main questions to address now are:

- What are the 'right'

  vertex algebras / Yangians $\longleftrightarrow$ $\mathbb{C}^2/\mathbb{Z}_2$, $T^*\mathbb{P}^1$?

- $\exists$ a framework that will allow us to address the previous question simultaneously for $\mathbb{C}^2/\mathbb{Z}_2$ and $T^*\mathbb{P}^1$?

Attention ⚠:

To the surfaces $\mathbb{C}^2/\mathbb{Z}_2$ and $T^*\mathbb{P}^1$ we can associate:

abelian categories: $\mathrm{Coh}_0([\mathbb{C}^2/\mathbb{Z}_2])$ and $\mathrm{Coh}_{\mathbb{P}^1}(T^*\mathbb{P}^1)$

derived categories: $D^b\mathrm{Coh}_0([\mathbb{C}^2/\mathbb{Z}_2]) \overset{\text{McKay}}{\simeq} D^b\mathrm{Coh}_{\mathbb{P}^1}(T^*\mathbb{P}^1)$

$$\Downarrow$$

$$\boxed{\mathrm{Coh}_0([\mathbb{C}^2/\mathbb{Z}_2]),\ \mathrm{Coh}_{\mathbb{P}^1}(T^*\mathbb{P}^1) \subset D^b\mathrm{Coh}_0([\mathbb{C}^2/\mathbb{Z}_2])}$$

$\Longrightarrow$ We need to translate this property to the representation-theoretic side

4. How to: algebra $\longleftrightarrow$ $\mathcal{A}$ abelian category

Let $\mathcal{A}$ be a "nice" abelian category.

E.g.: $\mathcal{A} = \mathrm{Coh}_0([\mathbb{C}^2/\mathbb{Z}_2]),\ \mathrm{Coh}_{\mathbb{P}^1}(T^*\mathbb{P}^1)$

Theorem (Diaconescu-Porta-S.-Schiffmann-Vasserot '25)

Let $\mathcal{M}_{\mathcal{A}}$ be the derived moduli stack of families of objects of $\mathcal{A}$.

Then, the Borel-Moore homology of $\mathcal{M}_{\mathcal{A}}$,

$$\boxed{\mathrm{COHA}_{\mathcal{A}} := H_*^{BM}(\mathcal{M}_{\mathcal{A}})}$$

is endowed with the structure of an associative algebra, induced by the algebra structure of $\mathcal{M}_A$

(seen as an object of Gaitsgory-Rozenblyum's category of correspondences).

$\Longrightarrow$ COHA_A = cohomological Hall algebra of A

Remark

$\exists$ an T-equivariant COHA w.r.t. the action of a torus T

Example

Dynkin: $\mathbb{Z}_2 \subset SL(2,\mathbb{C}) \longleftrightarrow A_1 : \bullet \rightsquigarrow A_1^{(1)}:$

$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$$

SV: $\mathrm{COHA}^T_{\mathbb{C}^2/\mathbb{Z}_2} := \mathrm{COHA}^T$ of $\mathrm{Coh}_0([\mathbb{C}^2/\mathbb{Z}_2])$

$$\simeq \mathbb{Y}^{MO,-}_{A_1^{(1)}} = \text{'deformation' of } U(\mathfrak{n}^-)$$

Here, $\mathfrak{n}^-$ is the standard negative half of the elliptic Lie algebra

$$\mathfrak{g}_{ell} := \mathrm{u.c.e.}\left(\widehat{\mathfrak{sl}}(2)\otimes\mathbb{C}[t]\right)$$

$\Longrightarrow \mathcal{Y}^{MO,-}_{A_1^{(1)}}$ = negative part of MO Yangian of $A_1^{(1)}$

Attention ⚠:

$$\begin{tikzcd}
\mathrm{COHA}^T_{\mathbb{C}^2/\mathbb{Z}_2} \arrow[r, leftrightarrow] & \mathrm{Coh}_0([\mathbb{C}^2/\mathbb{Z}_2]) \arrow[dr, hook] & \\
 & & D^b\mathrm{Coh}_0([\mathbb{C}^2/\mathbb{Z}_2]) \\
\mathrm{COHA}^T_{T^*\mathbb{P}^1} \arrow[r, leftrightarrow] & \mathrm{Coh}_{\mathbb{P}^1}(T^*\mathbb{P}^1) \arrow[ur, hook] &
\end{tikzcd}$$

5. Family of Yangians over the Bridgeland's space

We would like to 'relate' $\mathrm{COHA}_{\mathbb{C}^2/\mathbb{Z}_2}$ and $\mathrm{COHA}_{T^*\mathbb{P}^1}$

$\Longrightarrow$ We need to vary $\mathcal{A} \subset D^b\mathrm{Coh}_0([\mathbb{C}^2/\mathbb{Z}_2])$ over

$$\boxed{\begin{aligned}\mathrm{Stab} &= \mathrm{Stab}(D^b\mathrm{Coh}_0([\mathbb{C}^2/\mathbb{Z}_2])) \\ &:= \text{Bridgeland's space of stability conditions} \\ &= \left\{ (\mathcal{A}, Z = \text{stability function on } \mathcal{A}) \text{ satisfying some technical conditions} \right\}\end{aligned}}$$

Theorem (S.-Schiffmann-Shimpi)

$\forall$ abelian category $\mathcal{A}$ associated to a point of Stab $\exists$ a unital associative algebra

$$\mathrm{COHA}^T_{\mathcal{A}}$$

which is a 'deformation' of $\hat{U}(n_{\mathcal{A}})$ (completion), where $n_{\mathcal{A}}$ is a half of the elliptic Lie algebra $\mathfrak{g}_{ell}$.

Moreover, all halves of $\mathfrak{g}_{ell}$ arise in this way.

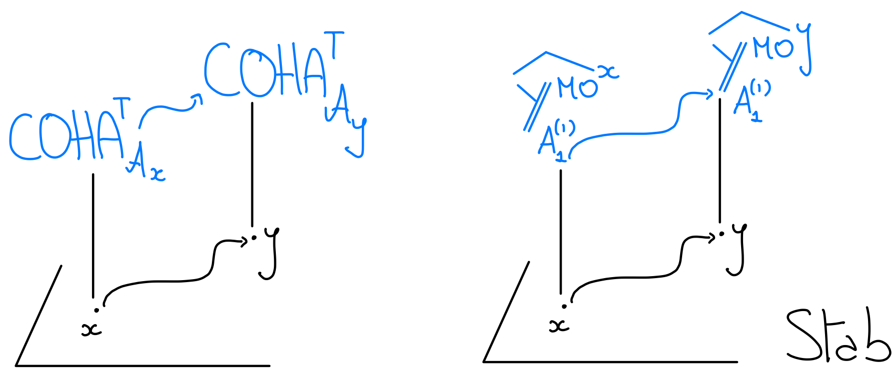
Conjecture

$\forall x \in \mathrm{Stab}$, $\exists$ a completed MO Yangian $\widehat{\mathbb{Y}^{MO}_{A_1^{(1)}}}^{x}$ such that

$$\widehat{\mathbb{Y}^{MO}_{A_1^{(1)}}}^{x,+} \simeq \mathrm{COHA}^T_{\mathcal{A}_x}$$

where $\mathcal{A}_x$ is the abelian category associated to x.

Pictorially:



Remark: These results generalize:

- $\mathbb{Z}_2 \rightsquigarrow G \subset SL(2,\mathbb{C})$ finite subgroup

- $\begin{cases} Q_{fin} = A_1 = \cdot \rightsquigarrow Q_{fin} = \text{finite ADE quiver} \\ \mathfrak{sl}(2) \rightsquigarrow \mathfrak{g}_{Q_{fin}} = \text{simple Lie algebra} \end{cases}$

- $$\begin{tikzcd} \mathbb{C}^2/\mathbb{Z}_2 & T^*\mathbb{P}^1 \arrow[l] & \rightsquigarrow & \mathbb{C}^2/G & & & \widetilde{X} \arrow[lll, "\text{partial resolution}"', "\widetilde{\pi}"] & X \arrow[l] \arrow[llll, bend left, "\text{minimal resolution}", "\pi"'] \end{tikzcd}$$

- $\mathbb{P}^1 \subset T^*\mathbb{P}^1 \rightsquigarrow \widetilde{C} := \widetilde{\pi}^{-1}(0) \subset \widetilde{X}, \; C := \pi^{-1}(0) \subset X$

Returning back to the summary table:

		$\mathbb{C}^2/\mathbb{Z}_2 \xleftarrow{\text{resolution}} X = T^*\mathbb{P}^1$
Surface	$\mathbb{C}^2/\mathbb{Z}_2$	$X = T^*\mathbb{P}^1$
Geometry $r=1$:	$\mathrm{Hilb}^n([\mathbb{C}^2/\mathbb{Z}_2])$	$\mathrm{Hilb}^n(T^*\mathbb{P}^1)$
$r\geq 2$:	$\mathcal{M}_{\mathbb{C}^2/\mathbb{Z}_2}(r)$	$\mathcal{M}_{T^*\mathbb{P}^1}(r)$
Repr. Th. $r=1$:	$\widehat{\mathfrak{sl}}(2)$	$\widehat{\mathfrak{sl}}(2)$
$r\geq 2$:	??	??
Yangians:	$\mathbb{Y}^{MO}_{A_1^{(1)}}$	$\widehat{\mathbb{Y}^{MO}_{A_1^{(1)}}}$

This provides a geometric framework to study recent research directions:

- "affinization procedure" in the theory of Yangians

(cf. recent works by Feigin, Jimbo, Mukhin, Miwa, ...)

- "gluing procedure" in the theory of vertex algebras (cf. recent works by Feigin, Gukov, Creutzig, Linshaw, ...)
- Alday-Gaiotto-Tachikawa conjectures for enumerative invariants of $T^*\mathbb{P}^1$