Yangians and Cohomological Hall algebras of curves

Plan

1. Introduction: moduli spaces and vertex algebras

2. Cohomological Hall algebras

3. COHA of Higgs sheaves on $\mathbb{P}^1$ (and vertex algebras?)

1. Introduction: moduli spaces and vertex algebras

Fix $r, n \in \mathbb{Z}$, $r \geq 1$, $n \geq 0$. Consider

$\mathcal{M}(r,n) :=$ moduli space of framed sheaves
$\left(\mathcal{E} \text{ torsion-free}, \ \phi : \mathcal{E}_{|\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r} \text{ framing}\right)$
on $\mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$ of $\mathrm{rk} = r$, $c_1 = 0$, $c_2 = n$

ADHM description of $\mathcal{M}(r,n)$

$$\simeq \left\{ (A,B,i,j) : \begin{array}{l} \blacktriangleright\ [A,B] + ij = 0 \\ \blacktriangleright\ \nexists\ 0 \subsetneq S \subsetneq \mathbb{C}^n \text{ s.t. } A(S) \subseteq S, B(S) \subseteq S, \\ \qquad \mathrm{Im}(i) \subseteq S \end{array} \right\} \Big/ GL_n(\mathbb{C})$$

via conjugation: $(gAg^{-1}, gBg^{-1}, gi, jg^{-1})$

$$\begin{tikzcd} \mathbb{C}^n \arrow[loop left, "A"] \arrow[loop right, "B"] \arrow[d, bend right, "j"'] \\ \mathbb{C}^r \arrow[u, bend right, "i"'] \end{tikzcd}$$

= smooth quasi-projective variety of dimension $2rn$

Rmk

- $\mathcal{M}(r,n) \supset$ moduli space of $SU(r)$-instantons on $\mathbb{R}^4$ of instanton charge n

- $\mathcal{M}(1,n) \simeq \mathrm{Hilb}^n(\mathbb{C}^2) =$ Hilbert scheme of n pts on $\mathbb{C}^2$

- $\mathcal{M}(r,n) =$ Nakajima quiver variety associated to 1-loop quiver

torus of $\mathbb{P}^2$

- $\exists\ T = \underset{\substack{\rotatebox{90}{$\in$} \\ (t_1,t_2)}}{(\mathbb{C}^*)^2} \times \underset{\substack{\rotatebox{90}{$\in$} \\ D}}{(\mathbb{C}^*)^r} \curvearrowright \mathcal{M}(r,n)$:

$D =$ diagonal matrix

$$(t_1,t_2,D)\cdot(A,B,i,j) = (t_1A,\ t_2B,\ iD^{-1},\ t_1t_2Dj)$$

The main player on the geometric side is:

Def. $\mathbb{L}_n^{(r)} := H_T^*(\mathcal{M}(r,n))$ module over $H_T^*(pt) = \mathbb{C}[\varepsilon_1,\varepsilon_2,a_1,\dots,a_r]$

$\mathbb{L}^{(r)} := \bigoplus_{n\geq 0} \mathbb{L}_n^{(r)}$, $\mathbb{L}_K^{(r)} := \mathbb{L}^{(r)} \otimes_{H_T^*(pt)} K$ $\quad (K = \mathrm{Frac}(H_T^*(pt)))$

The main player from the algebraic side is:

- $\mathcal{W}(\mathfrak{gl}(1)) :=$ (vertex algebra assoc. to) Heisenberg algebra

$$= \langle b_\ell, c : \ell \in \mathbb{Z}\setminus\{0\} \rangle / [b_\ell, c] = 0,\quad [b_\ell, b_{-k}] = \ell\,\delta_{\ell,k}\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)c$$

For $r \geq 2$:

- $\mathcal{W}(\mathfrak{gl}(r)) := \mathcal{W}(\mathfrak{sl}(r), \text{principal nilp.}) \otimes \mathcal{W}(\mathfrak{gl}(1))$

$=$ $\mathbb{Z}$-graded vertex algebra generated by

$$\widetilde{W}_i(z) = \sum_{\ell \in \mathbb{Z}} \widetilde{W}_{i,\ell}\, z^{-i-\ell} \quad \text{for } i = 1, 2, \ldots, r$$

<u>Ex</u>. $\mathcal{W}(\mathfrak{gl}(2)) =$ (vertex algebra assoc. to) $\begin{pmatrix}\text{Virasoro}\\ \text{algebra}\end{pmatrix} \otimes \begin{pmatrix}\text{Heisenberg}\\ \text{algebra}\end{pmatrix}$

<u>Thm</u> (Schiffmann-Vasserot, Maulik-Okounkov)
$\exists$ an action of $\mathcal{W}(\mathfrak{gl}(r))$ on $\mathbb{L}^{(r)}_K$ s.t. $\mathbb{L}^{(r)}_K \simeq$ Verma module with highest weight vector $[\mathcal{M}(r,0)]$ (= fundamental class of $\mathcal{M}(r,0)$), i.e., the module with basis

$$\left\{ \widetilde{W}_{i_1,-\ell_1} \cdots \widetilde{W}_{i_s,-\ell_s} [\mathcal{M}(r,0)] \;:\; s \geq 0,\ \ell_i \geq 1 \right\}$$

RmK

- This is part of the proof of the Alday-Gaiotto-Tachikawa conjecture for pure $SU(r)$-gauge theories on $\mathbb{C}^2$

- For $r=1$, one recovers Nakajima and Grojnowski's result: $\exists$ an action of the Heisenberg algebra on $\mathbb{L}_K^{(1)}$ s.t. $\mathbb{L}_K^{(1)} \simeq$ Fock space

Attention ⚠ :

- For $r=1$, Nakajima and Grojnowski constructed the action directly.

- For $r \geq 2$: Schiffmann-Vasserot and Maulik-Okounkov are unable to construct the action directly!

It is induced by the action of another algebra:

$$\boxed{\mathbb{Y}(\hat{gl}(1)) = \text{the affine Yangian of } gl(1)}$$

Def. (Schiffmann-Vasserot, Maulik-Okounkov)
The Yangian $\mathbb{Y}(\hat{\mathfrak{gl}}(1))$ is the unital associative $\mathbb{C}[\varepsilon_1,\varepsilon_2]$-algebra generated by

$$x_\ell^\pm, \quad h_\ell \quad \text{with } \ell \in \mathbb{Z}_{\geq 0}$$

$H^*_{(\mathbb{C}^*)^2}(pt)$

subject to "certain" relations.

Rmk

- $\exists$ an increasing $\mathbb{N}$-filtration $\{F_k\}_{k\geq 0}$ of $\mathbb{Y}(\hat{\mathfrak{gl}}(1))$ for which

$$\deg(x_\ell^\pm) = \ell = \deg(h_\ell) \quad \text{and} \quad \deg(\varepsilon_i) = 0$$

$\Longrightarrow \mathbb{Y}_0(\hat{\mathfrak{gl}}(1)) \subset \mathbb{Y}(\hat{\mathfrak{gl}}(1))$ generated by elements of degree zero.

- $$\exists \begin{cases} U(W_{1+\infty}) \xrightarrow{\sim} \mathbb{Y}_0(\hat{\mathfrak{gl}}(1)) \\ U(W_{1+\infty}) \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon_1,\varepsilon_2] \xrightarrow{\sim} \mathrm{gr}\, \mathbb{Y}(\hat{\mathfrak{gl}}(1)) \end{cases}$$

where $W_{1+\infty}$ = universal central extension (Lie algebra of differential operators on $\mathbb{C}^*$)

Thm (Schiffmann-Vasserot, Maulik-Okounkov)

1. $\exists$ a faithful representation of $\mathbb{Y}(\hat{gl}(1))_{loc}$ on $\mathbb{L}_K^{(r)}$ s.t.

$\mathbb{L}_K^{(r)}$ is generated by $[\mathcal{M}(r,0)]$

depends on r

2. $\exists$ an embedding $\mathbb{Y}(\hat{gl}(1))_{loc} \overset{\Phi}{\hookrightarrow} U(\mathcal{W}(gl(r)))$

as subalgebras of $End(\mathbb{L}_K^{(r)})$ s.t. $\exists$ an equivalence of categories:

$$\left\{ \begin{array}{c} \text{admissible} \\ U(\mathcal{W}(gl(r)))\text{-modules} \end{array} \right\} \xrightarrow[\Phi^*]{\sim} \left\{ \begin{array}{c} \text{admissible} \\ \mathbb{Y}(\hat{gl}(1))_{loc}\text{-modules} \end{array} \right\}$$

Rmk

The $\mathbb{Y}(\hat{gl}(1))_{loc}$ - action is constructed geometrically via:

- Maulik-Okounkov: theory of stable envelopes + R-matrix realiz.
- Schiffmann-Vasserot: explicit "Nakajima type" generators

- Schiffmann-Vasserot: action of the COHA of 0-dim. sheaves on $\mathbb{C}^2$ (= COHA of 1-loop quiver $\circlearrowleft\!\cdot$)

Attention ⚠:
This is a 2d COHA and **not** a 1d / 3d Kontsevich-Soibelman COHA

2. Cohomological Hall algebras

$$Q = (I, \Omega) \text{ quiver} \rightsquigarrow Q^{db} = (I, \underbrace{\Omega \sqcup \Omega^{op}}_{})$$

$$\Omega^{op} = \left\{ \underset{i}{\cdot} \xleftarrow{e^*} \underset{j}{\cdot} \;\middle|\; \underset{i}{\cdot} \xrightarrow{e} \underset{j}{\cdot} \in \Omega \right\}$$

Ex: $Q : \overset{s(e)}{\cdot} \xrightarrow[e]{} \overset{t(e)}{\cdot} \rightsquigarrow Q^{db} : \cdot \underset{e}{\overset{e^*}{\rightleftarrows}} \cdot$

Def A representation of Q^{db} is

$$E = \Big(V = \bigoplus_{i \in I} V_i \; ; \; \begin{tikzcd} \overset{V_i}{\cdot} \arrow[r, bend right, "x_e"'] & \overset{V_j}{\cdot} \arrow[l, bend right, "x_{e^*}"'] \end{tikzcd} \Big)$$

$\mathbb{C}$-vector space

linear map

linear map

Fix $\underline{d} \in \mathbb{N}^I$. Define the Representation Space:

$$\boxed{\mathrm{Rep}(Q^{db}, \underline{d}) := \bigoplus_{e \in \Omega} \mathrm{Hom}\left(\mathbb{C}^{d_{s(e)}}, \mathbb{C}^{d_{t(e)}}\right) \oplus \bigoplus_{e^* \in \Omega} \mathrm{Hom}\left(\mathbb{C}^{d_{t(e)}}, \mathbb{C}^{d_{s(e)}}\right)}$$

- $\exists\ GL(\underline{d}) := \prod_{i \in I} GL(d_i; \mathbb{C}) \curvearrowright \mathrm{Rep}(Q^{db}, \underline{d})$ by conjugation

- $\exists\ A := \mathbb{C}^* \times \mathbb{C}^* \curvearrowright \mathrm{Rep}(Q^{db}, \underline{d})$:

$$\begin{cases} (t_1, t_2) \cdot x_e = t_1 x_e & \text{for } e \in \Omega \\ (t_1, t_2) \cdot x_{e^*} = t_2 x_{e^*} \end{cases}$$

- $\exists\ \mu_{\underline{d}} : \mathrm{Rep}(Q^{db}, \underline{d}) \longrightarrow \mathrm{Lie}(GL(\underline{d})) = \mathfrak{gl}(\underline{d})$

$$(x_e, x_{e^*}) \longmapsto \sum_{e \in \Omega} [x_e, x_{e^*}]$$

- Set $\mu_{\underline{d}}^{-1}(0)^{nil} := \mu_{\underline{d}}^{-1}(0) \cap \{(x_e, x_{e^*}) : x_e \text{ is "strongly nilpotent"}\}$

Def. The (2-dim., nilp.) cohomological Hall algebra (COHA) of Q is

- as vector space,

Borel-Moore homology

$$\mathrm{COHA}_Q^A := \bigoplus_{\underline{d}\in\mathbb{N}^I} H_*^{A\times GL(\underline{d})}\left(\mu_{\underline{d}}^{-1}(0)^{nil}\right)$$

$$= \bigoplus_{\underline{d}\in\mathbb{N}^I} H_*^{A}\left(\mu_{\underline{d}}^{-1}(0)^{nil}/GL(\underline{d})\right)$$

$$= \bigoplus_{\underline{d}\in\mathbb{N}^I} \mathrm{COHA}_Q^A(\underline{d})$$

quotient stack

- Hall multiplication $m = \bigoplus_{\underline{d}_1,\underline{d}_2} m_{\underline{d}_1,\underline{d}_2}$,

$$\boxed{m_{\underline{d}_1,\underline{d}_2} : \mathrm{COHA}_Q^A(\underline{d}_1)\otimes \mathrm{COHA}_Q^A(\underline{d}_2) \longrightarrow \mathrm{COHA}_Q^A(\underline{d}_1+\underline{d}_2)}$$

given by $m_{\underline{d}_1,\underline{d}_2} := q_*\circ p^*$:

$$\begin{tikzcd}[column sep=small]
& \text{stack of s.e.s. } \{0 \to E_2 \to E \to E_1 \to 0\} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\mathcal{M}^{-1}_{\underline{d}_1}(0)^{nil} / GL(\underline{d}_1) \times \mathcal{M}^{-1}_{\underline{d}_2}(0)^{nil} / GL(\underline{d}_2) & & \mathcal{M}^{-1}_{\underline{d}_1+\underline{d}_2}(0)^{nil} / GL(\underline{d}_1+\underline{d}_2)
\end{tikzcd}$$

$p : 0 \to E_2 \longrightarrow E \longrightarrow E_1 \longrightarrow 0 \longmapsto (E_2, E_1)$

$q : 0 \to E_2 \longrightarrow E \longrightarrow E_1 \longrightarrow 0 \longmapsto E$

Rmk: $COHA^A_Q$ is a unital associative algebra over $H^*_A(pt) \simeq \mathbb{C}[\varepsilon_1, \varepsilon_2]$.

Thm (Schiffmann-Vasserot)

1. $\exists$ an algebra isomorphism of $\mathbb{C}[\varepsilon_1,\varepsilon_2]$-algebras

$$\Psi : \mathbb{Y}(\hat{gl}(1))^+ \xrightarrow{\sim} COHA^A_{1\text{-loop}}$$

2. $\exists \; (COHA^A_{1\text{-loop}})_{loc} \curvearrowright \mathbb{L}^{(r)}_K$

Attention ⚠

- We recover the whole $Y(\hat{gl}(1))$ by taking "Drinfeld double".

- $COHA_{1\text{-loop}}$ realizes only one specific half.

Rmk

In general, we have:

- $$\mu_{\underline{d}}^{-1}(0) \,/\, GL(\underline{d}) \xrightarrow{\ \sim\ } \underline{Rep}(\Pi_Q, \underline{d})$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \\ (x_e, x_{e^*}) : \sum_{e\in\Omega} [x_e, x_{e^*}] = 0 & \longmapsto & \underline{d}\text{-dim. module over} \end{array}$$

$$\Pi_Q := \mathbb{C}Q^{db} \Big/ \sum_{e\in\Omega} [e, e^*]$$

The preprojective algebra of Q

In particular, Q = 1-loop quiver:

$$\Pi_{1\text{-loop}} = \mathbb{C}\langle x,y\rangle \big/ _{xy-yx} \simeq \mathbb{C}[x,y] \rightsquigarrow \underline{Rep}(\Pi_{1\text{-loop}}, d) \simeq \underline{Coh}_{0\text{-dim}}(\mathbb{C}^2, d)$$

- $\exists$ a cohomological Hall algebra assoc. to $\underline{\mathrm{Rep}}(\Pi_Q)$
$\simeq \mathrm{COHA}_Q^A$ as algebras over $\mathbb{C}(\varepsilon_1, \varepsilon_2)$

- For any simply laced quiver Q without loops, there exists

$$\boxed{\mathbb{Y}_Q = \text{Yangian of } Q = \text{"deformation of" } U\big(\mathfrak{g}_Q[t] \oplus \underset{\text{extension}}{\text{central}}\big)}$$

<u>Thm</u> (Schiffmann-Vasserot, Yang-Zhao)
$\exists$ a surjective morphism of $\mathbb{C}[\varepsilon_1, \varepsilon_2]$-algebras

$$\boxed{\Psi : \mathbb{Y}_Q^+ \twoheadrightarrow \mathrm{COHA}_Q^A}$$

If Q = finite ADE, affine ADE, then Ψ is iso.

- In general, the theory of COHAs is a machinery :

$$\mathcal{A} = \left\{ \begin{array}{c} \text{objects on a} \\ \text{"2-dim. space"} \end{array} \right\} \rightsquigarrow \boxed{\begin{array}{c}\text{COHAs} \\ \text{machinery}\end{array}} \rightsquigarrow \mathrm{COHA}_{\mathcal{A}} = \text{unital associative algebra}$$

"= half of a whole quantum group"

We have already seen:

- $\mathcal{A} = \{$ finite-dimensional representations of $\Pi_Q\}$ $\rightsquigarrow$ COHA_Q
- $\mathcal{A} = \{$ 0-dim. sheaves on $\mathbb{C}^2\}$

More examples:

S smooth quasi-projective surface $/\mathbb{C}$

Kapranov-Vasserot, Yu Zhao for $rk=0$:

$$\boxed{\mathcal{A} = \{\text{properly supported coherent sheaves on } S\} \rightsquigarrow \mathrm{COHA}_S}$$

X smooth projective curve $/\mathbb{C}$

We define 3 COHAs assoc. to X

S.-Schiffmann, Minets for $rk=0$:

$$\boxed{\begin{array}{c} \mathcal{A} = \{\text{Higgs sheaves } (\underbrace{\mathcal{E}}_{\text{coherent sheaf}}, \phi : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_X) \text{ on } X\} \\ \rightsquigarrow \mathrm{COHA}^{Dol}_X = \text{Dolbeault COHA of } X \end{array}}$$

Porta-S.:

$$\boxed{\begin{array}{l} \mathcal{A} = \{ \text{flat bundles } (\mathcal{F}, \underbrace{\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega^1_X}_{\nabla^2 = 0}) \text{ on } X \} \\[2em] \rightsquigarrow \mathrm{COHA}^{dR}_X = \text{de Rham COHA of } X \end{array}}$$

Porta-S., Davison:

$$\boxed{\begin{array}{l} \mathcal{A} = \{ \rho : \pi_1(X) \longrightarrow GL_n(\mathbb{C}) \text{ for some } n \} \\[2em] \rightsquigarrow \mathrm{COHA}^{B}_X = \text{Betti COHA of } X \end{array}}$$

Thm (Porta-S.)

- COHA version of Riemann-Hilbert corr.: $\mathrm{COHA}^{dR}_X \simeq \mathrm{COHA}^{B}_X$
- ———"——— of non-abelian Hodge corr.: $\exists\, F^\bullet \subset \mathrm{COHA}^{dR}_X$ s.t.

$$\mathrm{gr}_{F^\bullet} \mathrm{COHA}^{dR}_X \simeq \mathrm{COHA}^{Dol}_X (\deg = 0)$$

3. COHA of Higgs sheaves on $\mathbb{P}^1$ (and vertex algebras?)

$\exists$ generalizations of the AGT conjecture to other surfaces.

Consider

- $X_2 = T^*\mathbb{P}^1 \xrightarrow{\pi} \mathbb{C}^2/\mathbb{Z}_2$ minimal resolution of A_1 singularity

- for $\vec{v}, \vec{w} \in \mathbb{Z}^2$, $\vec{w} \neq 0$,

$$\mathcal{M}_{X_2}(\vec{v},\vec{w}) := \text{moduli space of framed sheaves}$$

$$\left(\mathcal{E} \text{ torsion-free}, \ \phi : \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty} \otimes \left(\rho_0^{\oplus w_0} \oplus \rho_1^{\oplus w_1}\right)\right)$$

on $\overline{X}_2 = X_2 \cup \ell_\infty$ of K-theory class given by $\vec{v}$

stacky compactification

irreducible reprs of $\mathbb{Z}_2$

= smooth quasi-projective variety

RmK

$\mathcal{M}_{X_2}(\vec{v},\vec{w}) \simeq$ Nakajima quiver variety assoc. to $A_1^{(1)}$ with stability condition $\neq$ dominant stability condition $\underset{(1,1)}{\shortparallel}$

Conjecture (AGT conjecture for pure $SU(2)$-gauge theory on X_2)
Assume that $w_0+w_1=2$. $\exists$ an action:

$$\begin{pmatrix}\text{Super Virasoro}\\ \text{algebra}\end{pmatrix} \otimes \begin{pmatrix}\text{Heisenberg}\\ \text{algebra}\end{pmatrix}^{\otimes 2} \curvearrowright \mathbb{L}_K^{(\vec{w})} := \bigoplus_{\vec{v}} H_T^*\left(\mathcal{M}_{X_2}(\vec{v},\vec{w})\right)_{loc}$$

s.t. $\mathbb{L}_K^{(\vec{w})} \simeq$ Verma module

Attention ⚠ :
A possible approach is to mimic Schiffmann-Vasserot's approach, i.e., to use:

COHA of properly supp. sheaves on $T^*\mathbb{P}^1$ (which are set-theor. supp. on $\mathbb{P}^1 \subset T^*\mathbb{P}^1$) $\overset{\text{spectral corr.}}{\simeq}$ COHA of Higgs sheaves on $\mathbb{P}^1$ (which are nilpotent)

but first one need a description of $COHA_{T^*\mathbb{P}^1,\mathbb{P}^1} \simeq COHA_{\mathbb{P}^1}^{Dol,nil}$ by generators and relations

Thm (Diaconescu-Porta-S-Schiffmann-Vasserot)

$$\mathrm{COHA}^{\mathbb{C}^*\times\mathbb{C}^*}_{T^*\mathbb{P}^1,\mathbb{P}^1} \simeq \underbrace{\widehat{\mathbb{Y}}^+_{A_1^{(1)}}}_{\text{completion}} := \text{``}\varprojlim_{\ell\leq 0}\text{''}\, \mathbb{Y}^+_{A_1^{(1)},(\ell)}$$

where

- $$\mathbb{Y}^+_{A_1^{(1)},(\ell)} := \mathbb{Y}^+_{A_1^{(1)}} / \mathbb{I}_\ell$$

$$\mathbb{I}_\ell := \sum_{d_1-d_2>0} \mathbb{Y}^+_{A_1^{(1)}}\,\mathbb{Y}^+_{A_1^{(1)},\underline{d}} + \sum_{d_1-d_2/(d_1+d_2)<-\frac{1}{4\ell}} \mathbb{Y}^+_{A_1^{(1)},\underline{d}}\,\mathbb{Y}^+_{A_1^{(1)}}$$

grading w.r.t. $\underline{d}=(d_1,d_2)\in\mathbb{Z}^2$

- The transition maps are induced by the action of the braid group element corresponding to $-2\check{\omega}_1+2\check{\omega}_2$ ($\check{\omega}_1,\check{\omega}_2$ = fund. coweights of $A_1^{(1)}$)

Attention ⚠ : $\widehat{\mathbb{Y}}^+_{A_1^{(1)}}$ is a half of a new (!) Yangian

$$\widehat{\mathbb{Y}}_{A_1^{(1)}} \supset \mathbb{Y}_{A_1^{(1)}}$$

still to be defined by gens and rels

Notation:

- $$\mathcal{L}\mathfrak{g}_Q = \mathfrak{sl}(2)[s^{\pm 1}, t] \oplus \Big(\underbrace{\bigoplus_{\ell \geq 0} \mathbb{C} c_\ell \oplus \bigoplus_{\substack{\ell \geq 1, \\ k \in \mathbb{Z}, k \neq 0}} \mathbb{C} c_{k,\ell}}_{\text{central}} \Big)$$

with

$$\deg(X \otimes s^k t^\ell) = (\deg(X) + k, k) \in \mathbb{Z}^2, \quad \deg(c_{k,\ell}) = (k, k) \in \mathbb{Z}^2$$

$$\deg(c_\ell) = 0$$

- For $\ell \in \mathbb{Z}$: $\mathcal{L}\mathfrak{g}_\ell := \bigoplus_{\underline{d}} \mathcal{L}\mathfrak{g}_{\underline{d}}$,

$$\underline{d} = (d_1, d_2) \in \mathbb{Z}^2 \text{ s.t. } \begin{cases} d_1 < d_2 \\ \ell(d_1 - d_2) \leq d_2 < (\ell+1)(d_1 - d_2) \end{cases}$$

Thm (Diaconescu-Porta-S-Schiffmann-Vasserot)

$$\mathrm{COHA}_{T^*\mathbb{P}^1, \mathbb{P}^1} \simeq \text{“}\varprojlim_{\ell \leq 0}\text{''}\; U(\mathcal{L}m) / \left(\mathcal{L}\mathfrak{g}_{<\ell} \cdot U(\mathcal{L}m)\right) \text{ as top. algebras}$$

where $\mathcal{L}m := \bigoplus_{d_1 - d_2 \leq 0} \mathcal{L}\mathfrak{g}_{\underline{d}}$, $\mathcal{L}\mathfrak{g}_{<\ell} := \bigoplus_{k < \ell} \mathcal{L}\mathfrak{g}_k$

Rmk (Diaconescu-Porta-S-Schiffmann-Vasserot)
We have a similar description for $\mathrm{COHA}^{A}_{Y,C}$, where:

- $G \subset SL(2,\mathbb{C})$ finite group $\longleftrightarrow$ ADE quiver Q_{fin} $\subset$ affine ADE quiver Q
- $\pi : Y \longrightarrow X := \mathbb{C}^2/G$ resolution of singularities
- $C := \pi^{-1}(0) = C_1 \cup \dots \cup C_e$, $C_i \simeq \mathbb{P}^1$, $(C_i \cdot C_j) = -$ Cartan matrix of Q_{fin}
- Torus $A \subset GL(2,\mathbb{C})$ centralizing G ($A = \{1\}$, $\mathbb{C}^*$, or $\mathbb{C}^* \times \mathbb{C}^*$)

Rmk
$\exists$ conjectural descriptions of $\mathrm{COHA}^{Dol,nil}_{X}$ for genus$(X) \geq 1$.
For example,

$$\mathrm{COHA}^{Dol,nil}_{\substack{\text{elliptic}\\ \text{curve}}} \simeq \mathbb{Y}(\hat{\mathfrak{gl}}(1))^{+}$$

Attention ⚠ : $\mathbb{Y}(\hat{\mathfrak{gl}}(1))$ has not been properly defined yet!