Cohomological Hall algebras of 1-dimensional sheaves and Yangians

(Slides available on https://web.dm.unipi.it/sala/talks_trips/)

(based on

- joint work with D.-E. Diaconescu, M. Porta, O. Schiffmann, and Eric Vasserot, arXiv:2502.19445;
- joint work with O. Schiffmann and P. Shimpi, arXiv:2511.08576)

1. Overview

2-dimensional Cohomological Hall algebras (COHAs)

Construction of COHAs

Description of the "quantum nature" of a COHA

Quivers ⟵ Examples ⟶ Surfaces

1. 2d COHAs of quivers

1.1 Construction

Geometry				Algebra
Moduli stack	$\longrightarrow$	COHA	$\longrightarrow$	Associative algebra

Fix a quiver $Q = (I = \{\text{vertices}\}, \Omega = \{\text{edges}\})$, i.e., an oriented graph.

Examples

- Q = one-loop quiver : 
- $Q = A_N$-quiver : $\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \cdots \longrightarrow \overset{N-1}{\bullet} \longrightarrow \overset{N}{\bullet}$

Define

- the double quiver $Q^{db} = (I, \Omega^{db} := \Omega \sqcup \Omega^{op})$ where

$$\Omega^{op} := \{ e^* : j \longrightarrow i \mid e : i \longrightarrow j \in \Omega \}$$

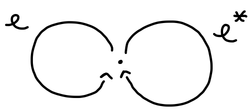
- the path algebra $\mathbb{C}Q^{db}$ of Q^{db} = associative algebra for which all the paths of Q^{db} of length $\ell \geq 0$ form a $\mathbb{C}$-basis

- the preprojective algebra Π_Q of Q = the quotient

$$\Pi_Q := \mathbb{C}Q^{db} \Big/ \sum_{e \in \Omega} [e, e^*]$$ (preprojective relations)

Example

Q = one-loop quiver : 

$\Longrightarrow Q^{db} =$ 

$\Longrightarrow \mathbb{C}Q^{db} = \mathbb{C}\langle e, e^* \rangle$

$$\Longrightarrow \Pi_Q = \mathbb{C}\langle e, e^* \rangle \Big/ [e, e^*] = \mathbb{C}[e, e^*]$$

- a finite-dimensional representation of Π_Q is a finite-dimensional right Π_Q-module, i.e., the datum

$$\left(V = \bigoplus_{i\in I} V_i, \left(x_e : V_{s(e)} \longrightarrow V_{t(e)}\right)_{e\in\Omega^{db}} \right)$$

such that $\sum_{e\in\Omega} [x_e, x_{e^*}] = 0$, with V_i's finite-dimensional.

Set

$$\boxed{\underline{\mathrm{Rep}}(\Pi_Q) := \text{moduli stack of finite-dimensional representations of } \Pi_Q}$$

Explicitly,

$$\underline{\mathrm{Rep}}(\Pi_Q) = \bigsqcup_{\underline{d}\in \mathbb{N}I} \underline{\mathrm{Rep}}(\Pi_Q;\underline{d})$$

(dimension vector)

for any $\underline{d}\in\mathbb{N}I$:

$$\underline{\mathrm{Rep}}(\Pi_Q;\underline{d}) = \text{quotient stack } C_{\underline{d}}/GL_{\underline{d}}$$

where

- $GL_{\underline{d}} := \prod_{i\in I} GL_{d_i}(\mathbb{C})$ acts by conjugation on:
- $C_{\underline{d}}$ = commuting variety :=

$$\left\{ (A_e)_{e\in\Omega^{db}} \in \bigoplus_{e\in\Omega^{db}} \mathrm{Hom}\left(\mathbb{C}^{d_{s(e)}}, \mathbb{C}^{d_{t(e)}}\right) : \sum_{e\in\Omega} [A_e, A_{e^*}] = 0 \right\}$$

Example: Q = one-loop quiver

For $d \in \mathbb{N}$, we get

$$\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}; d) = C_d / GL_d(\mathbb{C})$$

where

$$C_d = \left\{ (A_1, A_2) \in \mathrm{Mat}_d(\mathbb{C})^{\times 2} : [A_1, A_2] = 0 \right\}$$

is the "usual" commuting variety.

The moduli stack $\underline{\mathrm{Rep}}(\Pi_Q)$ admits an action of the torus $T_{max} := (\mathbb{C}^*)^{\Omega} \times \mathbb{C}^*$ induced by

$$\left((t_e)_{e\in\Omega}, t \right) \cdot \left(A_e, A_{e^*} \right)_{e\in\Omega} := \left(t_e A_e, t_e^{-1} t A_{e^*} \right)_{e\in\Omega}$$

Example: Q = one-loop quiver

- $T_{max} = \mathbb{C}^* \times \mathbb{C}^*$
- $(t_1, t) \cdot (A_1, A_2) = (t_1 A_1, t_1^{-1} t A_2)$

Schiffmann-Vasserot, Yang-Zhao:

Fix a subtorus $T \subseteq T_{max} = (\mathbb{C}^*)^{\Omega} \times \mathbb{C}^*$. Then

$$\boxed{\exists\ COHA^T_Q = T\text{-equivariant cohomological Hall algebra associated to finite-dimensional reps of } \Pi_Q}$$

= unital associative algebra structure on

$$\underbrace{H^T_*}\left(\underline{Rep}(\Pi_Q)\right) = \bigoplus_{\underline{d} \in \mathbb{N}I} \underbrace{H^{GL_{\underline{d}} \times T}_*}(C_{\underline{d}})$$

(equivariant Borel-Moore homology)

where the multiplication $p_* \circ q^!$ is induced by

$$\underline{Rep}(\Pi_Q) \times \underline{Rep}(\Pi_Q) \xleftarrow{q} \underline{Rep}^{ext}(\Pi_Q) \xrightarrow{p} \underline{Rep}(\Pi_Q)$$

(stack of extensions)

Here,

$$\begin{array}{llcl} p: & 0 \to E_2 \to E \to E_1 \to 0 & \longmapsto & E \\ q: & \text{------------}''\text{------------} & \longmapsto & (E_2, E_1) \end{array}$$

Remarks

1. p is a proper representable map $\Longrightarrow \exists\ p_*$

2. $\underline{\mathrm{Rep}}(\Pi_Q)$ is singular $\Longrightarrow$ Technically nontrivial to construct $q^! =$ Gysin pullback. It can be defined either

- using the explicit presentation of $\underline{\mathrm{Rep}}(\Pi_Q)$ as a quotient stack and Fulton's construction in equivariant BM homology, or (viewpoint of Schiffmann-Vasserot, Yang-Zhao)

- observing that $\exists\ \mathbb{R}\underline{\mathrm{Rep}}(\Pi_Q), \mathbb{R}\underline{\mathrm{Rep}}^{ext}(\Pi_Q) =$ derived enhancements $\Longrightarrow \mathbb{R}q$ is quasi-smooth $\Longrightarrow \exists\ q^!$ (viewpoint of Varagnolo-Vasserot, Diaconescu-Porta-S.)

3. $H_*^T \rightsquigarrow G_0^T =$ T-equivariant G_0-theory; T-equivariant Chow groups; ...

4. $H_*^T \rightsquigarrow D^b\mathrm{Coh}(-)$; $\mathrm{Coh}^b(-)$ = dg-enhancement of $D^b\mathrm{Coh}(-)$
(following Păduraru, Varagnolo-Vasserot, Diaconescu-Porta-S.)

Attention ⚠:

1. $\underline{\mathrm{Rep}}(\Pi_Q) \rightsquigarrow \underline{\mathrm{Rep}}(Q)$ = moduli stack of finite-dimensional repr.s of Q

$\Longrightarrow \exists$ 1-dimensional COHA $H_*^T(\underline{\mathrm{Rep}}(Q))$

Here,

- 1-dimensional "=" $\mathrm{Ext}^i(M,N) = 0 \ \forall i \geq 2$
  $\forall M,N$ repr.s of Q
- 2-dimensional "=" $\mathrm{Ext}^i(M,N) = 0 \ \forall i \geq 3$
  $\forall M,N$ repr.s of Π_Q

Although the construction of the 1d COHA is easier than the —"— the 2d COHA since

$\underline{\mathrm{Rep}}(Q)$ is smooth, but its algebraic structure is "obscure".

2. $\exists$ also a 3-dimensional COHA due to Kontsevich-Soibelman

Returning back to 2d COHAs, we may define also a "nilpotent variant":

Schiffmann-Vasserot:

$\exists$ $COHA_Q^{T,nil}$ = T-equivariant COHA associated to strongly semi-nilpotent finite-dimensional reprs of Π_Q

Here, strongly semi-nilpotent means

- A_e and A_{e^*} are nilpotent, if Q is without edge-loops,
- if Q = one-loop quiver, A_1 is nilpotent while A_2 not
- for arbitrary quivers: see Bozec-Schiffmann-Vasserot

Now, the questions I would like to address are:

- Q1: Do $COHA_Q^T$ and $COHA_Q^{T,nil}$ admit a presentation by gen.s and rel.s?
- Q2: What is the "quantum" nature of $COHA_Q^T$ and $COHA_Q^{T,nil}$?

A partial answer to Q1 was provided by Schiffmann-Vasserot: they constructed a family of generators of $COHA_Q^{T,nil}$.

Before addressing Q2, let me introduce the relevant quantum group to consider.

1.2 Intermezzo: Yangians

Fix a subgroup $T \subseteq T_{max} = (\mathbb{C}^*)^{\Omega} \times \mathbb{C}^*$.
The relevant quantum group to consider is:

$$\boxed{\mathbb{Y}_{Q,T}^{MO} = \text{Maulik-Okounkov Yangian of } Q}$$

Key facts:

- $\mathbb{Y}^{MO}_{Q,T}$ = Hopf algebra$/H^*_T(pt)$
- $\mathbb{Y}^{MO}_{Q,T} = \mathrm{Sym}\big(\underbrace{\mathfrak{g}^{MO}_{Q,T}[u]}_{\text{current Lie algebra}}\big)$ as graded vector spaces

$$\Longrightarrow \mathbb{Y}^{MO}_{Q,T} = \text{deformation of } U\big(\mathfrak{g}^{MO}_{Q,T}[u]\big)$$

- Botta-Davison: $\mathfrak{g}^{MO}_{Q,T} \simeq \underbrace{\mathfrak{g}^{MO}_{Q}}_{\text{Maulik-Okounkov Lie alg.}} \otimes H^*_T(pt)$

Examples

1. Q = finite ADE quivers (e.g. A_N: $\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \cdots \overset{N-1}{\bullet} \longrightarrow \overset{N}{\bullet}$)

McBreen: $\mathbb{Y}^{MO}_{Q,\mathbb{C}^*} = \mathbb{Y}^{D}_{Q,\mathbb{C}^*}$ as Hopf algebras

Attention ⚠: $\exists$ another Yangian:

$$\boxed{\mathbb{Y}^{D}_{Q,\mathbb{C}^*} = \text{Drinfel'd Yangian}}$$

Key facts:

- $Y^{D}_{Q,\mathbb{C}^*}$ = Hopf algebra$/H^*_{\mathbb{C}^*}(pt)$, given by gen.s and rel.s
- Nakajima, Varagnolo, Guay:

definition of $Y^{D}_{Q,\mathbb{C}^*}$ for any quiver without ede-loops

2. Q = affine ADE quivers

$$
\left(\text{e.g. } A^{(1)}_N : \quad
\begin{tikzcd}
\overset{1}{\bullet} \arrow[r] & \overset{2}{\bullet} \cdots & \overset{N-1}{\bullet} \arrow[r] & \overset{N}{\bullet} \\
 & \underset{0}{\bullet} \arrow[ul] \arrow[urr] & &
\end{tikzcd}
\right)
$$

Jindal in type A, Diaconescu-Porta-S-Schiffmann-Vasserot:

$$
\boxed{Y^{MO}_{Q,(\mathbb{C}^*)^2} = Y^{D}_{Q,(\mathbb{C}^*)^2}}
$$

where

- $Y^{D}_{Q,(\mathbb{C}^*)^2}$ is given by gen.s and rel.s

(definition given by Kodera, Bershtein-Tsymbaliuk)

- Guay-Regelskis-Wendlandt, DPSSV:
∃ a filtration $F^{\cdot} \subset Y^{D}_{Q,(\mathbb{C}^*)^2}$ such that

$$\mathrm{gr}_{F^{\cdot}} Y^{D}_{Q,(\mathbb{C}^*)^2} \simeq U(\mathfrak{g}_{ell}) \otimes H^*_{(\mathbb{C}^*)^2}(pt)$$

where $\mathfrak{g}_{ell} := \text{u.c.e.}\left(\mathfrak{g}_{Q_{fin}}[s^{\pm 1}, u]\right)$ (Q_{fin} = finite ADE quiver)

3. Q = one-loop quiver.
Maulik-Okounkov, Schiffmann-Vasserot:

$$Y^{MO}_{1\text{-loop},(\mathbb{C}^*)^2} = SH^{c} := \text{SV's degenerate spherical DAHA of type } GL_\infty$$

Key facts:

- SH^c is defined by generators and relations
- ∃ a filtration $F^{\cdot} \subset SH^c$ such that

$$\mathrm{gr}_{F^{\cdot}} SH^c \simeq U(W_{1+\infty}) \otimes H^*_{(\mathbb{C}^*)^2}(pt)$$

where $W_{1+\infty} :=$ u.c.e. (Lie algebra of regular differential operators on $\mathbb{C}^*$)

1.2 "Quantum" nature of the 2d COHAs of quivers

Now, let's recall the main result relating COHA of quivers and Yangian.

Theorem (SV, Neguț for $Q = \hat{G}$; BD, SV)
Let Q be a quiver and let $T \subseteq T_{max} = (\mathbb{C}^*)^{\Omega} \times \mathbb{C}^*$ be a torus.
$\exists$ an isomorphism of $H_T^*(pt)$-algebras

$$\mathrm{COHA}_Q^T \xrightarrow{\sim} \mathbb{Y}_{Q,T}^{MO,+}$$

$$\mathrm{COHA}_Q^{T,nil} \xrightarrow{\sim} \mathbb{Y}_{Q,T}^{MO,-}$$

where

$$\mathbb{Y}_{Q,T}^{MO,+} \otimes_{H_T^*(pt)} \mathbb{Y}_{Q,T}^{MO,0} \otimes_{H_T^*(pt)} \mathbb{Y}_{Q,T}^{MO,-} \xrightarrow{\sim} \mathbb{Y}_{Q,T}^{MO}$$

is a triangular decomposition.

2. COHAs of surfaces

2.1 Construction

$$\boxed{\begin{array}{c}\text{Geometry}\\ \text{Moduli stack}\end{array}} \longrightarrow \text{COHA} \longrightarrow \boxed{\begin{array}{c}\text{Algebra}\\ \text{Associative algebra}\end{array}}$$

S = smooth quasi-projective surface $/\mathbb{C}$

T = (possibly trivial) torus $\curvearrowright S$

$\underline{\mathrm{Coh}}_{ps}(S)$ = moduli stack of properly supported coherent sheaves on S

Remark

We can also define

- $\underline{\mathrm{Coh}}_0(S) \subset \underline{\mathrm{Coh}}_{ps}(S) \longleftrightarrow$ 0-dim. sheaves
- $\underline{\mathrm{Coh}}_{\leq 1}(S) \subset \underline{\mathrm{Coh}}_{ps}(S) \longleftrightarrow$ sheaves of dim ≤ 1

Attention ⚠

$\exists$ a derived enhancement

$$\underline{\mathbb{R}\mathrm{Coh}}_{ps}(S) \times \underline{\mathbb{R}\mathrm{Coh}}_{ps}(S) \xleftarrow{\mathbb{R}q} \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{ps}(S) \xrightarrow{\mathbb{R}p} \underline{\mathbb{R}\mathrm{Coh}}_{ps}(S)$$

such that $\mathbb{R}q$ is quasi-smooth $\Longrightarrow \exists (\mathbb{R}q)^!$

Kapranov-Vasserot, Yu Zhao in dim = 0, Porta-S. :

$\boxed{\exists\ \mathrm{COHA}_S^{(T)} = \text{(T-equivariant) COHA associated to properly supported sheaves on } S}$

= unital associative algebra structure on

$$H_*^{(T)}\left(\underline{\mathbb{R}\mathrm{Coh}}_{ps}(S)\right) = H_*^{(T)}\left(\underline{\mathrm{Coh}}_{ps}(S)\right)$$

with multiplication $(\mathbb{R}p)_* \circ (\mathbb{R}q)^!$.

Remark

- $\exists\ \mathrm{COHA}^{(T)}_{S,0\text{-dim}}$ associated to $\underline{\mathrm{Coh}}_0(S)$

- $\exists\ \mathrm{COHA}^{(T)}_{S,\leq 1}$ associated to $\underline{\mathrm{Coh}}_{\leq 1}(S)$

2.2 "Quantum" nature

Important and challenging questions are:

- **Q1:** Do $\mathrm{COHA}^{(T)}_{S}$, $\mathrm{COHA}^{(T)}_{S,0\text{-dim}}$, and $\mathrm{COHA}^{(T)}_{S,\leq 1}$ admit a presentation by gen.s and rel.s?

- **Q2:** Do COHA^{T}_{S}, $\mathrm{COHA}^{T}_{S,0\text{-dim}}$, and $\mathrm{COHA}^{T}_{S,\leq 1}$ geometrically realize halves of Yangians?

Example: $S=\mathbb{C}^2$

$$\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}) \xrightarrow{\ \sim\ } \underline{\mathrm{Coh}}_0(\mathbb{C}^2)$$

$$A_1 \circlearrowright \mathbb{C}^d \circlearrowleft A_2 \longmapsto \mathbb{C}^d = \mathbb{C}[A_1, A_2]\text{-module}$$

$$\Longrightarrow \mathrm{COHA}^{(T)}_{\mathbb{C}^2, 0\text{-dim}} \simeq \mathrm{COHA}^{(T)}_{1\text{-loop}}$$

In dim=0, we have a complete characterization:

Theorem (Neguţ, Mellit-Minets-Schiffmann-Vasserot)
$COHA^{(T)}_{S,0\text{-dim}}$ can be described explictly in terms of generators and relations.
In particular, if $\omega_S \simeq \mathcal{O}_S$:

$$COHA_{S,0\text{-dim}} \simeq U(W^+_{1+\infty}(S))$$

where $W_{1+\infty}(S) = W_{1+\infty}$-algebra associated to $H^*(S)$.

Now, we will address Q1 and Q2 for $COHA^{(T)}_{S,\leq 1}$

3. COHA of 1-dim. sheaves and Yangians

First, we will introduce a "nilpotent" version of $\mathrm{COHA}^{(T)}_{S,\leq 1}$.

S = smooth quasi-projective surface$/\mathbb{C}$

T = (possibly trivial) torus $\curvearrowright S$

$C \subset S$ T-invariant closed reduced subscheme

Consider

$\boxed{\underline{\mathrm{Coh}}_C(S)}$ = moduli stack of coherent sheaves on S set-theoretically supported on C

→ sheaf analog of nilpotency

Example

$S = T^*X$, X = smooth projective curve$/\mathbb{C}$

$C = X \subset T^*X$ zero section

$\Longrightarrow$ $\underline{\mathrm{Coh}}_X(T^*X)$ = moduli stack of Higgs sheaves $(\mathcal{F}, \phi : \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega^1_X)$ on X, such that ϕ is nilpotent

Theorem 1 (Diaconescu-Porta-S-Schiffmann-Vasserot)

1. $\exists$ a unital associative algebra structure on

$$H_*^{(T)}(\underline{\mathrm{Coh}}_C(S))$$

$\Longrightarrow$ $\mathrm{COHA}^{(T)}_{S,C} =: \mathbb{HA}^{(T)}_{S,C}$

2. $\mathbb{HA}^{(T)}_{S,C}$ depends "locally" on C, i.e., given (S_1, C_1) and (S_2, C_2) such that the formal completions $(\widehat{S_1})_{C_1} \simeq (\widehat{S_2})_{C_2}$ we have:

$$\mathbb{HA}^{(T)}_{S_1,C_1} \simeq \mathbb{HA}^{(T)}_{S_2,C_2}$$

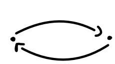
Now, we discuss a relation between $HA^T_{S,C}$ and Yangians when

$$\boxed{S = \text{minimal resolution of ADE singularity}}$$

- $G \subset SL(2,\mathbb{C})$ finite subgroup
  $\updownarrow$
  Q_{fin} = finite ADE quiver, Q = affine ADE quiver
- $\pi : S \longrightarrow \mathbb{C}^2/G$ Kleinian resolution of singularities
- $C := \pi^{-1}(0)$. Then, $C_{red} = C_1 \cup \cdots \cup C_e$, $C_i \simeq \mathbb{P}^1$

  $(C_i \cdot C_j) = -$ Cartan matrix of Q_{fin}
- torus $T \subset GL(2,\mathbb{C})$ centralizing G
  (T = trivial, $\mathbb{C}^*$, or $\mathbb{C}^* \times \mathbb{C}^*$)

Example

$G = \mathbb{Z}_2 \Longrightarrow Q_{fin} = A_1 = \cdot$, $Q = A_1^{(1)}$: 

$\Longrightarrow S = T^*\mathbb{P}^1 \supset C = \mathbb{P}^1 = \text{zero section}$

Recall the derived McKay correspondence:

$$\tau : D^b(\mathrm{Coh}(S)) \xrightarrow{\sim} D^b(\mathrm{Mod}(\Pi_Q))$$

$$\Downarrow$$

$$\begin{cases} K_0(\mathrm{Coh}_C(S)) \xrightarrow{\sim} K_0(\mathrm{nilp}(\Pi_Q)) \simeq \text{root lattice } Y \text{ of } Q \\ \mathrm{Pic}(S) \xrightarrow{\sim} \text{coweight lattice } \check{X}_{fin} \text{ of } Q_{fin} \end{cases}$$

Recall

$$\mathfrak{g}_{ell} := \text{universal central extension of } \mathfrak{g}_{Q_{fin}}[s^{\pm 1}, u]$$

$$\simeq \mathfrak{g}_{Q_{fin}}[s^{\pm 1}, u] \oplus K \quad \Big(K = \underbrace{\bigoplus_{\ell \in \mathbb{N}} \mathbb{Q} c_\ell \oplus \bigoplus_{\substack{k \in \mathbb{Z}_{\neq 0} \\ \ell \in \mathbb{N}_{\geq 1}}} \mathbb{Q} c_{k,\ell}}_{\text{central elements}} \Big)$$

is graded w.r.t. $Y \times \mathbb{Z}\delta_u$.

Example

For $G = \mathbb{Z}_2$: $\mathfrak{g}_{ell} = \mathfrak{sl}(2)[s^{\pm 1}, u] \oplus K$

Define

$$\mathfrak{g}_{ell}^{+} := \text{Lie subalgebra of } \mathfrak{g}_{ell} \text{ associated to the monoid } K_0(\mathrm{Coh}_c(S))^{+} \subset Y \text{ of effective classes}$$

$$= \mathfrak{n}^{-}_{\mathbb{Q}_{fin}}[s^{\pm 1}, u] \oplus s^{-1}\mathfrak{h}_{\mathbb{Q}_{fin}}[s^{-1}, u] \oplus \bigoplus_{k<0} \mathbb{Q}\, c_{k,\ell}$$

where $\mathfrak{g}_{\mathbb{Q}_{fin}} = \mathfrak{n}^{-}_{\mathbb{Q}_{fin}} \oplus \mathfrak{h}_{\mathbb{Q}_{fin}} \oplus \mathfrak{n}^{+}_{\mathbb{Q}_{fin}}$ is a triangular decomposition.

Theorem 2 (Diaconescu-Porta-S-Schiffmann-Vasserot)

- $\exists$ a canonical algebra isomorphism

$$\mathbf{HA}_{S,C} \simeq \widehat{U}(\mathfrak{g}_{ell}^{+})$$

(completion)

- $\exists$ a canonical algebra isomorphism

$$\mathbf{HA}^{T}_{S,C} \simeq \mathbb{Y}_{S,C}$$

where $\mathbb{Y}_{S,C}$ is a deformation of $\widehat{U}(\mathfrak{g}_{ell}^{+})$, i.e., $\exists$ a filtration $F^{\bullet} \subset \mathbb{Y}_{S,C}$ such that $\mathrm{gr}_{F^{\bullet}} \mathbb{Y}_{S,C} \simeq \widehat{U}(\mathfrak{g}_{ell}^{+}) \otimes H_T^{*}(pt)$

Let us recall the questions I have stated. First:

Q: Does $HA^{T}_{S,C}$ geometrically realizes a half of a Yangian?

Conjecture
$HA^{T}_{S,C} \simeq \mathbb{Y}_{S,C}$ realizes a half of a completion of $\mathbb{Y}^{MO}_{Q,T}$

On the other hand, regarding the question:

Q: Does $HA^{T}_{S,C}$ admit a presentation by gen.s and rel.s?

We have a partial answer. First, let's introduce some notation.

For $i \in I_{fin}$, $n \in \mathbb{N}$, $d \in \mathbb{Z}$, set

$\mathcal{C}_{i,d}$:= irreducible component containing the sheaf $\mathcal{O}_{C_i}(d-1)$,

$\mathcal{Y}_{i,n}$:= irreducible component containing the 0-dimen=

sional sheaves on S of length n whose support is a single point $x \in C_i$.

Theorem 3 (Diaconescu-Porta-S-Schiffmann-Vasserot)
$HA^T_{S,C}$ is generated by the fundamental classes of $\mathcal{Y}_{i,n}$ and $\mathcal{C}_{i,d}$ for any $i \in I_{fin}$, $n \in \mathbb{N}$, $d \in \mathbb{Z}$
(as an algebra over a certain ring of poly.s in infinitely many variables).

4. Proof of Theorem 2

Note that derived McKay correspondence

$$\tau : D^b(\mathrm{Coh}_c(S)) \xrightarrow{\sim} D^b(\mathrm{nilp}(\Pi_Q))$$

is not t-exact w.r.t. the standard t-structures, i.e.,

$$\mathrm{Coh}_c(S) \cancel{\xrightarrow{\sim}} \mathrm{nilp}(\Pi_Q)$$

$$\Downarrow$$

$$\underline{\mathrm{Coh}}_c(S) \ \cancel{\simeq}\ \Lambda_Q = \text{stack of nilpotent repr.s of } \Pi_Q$$

$$\Downarrow$$

$$\mathrm{HA}^T_{S,C} \ \cancel{\simeq}\ \mathrm{COHA}^{T,\mathrm{nil}}_Q$$

Attention ⚠: The relation between $\mathrm{Coh}_c(S)$ and $\mathrm{nilp}(\Pi_Q)$ is more subtle:

$$\boxed{\mathrm{Coh}_c(S) = \text{"limit" of } \mathrm{nilp}(\Pi_Q)}$$

More precisely,

- hearts of bounded t-structures on $\mathcal{C}=D^b(\mathrm{nilp}(\Pi_Q))$ form a partial ordered set:

$$\boxed{H_1:=\mathcal{C}_1^{\geq 0}\cap\mathcal{C}_1^{\leq 0}\leq H_2:=\mathcal{C}_2^{\geq 0}\cap\mathcal{C}_2^{\leq 0} \iff \mathcal{C}_1^{\geq 0}\subseteq\mathcal{C}_2^{\geq 0}}$$

- $\check{\theta}\in\check{X}_{fin}$ strictly dominant (i.e. $\check{\theta}(\alpha_i)>0,\forall i$) $\rightsquigarrow$ $\mathcal{L}_{\check{\theta}}$ π-ample

Set

$$\boxed{\tilde{L}_{\check{\theta}}:=\tau\circ(\mathcal{L}_{\check{\theta}}^{-1}\otimes -)\circ\tau^{-1}: D^b(\mathrm{nilp}(\Pi_Q))\longrightarrow D^b(\mathrm{nilp}(\Pi_Q))}$$

- Shimpi: $\boxed{\inf_{n\geq 0}\tilde{L}_{\check{\theta}}^n(\mathrm{nilp}(\Pi_Q))=\mathrm{Coh}_C(S)}$

Now we translate Shimpi's result in the theory of COHAs.

We will use:

- braid group actions on bounded derived cats

▶ Bridgeland stability conditions

Recall that

▶ the extended affine braid group

$$\boxed{B_{ex} \simeq \left(B_{fin} \cup \{ L_{\check{\lambda}} : \lambda \in \check{X}_{fin} \}\right) / \text{rels}}$$

▶ $\check{X}_{fin} \xrightarrow{\sim} \mathrm{Pic}(S)$, $\check{\lambda} \longmapsto \mathcal{L}_{\check{\lambda}}$

Lemma

$\exists$ a group homomorphism

$$\rho : B_{ex} \longrightarrow \mathrm{Aut}(D^b(\mathrm{Coh}_c(S)))$$

such that $\rho(L_{\check{\lambda}}) = \mathcal{L}_{\check{\lambda}} \otimes -$ $\forall$ $\lambda \in \check{X}_{fin}$

Remark

By the derived McKay correspondence

$$L_{\check{\lambda}} = \tilde{L}_{\check{\lambda}} \in \mathrm{Aut}(D^b(\mathrm{nilp}(\Pi_Q)))$$

Now, fix a strictly dominant coweight $\check{\theta} \in \check{X}_{fin} \hookrightarrow \check{X}$.

$\check{\theta}$ defines a King's stability condition on $\mathrm{nilp}(\Pi_Q)$:

$$Z_{\check{\theta}} : K_0(\mathrm{nilp}(\Pi_Q)) \simeq \mathbb{Z}I \longrightarrow \mathbb{C}$$
$$\underline{d} \longmapsto -(\check{\theta}, \underline{d}) + (\check{\rho}, \underline{d})\, i$$

$\check{\rho}(\alpha_i) = 1$

$$\Longrightarrow \boxed{\exists\ (Z_{\check{\theta}}, \mathcal{P}_{\check{\theta}}) = \text{Bridgeland's stability condition on } D^b(\mathrm{nilp}(\Pi_Q))}$$

Here, $\mathcal{P}_{\check{\theta}}$ = slicing = family of full additive subcats

$$\mathcal{P}_{\check{\theta}}(\phi) \subset D^b(\mathrm{nilp}(\Pi_Q)) \quad \forall \phi \in \mathbb{R}$$

such that

- $\mathcal{P}_{\check{\theta}}(\phi + 1) = \mathcal{P}_{\check{\theta}}(\phi)[1] \quad \forall \phi \in \mathbb{R}$;
- $\forall\, \phi_1 > \phi_2$ and $F_j \in \mathcal{P}_{\check{\theta}}(\phi_j)$, $\mathrm{Hom}(F_1, F_2) = 0$;
- $\forall\, 0 \neq E \in D^b(\mathrm{nilp}(\Pi_Q))$, $\exists\ \phi_1 > \cdots > \phi_n$ and triangles

$$\begin{tikzcd}[column sep=small]
0=E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& F_1 \in \mathcal{P}_{\check{\theta}}(\phi_1) \arrow[ul, dashed] & & & & F_n \in \mathcal{P}_{\check{\theta}}(\phi_n) \arrow[ul, dashed] &
\end{tikzcd}$$

(Harder-Narasimhan type filtration)

Set $\mathcal{P}_{\check{\theta}}(I) := \langle \mathcal{P}_{\check{\theta}}(\phi) : \phi \in I \rangle$ $\forall$ interval $I \subset \mathbb{R}$

Lemma

1. $\forall k \in \mathbb{Z}$, $\mathbb{L}_{\check{\theta}}^{-2k} : \mathrm{nilp}(\Pi_Q) \xrightarrow{\sim} \mathcal{P}_{\check{\theta}}((\nu_{-k}, \nu_{-k}+1])$

with $\nu_\ell := \frac{1}{\pi} \arctan(2h\ell)$

Coxeter number

2. $\mathcal{P}_{\check{\theta}}((-\frac{1}{2}, \frac{1}{2}]) \simeq \mathrm{Coh}_c(S)$

For $\ell, k \in \mathbb{N}$, $k \geq \ell$, set

$$\boxed{\Lambda_Q^{\ell,k} := \text{derived moduli stack of objects in } \mathcal{P}_{\check{\theta}}((\nu_{-\ell}, \nu_{-k}+1])}$$

Attention: $\mathbb{L}_{\check{\theta}}^{2k} : \Lambda_Q^{\ell,k} \xrightarrow{\sim} \Lambda_Q^{\ell-k,0} \simeq \Lambda_Q^{>\nu_{-\ell+k}}$ = HN stratum

Lemma

1. The vector space

$$\boxed{HA^{(T)}_{\check{\Theta}} := \lim_{\ell} \operatorname{colim}_{k \geq \ell} H^{(T)}_{*}\left(\Lambda_Q^{\ell,k}\right)}$$

has the structure of an unital associative algebra with multiplication induced from that of $COHA_Q^{(T),nil}$

2. $\exists$ an algebra isomorphism $HA^{(T)}_{S,C} \simeq HA^{(T)}_{\check{\Theta}}$

Corollary

$HA^{(T)}_{\check{\Theta}}$ does not depend on the specific choice of strictly dominant finite coweight.

Finally, a careful analysis of the compatibility between the action of B_{ex} on Yangians and on $COHA_Q^{T,nil}$ yields:

Lemma

$\exists$ an algebra isomorphism $HA^{T}_{\check{\Theta}} \xrightarrow{\sim} \mathbb{Y}_{S,C}$, where

$$\boxed{\mathbb{Y}_{S,C} := \lim_{\ell} T^{2\ell}_{\check{\theta}}(\mathbb{Y}^-_Q) / T^{2\ell}_{\check{\theta}}(J)}$$

where $J := \sum_{\mu_{\check{\theta}}(\underline{d})>0} \mathbb{Y}^-_Q \cdot \mathbb{Y}^-_{Q,-\underline{d}}$. (Here, $\mathbb{Y}^-_Q = \mathbb{Y}^{MO,-}_{Q,(\mathbb{C}^*)^2}$; $\mu_{\check{\theta}}(\underline{d}) := \frac{(\check{\theta},\underline{d})}{(\check{\rho},\underline{d})}$.)

The proof of Thm 2 follows from the above lemmas.

$\square$

Let's address what happens when we drop "strictly" from $\check{\theta}$.

Fix $J \subseteq I_{fin}$ and let $\check{\theta} \in \check{X}_{fin}$ be any dominant coweight such that

$$\check{\theta}(\alpha_i) = 0 \quad \forall\, i \in I_f \setminus J$$

simple root

Remark: Shimpi :

$$\boxed{\inf_{n\geq 0} L^n_{\check{\theta}}(\mathrm{nilp}(\Pi_Q)) = P_C(S/S_J)}$$

where

- contraction of C_i, for $i \in I_{fin} \setminus J$:

$$\begin{tikzcd} S \arrow[rr, "\pi_J"] \arrow[dr, "\pi"'] & & S_J \arrow[dl] \\ & \mathbb{C}^2/G & \end{tikzcd}$$

- $P_c(S/S_J) \subset D^b(\mathrm{Coh}_c(S))$ = Van der Bergh's category of perverse coherent sheaves set-theoretically supported on C

$\Longrightarrow \exists$ COHA$^{(T)}$ of $P_c(S/S_J) := HA_J^{(T)}$

Theorem (S-Schiffmann-Shimpi)

- $\exists$ a canonical algebra isomorphism

$$HA_J \simeq \hat{U}(\mathfrak{g}^J_{ell})$$

where $\mathfrak{g}^J_{ell}$ is a positive half of $\mathfrak{g}_{ell}$.

- $\exists$ a canonical algebra isomorphism

$$HA_J^T \simeq \mathbb{Y}_J$$

where $\mathbb{Y}_J$ is a deformation of $\hat{U}(\mathfrak{g}^J_{ell})$