From AGT conjectures

to Cohomological Hall Algebras

Plan

- Instanton counting
- Alday-Gaiotto-Tachikawa conjectures and Cohomological Hall algebras
- Cohomological Hall algebras of ADE singularities

Disclaimer ⚠: I will provide mathematical def.s for the physics quantities I introduce, setting aside their origin from physics.

1. Instanton counting

We introduce the instanton moduli space.

Fix $r,n\in\mathbb{Z}$, $r\geq 1$, $n\geq 0$. Consider

$$\mathcal{M}(r,n) := \left\{ (A,B,i,J) : \begin{array}{l} 1)\ [A,B]+iJ=0 \quad \text{(Preprojective rel.)} \\ 2)\ \nexists\ 0\subsetneq S\subsetneq \mathbb{C}^n \text{ s.t. } \quad \text{(Stability rel.)} \\ \quad A(S)\subseteq S,\ B(S)\subseteq S,\ \mathrm{Im}(i)\subseteq S \end{array}\right\} \Big/ GL_n(\mathbb{C})$$

via conjugation: $(gAg^{-1}, gBg^{-1}, gi, Jg^{-1})$

$$\begin{tikzcd} \mathbb{C}^n \arrow[loop left, "A"] \arrow[loop right, "B"] \arrow[d, bend right, "J"'] \\ \mathbb{C}^r \arrow[u, bend right, "i"'] \end{tikzcd}$$

= smooth (quasi-projective) variety over $\mathbb{C}$ of dimension $2rn$

$$\supset \mathcal{M}^{reg}(r,n) := \left\{ (A,B,i,J) : \begin{array}{l} (A,B,i,J) \text{ satisfies preprojective and stability rels} \\ ({}^tA, {}^tB, {}^tJ) \text{ satisfies stability rel} \end{array}\right\} \Big/ GL_n(\mathbb{C})$$

open subset

$$\simeq \mathcal{M}^{ASD}(r,n) = \left\{ \begin{array}{l} \text{framed } SU(r)\text{-instantons on } S^4 \\ \text{of instanton charge } n \end{array} \right\} \Big/ \begin{array}{l}\text{gauge} \\ \text{trans.} \\ \text{trivial at } \infty\end{array}$$

Donaldson

<u>Remark</u> (Two different compactifications)

1. $S^4 = \mathbb{R}^4 \cup \{\infty\}$ (1-point compactification)

$\Longrightarrow \mathcal{M}^{ASD}(r,n)$ = moduli space of $SU(r)$-instantons 'framed' at ∞

2. fix $\mathbb{R}^4 \simeq \mathbb{C}^2$ (complex structure)

$\Longrightarrow \mathbb{P}^2_{\mathbb{C}} = \mathbb{C}^2 \cup \ell_\infty$
(compactification by adding a line at infinity)

$\Longrightarrow \mathcal{M}^{reg}(r,n) \simeq$ moduli space of rank r vector bundles $\mathcal{E}$ on $\mathbb{P}^2_{\mathbb{C}}$, with $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = n$, framed at ℓ_∞, i.e., $\mathcal{E}\big|_{\ell_\infty} \simeq \mathcal{O}_{\ell_\infty}^{\oplus r}$.

Also, $M^{\cancel{reg}}(r,n) \simeq$ moduli space of rank r ~~vector bundles~~ torsion-free sheaves $\mathcal{E}$ on $\mathbb{P}^2_{\mathbb{C}}$, with $c_1(\mathcal{E})=0$, $c_2(\mathcal{E})=n$, framed at ℓ_∞

Goal: define (Nekrasov's) partition function as

$$\sum_{n\geq 0} \int_{M(r,n)} \text{"cohomology class"} \text{ (depending on the gauge theory)}$$

$\Longrightarrow$ instanton counting = computation of such a partition function

Problem ⚠: $M(r,n)$ is NOT compact

$\Longrightarrow$ integration $\int_{M(r,n)}$ is not well-defined

Solution: we need to work equivariantly (Nekrasov's Ω-deformation) torus of $\mathbb{C}^2$

- $\exists\ T = \underset{(t_1,t_2)}{\underset{\psi}{(\mathbb{C}^*)^2}} \times \underset{D}{\underset{\psi}{(\mathbb{C}^*)^r}} \curvearrowright \mathcal{M}(r,n)$:

D = diagonal matrix

$$\boxed{(t_1,t_2,D)\cdot(A,B,i,j) = (t_1A,\, t_2B,\, iD^{-1},\, t_1t_2Dj)}$$

- $\mathcal{M}(r,n)^T$ is compact: it is a finite set of isolated points

$$= \left\{ \vec{Y} = (Y_1,\dots,Y_r) : \begin{array}{l} \blacktriangleright\ Y_i \text{ Young diagram} \\ \blacktriangleright\ |\vec{Y}| := \sum_{i=1}^{n} |Y_i| = n \end{array} \right\}$$

- $H_T^*(\mathcal{M}(r,n))$ module over $H_T^*(pt) = \mathbb{C}[\varepsilon_1,\varepsilon_2,a_1,\dots,a_r]$

$i : \mathcal{M}(r,n)^T \hookrightarrow \mathcal{M}(r,n)$ inclusion.

$$\Longrightarrow \int_{\mathcal{M}(r,n)} \alpha = \int^{\text{equiv}}_{\mathcal{M}(r,n)^T} \alpha := \sum_{\vec{Y}} \frac{i^*_{\vec{Y}}(\alpha)}{\text{euler}_T(T_{\vec{Y}}\mathcal{M}(r,n))} \in H_T^*(\mathcal{M}(r,n)^T)_{loc}$$

where

$$H_T^*(\mathcal{M}(r,n)^T)_{loc} := H_T^*(\mathcal{M}(r,n)^T) \otimes_{\mathbb{C}[\varepsilon_1,\varepsilon_2,a_1,\dots,a_r]} \mathbb{C}(\varepsilon_1,\varepsilon_2,a_1,\dots,a_r)$$

<u>Def.</u>

(Instanton part of) Nekrasov's partition function of the pure $\mathcal{N}=2$ SUSY $SU(r)$-gauge theory on $\mathbb{C}^2$ is

$$Z_{\mathbb{C}^2}^{inst,pure}(\varepsilon_1,\varepsilon_2,a_1,\dots,a_r;q) := \sum_{n\geq 0} q^n \int_{\mathcal{M}(r,n)} 1$$

equivariant fundamental class

$$= \sum_{\vec{Y}} \frac{1}{euler_T(T_{\vec{Y}}\mathcal{M}(r,n))}$$

<u>Attention</u> ⚠ : We can rewrite this partition function as:

$$\boxed{Z_{\mathbb{C}^2}^{inst,pure} = \sum_{n\geq 0} q^n \left([\mathcal{M}(r,n)]_T, [\mathcal{M}(r,n)]_T\right)}$$

where $[\mathcal{M}(r,n)]_T :=$ fundamental class of $\mathcal{M}(r,n)$

$$(\alpha,\beta) := \int_{\mathcal{M}(r,n)} \alpha \cup \beta$$

Remark

One can define Nekrasov's partition functions $Z^{inst, Q}_{\mathbb{C}^2}$ for any quiver gauge theory

The original Alday-Gaiotto-Tachikawa conjecture states that

$$\boxed{Z^{inst, A_1}_{\mathbb{C}^2} = \text{4-point conformal block of the } A_{r-1} \text{ conformal Toda field theory}}$$

with 4 masses in the fundamental representation

Remark: this was checked by AGT by instanton expansion up to order 11.

Now, we state a mathematical reformulation of the AGT conjecture.

First note that the "symmetry algebra" of the A_{r-1} Toda CFT is:

$$\boxed{\mathcal{W}(\mathfrak{gl}(r)) := \mathcal{W}(\mathfrak{sl}(r)) \otimes \mathrm{Heis}}$$

where

$$\mathcal{H}eis = \text{infinite-dimensional Heisenberg algebra}$$

$$\mathcal{W}(sl(r)) = \mathbb{Z}\text{-graded vertex algebra generated by}$$

$$\widetilde{W}_i(z) = \sum_{l\in\mathbb{Z}} \widetilde{W}_{i,l}\, z^{-l-i} \quad \text{for } i=2,\dots,r$$

Example

For $r=2$, A_1 Toda CFT = Liouville CFT

$$\updownarrow$$

$$\mathcal{W}(gl(2)) = \text{Virasoro} \otimes \mathcal{H}eis$$

A mathematical formulation of the AGT conjecture for pure gauge theories (stated by Gaiotto) is:

Theorem (Schiffmann-Vasserot '12; Maulik-Okounkov; '12)

1. $\exists$ a representation of $\mathcal{W}(gl(r))$ on

$$\mathbb{L}^{(r)}_{loc} := \bigoplus_{n\geq 0} H^*_T(\mathcal{M}(r,n))_{loc}$$

identifying it with the Verma module of highest weight

$\frac{\varepsilon_1}{\varepsilon_2}\Big((a_1,\ldots,a_r)+\big(1+\frac{\varepsilon_2}{\varepsilon_1}\big)(0,-1,\ldots,-r)\Big).$

2. The Gaiotto state $G=\sum_{n\geq 0}[\mathcal{M}(r,n)]_T \in \prod_{n\geq 0} H_T^*(\mathcal{M}(r,n))_{loc}$ is a Whittaker vector, i.e.,

$$W_{i,\ell}(G)=\begin{cases}\varepsilon_2^{1-r}\varepsilon_1^{-1}G & \text{if } i=r,\ \ell=1\\ 0 & \text{otherwise}\end{cases}$$

Remark (Proofs of AGT conjectures for quiver gauge theories)

- Neguț, '15: $\mathcal{N}=2^*$ $SU(2)$ gauge theory
- Mironov-Morozov-Shakirov, '11: A_1 $SU(2)$ gauge theory with 4 masses and $\varepsilon_1+\varepsilon_2=0$
- Ghosal-Remy-Sun-Sun, '20: A_1 $SU(2)$ gauge theory with 4 masses
- Yuan-Hu-Huang-Zhang, '24: A_n $SU(r)$ gauge theory with $2n+2$ masses and $\varepsilon_1+\varepsilon_2=0$

Let us return to SV and MO:

Attention ⚠ : They did not construct directly the representation of $\mathcal{W}(\mathfrak{gl}(r))$, rather...

Theorem (Schiffmann-Vasserot '12; Maulik-Okounkov; '12)

1. $\exists$ a faithful representation of $Y(\hat{\mathfrak{gl}}(1))_{loc(r)}$ on $\mathbb{L}^{(r)}_{loc}$ such that it is generated by $[M(r,0)]_T$.

2. $\exists$ an embedding

$$Y(\hat{\mathfrak{gl}}(1))_{loc(r)} \hookrightarrow U(\mathcal{W}(\mathfrak{gl}(r)))$$

as subalgebras of $\mathrm{End}(\mathbb{L}^{(r)}_{loc})$ such that $\exists$ an equivalence of categories:

$$\{\text{admissible } U(\mathcal{W}(\mathfrak{gl}(r)))\text{-modules}\}$$
$$\cong$$
$$\{\text{admissible } Y(\hat{\mathfrak{gl}}(1))_{loc(r)}\text{-modules}\}$$

Here,

- $Y(\widehat{\mathfrak{gl}}(1))$ = deformation of $U(\mathcal{W}_{1+\infty})$
- $\mathcal{W}_{1+\infty}$ = universal central extension of the Lie algebra of regular differential operators on the circle

Attention ⚠: Note that $Y(\mathfrak{g})$, with $\mathfrak{g}$ = KM Lie algebra, is deformation of

$$U(\underbrace{\mathfrak{g}[z]}_{\text{Lie algebra of currents}})$$

- $Y(\widehat{\mathfrak{gl}}(1))_{\mathrm{loc}(r)}$ = certain localization of $Y(\widehat{\mathfrak{gl}}(1))$

Remark: the two reprs are different

$$\underbrace{Y(\widehat{\mathfrak{gl}}(1))}_{\text{always the same}}{}_{\mathrm{loc}(r)} \xrightarrow{\text{faithful}} \mathbb{L}^{(r)}_{\mathrm{loc}} \xleftarrow{\text{Verma}} \mathcal{W}(\mathfrak{gl}(r))$$

We can replace the RHS by a vertex algebra that does not depend on r:

Conjecture

$$Y(\hat{gl}(1)) \simeq U(\text{universal 2-parameter } \mathcal{W}_\infty\text{-algebra})$$

- all $\mathcal{W}(gl(r))$ can be obtained as quotients of it.
- its existence was conjectured by Gaberdiel and Gopakumar. It was constructed by Linshaw.

Remark: equivalent geometric realizations of $Y(\hat{gl}(1))$

- Maulik-Okounkov: R-matrix realization, as subalgebra of $\mathrm{End}(\mathbb{L}^{(r)}_{loc})$
- Schiffmann-Vasserot: as the subalgebra of $\mathrm{End}(\mathbb{L}^{(r)}_{loc})$ generated by Nakajima type operators
- Schiffmann-Vasserot: as a cohomological Hall algebra

Attention ⚠: I will recall the 3rd approach, which is more flexible for generalizations

2. Cohomological Hall algebras

The relevant COHA for us is the

2d COHA of the 1-loop quiver $\mathrm{COHA}_{1\text{-loop}}$

which is:

- as a vector space,

$$\mathrm{COHA}_{1\text{-loop}} = \bigoplus_{n\geq 0} H_*^{GL(n)}\Big(\underbrace{\{(A,B)\in \mathrm{Mat}(n) : [A,B]=0\}}_{\text{commuting variety } \mathcal{C}_n}\Big)$$

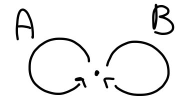
GL(n) acts by conjugation

$$= \bigoplus_{n\geq 0} H_*^{BM}\left(\mathcal{C}_n / GL(n)\right)$$

(stack quotient)

$$= \bigoplus_{n\geq 0} H_*^{BM}\left(\underline{\mathrm{Rep}}_d\left(\Pi_{1\text{-loop}}\right)\right)$$

$$\downarrow$$

a point $(A,B) \in \mathcal{C}_n \longleftrightarrow$ representation 

$+ \; [A,B]=0$

$$\longleftrightarrow \text{ module over } \Pi_{1\text{-loop}} := \mathbb{C}\langle x,y\rangle / [x,y] = \mathbb{C}[x,y]$$

$$\Longrightarrow \boxed{\mathrm{COHA}_{1\text{-loop}} = H_*^{BM}\left(\underline{\mathrm{Rep}}_d\left(\Pi_{1\text{-loop}}\right)\right)}$$

- the multiplication m is induced by the "Hall convolution diagram":

$$\begin{tikzcd}
 & \underline{\mathrm{Rep}}^{ext}_{d_1,d_2}\left(\Pi_{1\text{-loop}}\right) \arrow[dl, "q_{d_1,d_2}"'] \arrow[dr, "p_{d_1,d_2}"] & \\
\underline{\mathrm{Rep}}_{d_2}\left(\Pi_{1\text{-loop}}\right) \times \underline{\mathrm{Rep}}_{d_1}\left(\Pi_{1\text{-loop}}\right) & & \underline{\mathrm{Rep}}_{d_1+d_2}\left(\Pi_{1\text{-loop}}\right)
\end{tikzcd}$$

$$\boxed{m = \bigoplus_{d_1,d_2} \left(p_{d_1,d_2}\right)_* \circ \left(q_{d_1,d_2}\right)^!}$$

Here,

$$p_{d_1,d_2} : 0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0 \longmapsto M$$

$$q_{d_1,d_2} : \qquad \text{———} \,//\, \text{———} \qquad \longmapsto (M_2, M_1)$$

Remark

- $\exists$ COHA_Q for any quiver Q (due to SV, Yang-Zhao)
- $\exists$ a T-equivariant version COHA_Q^T w.r.t. a torus T

Theorem

- Schiffmann-Vasserot: $\left(\mathrm{COHA}_{1\text{-loop}}^{\mathbb{C}^* \times \mathbb{C}^*}\right)_{loc} \simeq Y^+(\hat{\mathfrak{gl}}(1))_{loc}$
- Schiffmann-Vasserot, Botta-Davison: $\left(\mathrm{COHA}_Q^T\right)_{loc} \simeq \left(Y^+(\mathfrak{g}_Q^{MO})\right)_{loc}$

Maulik-Okounkov graded Lie algebra

Attention ⚠ : one recovers the full Yangian by "doubling" it

Let's go back to gauge theories. Note that:

$$M(r,n) = \text{Nakajima quiver variety of the 1-loop quiver}$$

We can replace:

$$\boxed{\begin{array}{l} \text{1-loop quiver} \rightsquigarrow \text{affine ADE quiver } Q \\ \mathcal{M}(r,n) \rightsquigarrow \mathcal{M}_Q^{\theta}(\vec{v},\vec{w}) \end{array}}$$

Attention ⚠

$\mathcal{M}_Q^{\theta}(\vec{v},\vec{w})$ depends on a stability parameter θ:

- $\exists\, \theta^{res}$ such that

$$\mathcal{M}_Q^{\theta^{res}}(\vec{v},\vec{w}) = \text{instanton moduli space of } U(\underbrace{r}_{\sum w_i}) \text{ gauge theories on } S_{\Gamma}$$

where

- $\Gamma \subset SL(2,\mathbb{C})$ is the finite group associated to the finite quiver Q^{fin}
- $S_{\Gamma} \xrightarrow{\pi} \mathbb{C}^2/\Gamma$ is the minimal resolution of the ADE singularity, $C = \pi^{-1}(0) = C_1 \cup \dots \cup C_e$, $C_i \simeq \mathbb{P}^1$

- $\exists\, \theta^{orb}$ such that

$$\mathcal{M}_Q^{\theta^{orb}}(\vec{v},\vec{w}) = \text{instanton moduli space of } U(r) \text{ gauge theories on the orbifold } [\mathbb{C}^2/\Gamma]$$

We have that

$$\boxed{\begin{array}{ll} \mathcal{M}_Q^{\theta^{res}}(\vec{v},\vec{w}) \not\simeq \mathcal{M}_Q^{\theta^{orb}}(\vec{v},\vec{w}) & \text{(as algebraic varieties)} \\ \mathcal{M}_Q^{\theta^{res}}(\vec{v},\vec{w}) \simeq \mathcal{M}_Q^{\theta^{orb}}(\vec{v},\vec{w}) & \text{(as diff. manifolds)} \end{array}}$$

$$\Longrightarrow \boxed{\begin{array}{l} Z^{inst,pure}_{S_\Gamma} = \bar{Z}^{inst,pure}_{[\mathbb{C}^2/\Gamma]} \\ Z^{inst,quiver}_{S_\Gamma} \neq \bar{Z}^{inst,quiver}_{[\mathbb{C}^2/\Gamma]} \end{array}}$$

In this case, AGT conjectures have been formulated only for $\Gamma = \mathbb{Z}_p$:

- Belavin-Feigin: $\bar{Z}^{inst,A_1}_{[\mathbb{C}^2/\mathbb{Z}_2]} \longleftrightarrow$ conformal blocks of $\mathcal{N}=1$ super Liouville CFT

(Later, generalized by Belavin-Bershtein-Feigin-Litvinov-Tarnopolsky)

From a mathematical viewpoint, the first step consists of geometrically realizing representations of Yangians.

We have the following:

Theorem

- Maulik-Okounkov: $\exists$ a faithful repr. of $Y(\mathfrak{g}_Q^{KM})_{loc}$ on
$$\mathbb{L}_{loc}^{(\theta^{orb}, \vec{w})} := \bigoplus_{\vec{v}} H_T^*\left(\mathcal{M}_Q^{\theta^{orb}}(\vec{v}, \vec{w})\right)_{loc}$$
- Schiffmann-Vasserot: $\exists$ a faithful repr. of $\left(COHA_Q^T\right)_{loc}$ on $\mathbb{L}_{loc}^{(\theta^{orb}, \vec{w})}$

Attention ⚠: At the moment, an equivalent result for $\mathcal{M}_Q^{\theta^{res}}(\vec{v}, \vec{w})$ is missing.

A more fundamental question is: what kind of subalgebra of $End\left(\mathbb{L}_{loc}^{(\theta^{res}, \vec{w})}\right)$ realizes $Y(\mathfrak{g}_Q^{KM})$?

Since instantons on S_Γ have non-trivial first Chern classes one needs to construct an algebra of operators in cohomology the changes c_1

Since c_1 is expressed in terms of the C_i's, the correct COHA to consider is:

$\boxed{\mathrm{COHA}^{T}_{S_\Gamma, C} = \text{COHA of sheaves on } S_\Gamma \text{ set-theoretically supported on } C}$

The main characterization is:

<u>Theorem</u> (Diaconescu-Porta-S.-Schiffmann-Vasserot)

$$\boxed{\mathrm{COHA}^{T}_{S_\Gamma, C} \simeq \widehat{\mathbb{Y}}_Q := \bigoplus_{\underline{d}} \widehat{\mathbb{Y}}_{Q,\underline{d}}}$$

i.e., it is a completion of half of the Yangian:

$$\widehat{\mathbb{Y}}_{Q,\underline{d}} := \lim_{\ell \leq 0} \left(T^{-\ell}_{-2\check{\theta}} \left(\mathbb{Y}^{+}_{t^{\ell}_{-2\check{\theta}}(\underline{d})} \right) \Big/ T^{-\ell}_{-2\check{\theta}} \left(J_{\check{\theta}, t^{\ell}_{-2\check{\theta}}(\underline{d})} \right) \right)$$

$$J_{\check{\theta},\underline{d}} := \sum_{\mu_{\check{\theta}}(\underline{d})>0} \mathbb{Y}^{+}_{Q} \, \mathbb{Y}^{+}_{\underline{d}} \subset \mathbb{Y}^{+}_{\underline{d}}$$

where

- $\check{\theta}$ = dominant coweight of Q_{fin}
  (e.g. $\check{\theta} = \check{\rho}_0 = \sum$ fundamental coweights of Q_{fin})

- $\mu_{\check{\theta}}(\underline{d}) = \dfrac{(\check{\theta}, \underline{d})}{(\check{\rho}, \underline{d})}$ (Here, $\check{\rho} = \sum$ fundamental coweights of Q)

- $\begin{cases} t_{-2\check{\theta}} \in W_Q^{ex} = \text{extended affine Weyl group} \\ T_{-2\check{\theta}} \in B_Q^{ex} = \text{——— '' ——— braid group} \end{cases}$

Open questions:

- how do we define a completion of the whole $Y(\mathfrak{g}_Q^{KM})$?
- what is the role of this completion w.r.t. AGT ?