Cohomological Hall algebras, their representations, and Nakajima operators

Plan:

1. Geometric representations of quantum groups
2. COHAs and their representations
3. Nakajima type operators
4. Examples

1. Geometric Representations of Quantum Groups

Let us recall the construction of an action of the elliptic Hall algebra on the K-theory of Hilbert schemes of pts on a smooth surface.

Definition of the elliptic Hall algebra

$$U_{q_1,q_2}(\hat{\hat{\mathfrak{gl}}}(1)) = \mathbb{C}(q_1,q_2)\langle e_n, f_n, h^{\pm}_m \rangle_{n\in\mathbb{Z}, m\in\mathbb{N}} / \text{rels}$$

where the relations for $e(z) := \sum_{n\in\mathbb{Z}} e_n z^{-n}$, $f(z) := \sum_{n\in\mathbb{Z}} f_n z^{-n}$, and $h^{\pm}(z) := \sum_{m\geq 0} h^{\pm}_m z^{-m}$ are

quadratic relations
$$\begin{cases} e(z)e(w)\tilde{\zeta}\left(\frac{w}{z}\right) = e(w)e(z)\tilde{\zeta}\left(\frac{z}{w}\right) \\ h^{\pm}(z)e(w) = e(z)h^{\pm}(w)\dfrac{\tilde{\zeta}\left(\frac{z}{w}\right)}{\tilde{\zeta}\left(\frac{w}{z}\right)} \end{cases}$$

with $\tilde{\zeta}(t) = (1-tq_1q_2)(1-t^{-1}q_1q_2)\boxed{\dfrac{(1-tq_1)(1-tq_2)}{(1-tq_1q_2)(1-t)}}$

zeta function of an elliptic curve

Similar rels for $f(z) \longleftrightarrow e(w)$, commutator relation $[e(z), f(w)]$, and

cubic rels
$$\begin{cases} \sum_{\sigma \in S_3} \left[e_{n_{\sigma(1)}}, \left[e_{n_{\sigma(2)-1}}, e_{n_{\sigma(3)+1}} \right] \right] = 0 \quad \forall n_1, n_2, n_3 \in \mathbb{Z} \\ \sum_{\sigma \in S_3} \left[f_{n_{\sigma(1)}}, \left[f_{n_{\sigma(2)-1}}, f_{n_{\sigma(3)+1}} \right] \right] = 0 \quad \forall n_1, n_2, n_3 \in \mathbb{Z} \end{cases}$$

Now, S = smooth quasi-projective surface$/\mathbb{C}$.

$\mathrm{Hilb}(S)$ = the Hilbert scheme of pts on S
= moduli space of ideal sheaves I of zero-dimensional $\,^{\text{sub}}$schemes of S.

Consider the Hecke correspondence modifying sheaves at 1 pt:

$\mathcal{L}$ = Tautological line bundle

$$\boxed{\{(I', I, x) : 0 \to I' \to I \to \mathcal{O}_x \to 0\}}$$

$$\begin{tikzcd} & \{(I',I,x)\} \arrow[dl, "ev_1"'] \arrow[d, "ev_3"] \arrow[dr, "ev_2"] & \\ \mathrm{Hilb}(S) & S & \mathrm{Hilb}(S) \end{tikzcd}$$

$$ev_1 : (I', I, x) \longmapsto I', \quad ev_2 : (I', I, x) \longmapsto I, \quad ev_3 : (I', I, x) \longmapsto x$$

Now, fix $S = \mathbb{C}^2 \curvearrowleft \mathbb{C}^* \times \mathbb{C}^*$. Set

$K_0^{\mathbb{C}^* \times \mathbb{C}^*}(-) :=$ $\mathbb{C}^* \times \mathbb{C}^*$-equivariant (compl) Grothendieck group of coherent sheaves

$$\Longrightarrow K_0^{\mathbb{C}^* \times \mathbb{C}^*}(pt) \simeq \mathbb{C}[q_1^{\pm 1}, q_2^{\pm 1}]$$

<u>Theorem</u> (Schiffmann-Vasserot)

The elements of

$$\mathrm{End}\left(K_0^{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{Hilb}(S)) \otimes_{K_0^{\mathbb{C}^* \times \mathbb{C}^*}(pt)} \mathbb{C}(q_1, q_2)\right)$$

defined by

- $e_n := (1-q_1)(1-q_2)\, ev_{1*}\left(ev_2^*(-) \cdot \mathcal{L}^{\otimes n}\right)$
- $f_n := (1-q_1)(1-q_2)\, ev_{2*}\left(ev_1^*(-) \cdot \mathcal{L}^{\otimes n-1} \cdot (-\det(\mathcal{U}))\right)$
- $h^{\pm}(z) = \sum_{m \geq 0} h^{\pm}_m z^{-m}$ = multiplication by $\Lambda^{\bullet}\left(z(q_1q_2-1)\mathcal{U}^{\vee}\right)$

give rise to an action of $U_{q_1,q_2}(\widehat{\mathfrak{gl}}(1))$.

Here, $\mathcal{U}$ = restriction of the universal sheaf of $\mathrm{Hilb}(\mathbb{C}^2)$ to $\mathrm{Hilb}(\mathbb{C}^2) \times \{\text{origin}\}$.

Remark

Negut: $\exists$ actions of $U_{q_1,q_2}(\widehat{\mathfrak{gl}}(1))$ on the K-theory of

- $\mathrm{Hilb}(S)$ for S projective, and
- $\mathcal{M}^{st}_H(r, c_1)$ = moduli space of Gieseker H-stable sheaves of rank r, first Chern class c_1 on S

where S is such that either $\omega_S \simeq \mathcal{O}_S$ or $c_1(\omega_S)\cdot H<0$, and $g.c.d.(r, c_1\cdot H)=1$

(Attention ⚠: in this construction, $q_1 q_2 = [\omega_S] \in K_0(S)$)

Remark

Mellit-Minets-Schiffmann-Vasserot: prove a similar result in Borel-Moore homology and for S (quasi-)projective

$\Longrightarrow$ Key ingredient in the proof of the $P=W$ conj. by Hausel-Mellit-Minets-Schiffmann

Remark

The elliptic Hall algebra $U_{q_1,q_2}(\widehat{\mathfrak{gl}}(1))$ = quantum loop algebra of

 = one-loop quiver

Neguț-S-Schiffmann: $Q=(I=\{\text{vertices}\},\ E=\{\text{edges}\})$

$\Longrightarrow U_Q$ = quantum loop algebra of Q

$$= \mathbb{C}(q, q_e)_{e\in E} \langle e_{i,n}, f_{i,n}, h^{\pm}_{i,m} \rangle_{i\in I, n\in \mathbb{Z}, m\in \mathbb{N}} / \text{rel.s}$$

where rel.s = quadratic and cubic relations depending on $\tilde{\zeta}_Q(t)$ and commutator rel.

$\hookrightarrow \tilde{\zeta}_{\text{one-loop}} = \tilde{\zeta}$ seen before

Example

Q = affine ADE quiver $\rightsquigarrow U_Q$ = quantum toroidal algebra of type ADE

Q = quiver without edge-loops $\rightsquigarrow U_Q$ = "usual" quantum loop algebra of Q

The geometric operators $e_n, f_n, h^{\pm}_m$ defined before = generalization of Nakajima's operators which give rise to an action of U_Q on $K_0^{\mathbb{C}^*}$(quiver varieties).

Now, consider
S = smooth projective surface$/\mathbb{C}$, $D \xhookrightarrow{i} S$ effective divisor such that

$$D_{red} = \text{affine ADE configuration } Q \text{ of rational } (-2)\text{-curves } C_k$$

Goal of today's talk:

Theorem (Diaconescu-Porta-S-Zhao)
$\exists$ a moduli stack $\mathcal{M}$ such that its K-theory admits an action of U_Q via operators depending on Hecke correspondences modifying sheaves along C_k

$$\left\{0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow i_* \mathcal{O}_{C_k}(-1) \longrightarrow 0\right\}$$

In the remaining part of the talk, I will define $\mathcal{M}$ and the Nakajima type operators in this context.

The construction follows from the theory of COHAs and their representations.

2. COHA of surfaces and their representations

S = smooth projective surface$/\mathbb{C}$.

$\boxed{\begin{array}{l} \underline{\mathrm{Coh}}(S) = \text{moduli stack of coherent sheaves on } S \\ \underline{\mathbb{R}\mathrm{Coh}}(S) = \text{derived enhancement of } \underline{\mathrm{Coh}}(S) \end{array}}$

Consider the "convolution diagram":

$$\underline{\mathbb{R}\mathrm{Coh}}(S) \times \underline{\mathbb{R}\mathrm{Coh}}(S) \xleftarrow{ev_1 \times ev_3} \underline{\mathbb{R}\mathrm{Coh}}^{ext}(S) \xrightarrow{ev_2} \underline{\mathbb{R}\mathrm{Coh}}(S)$$

$\underline{\mathbb{R}\mathrm{Coh}}^{ext}(S)$ = stack of extensions

where:

$$ev_2: \; 0 \to E_1 \to E_2 \to E_3 \to 0 \longmapsto E_2$$

$$ev_1 \times ev_3: \; \text{———} \, '' \, \text{———} \longmapsto (E_1, E_3)$$

- ev_2 is proper representable
- $ev_1 \times ev_3$ is derived lci (i.e., the cotangent complex $\mathbb{L}_{ev_1 \times ev_3}$ is perfect of amplitude $[-1,1]$)

Theorem (Kapranov-Vasserot, Yu Zhao for 0-dim. sheaves)
$K_0(\underline{\mathrm{Coh}}(S))$ has the structure of an associative algebra, whose product is given by:

$$\boxed{\begin{aligned} m: \; & K_0(\underline{\mathrm{Coh}}(S)) \times K_0(\underline{\mathrm{Coh}}(S)) \xrightarrow{\boxtimes} \\ & K_0(\underline{\mathrm{Coh}}(S) \times \underline{\mathrm{Coh}}(S)) \xrightarrow{(ev_2)_* \circ (ev_1 \times ev_3)^*} K_0(\underline{\mathrm{Coh}}(S)) \end{aligned}}$$

$\Longrightarrow$ K-theoretical Hall algebra $KHA(S)$ of S

Remark

By replacing $K_0(\underline{\mathrm{Coh}}(S))$ with

- $H_*^{BM}(\underline{\mathrm{Coh}}(S))$ = Borel-Moore homology of $\underline{\mathrm{Coh}}(S)$
  $\Longrightarrow$ Cohomological Hall algebra of S
- $D^b_{coh}(\underline{\mathbb{R}\mathrm{Coh}}(S)) \Longrightarrow$ Categorified Hall algebra of S
  (Porta-S.)

Remark

- The Thms hold also for
  - S only quasi-proj.
  - $\underline{\mathrm{Coh}}_{ps}(S)$ = moduli stack of properly supported sheaves on S
- $\exists$ an equivariant version of the Thms w.r.t.

$$T = \text{torus} \curvearrowright S \rightsquigarrow T \curvearrowright \underline{\mathrm{Coh}}_{ps}(S)$$

- if τ is a Serre subcategory of $\mathrm{Coh}_{ps}(S)$ such that the moduli stack $\underline{\mathbb{R}\mathrm{Coh}}_\tau(S)$ of objects in τ is open in $\underline{\mathbb{R}\mathrm{Coh}}_{ps}(S)$, then $\exists$

$KHA_{\mathcal{T}}(S)$ = K-theoretical Hall algebra of $\mathcal{T}$

For example,

- $\mathcal{T} = Coh_0(S) = \{\mathcal{E} \in Coh_{ps}(S) : \dim supp(\mathcal{E}) = 0\} \subset Coh_{ps}(S)$
- $\mathcal{T} = Coh_{\leq 1}(S) = \{\mathcal{E} \in Coh_{ps}(S) : \dim supp(\mathcal{E}) \leq 1\} \subset Coh_{ps}(S)$

Now, I will describe how to construct repr.s of $KHA_{\mathcal{T}}(S)$.

From now on, assume that S is projective.

Fix a torsion pair $v = (\mathcal{T}, \mathcal{F})$ of $Coh(S)$, i.e.,

- $Hom(T, F) = 0 \quad \forall\, T \in \mathcal{T},\ F \in \mathcal{F}$;
- $\forall\, E\ \exists\ 0 \longrightarrow \underset{\in \mathcal{T}}{T} \longrightarrow E \longrightarrow \underset{\in \mathcal{F}}{F} \longrightarrow 0$

such that

- $\mathcal{T}$ is a Serre subcategory, and
- the moduli stacks $\mathbb{R}\underline{Coh}_{\mathcal{T}}(S)$ and $\mathbb{R}\underline{Coh}_{\mathcal{F}}(S)$ are open in $\mathbb{R}\underline{Coh}(S)$.

Fix a stack $\mathfrak{M} \subseteq \underline{\mathbb{R}\mathrm{Coh}}_{F}(S)$. Set $\mathfrak{X} := \underline{\mathbb{R}\mathrm{Coh}}_{\tau}(S)$.

Goal: define left or right action of $KA_{\tau}(S)$ on $K_0(\mathfrak{M})$

$\Longrightarrow$ Nakajima type operators will be defined via these actions

Consider

$$0 \longrightarrow \underset{\in \mathfrak{M}}{\mathcal{E}'} \longrightarrow \underset{\in \mathfrak{M}}{\mathcal{E}} \longrightarrow \underset{\in \mathfrak{X}}{\mathcal{G}} \longrightarrow 0$$

$$\begin{tikzcd}
\mathfrak{M} \times \mathfrak{X} \arrow[d] & \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S) \arrow[l, "ev_1^{\mathfrak{M}} \times ev_3^{\mathfrak{X}}"'] \arrow[r, "ev_2^{\mathfrak{M}}"] \arrow[d] & \mathfrak{M} \arrow[d] \\
\underline{\mathbb{R}\mathrm{Coh}}(S) \times \underline{\mathbb{R}\mathrm{Coh}}(S) & \underline{\mathbb{R}\mathrm{Coh}}^{ext}(S) \arrow[l, "ev_1 \times ev_3"] \arrow[r, "ev_2"'] & \underline{\mathbb{R}\mathrm{Coh}}(S)
\end{tikzcd}$$

If (RM): $\exists \; (ev_2^{\mathfrak{M}})_* \circ (ev_1^{\mathfrak{M}} \times ev_3^{\mathfrak{X}})^* \circ \boxtimes$

$\Longrightarrow$ $K_0(\mathfrak{M})$ is a right $KHA_{\tau}(S)$-module

If we exchange role of $ev_1 \longleftrightarrow ev_2$:

If (LM): $\exists \; (ev_1^{\mathfrak{M}})_* \circ (ev_3^{\mathfrak{X}} \times ev_2^{\mathfrak{M}})^* \circ \boxtimes$

$\Longrightarrow$ $K_0(\mathfrak{M})$ is a left $KHA_{\tau}(S)$-module

Attention: $K_0(\mathcal{M})$ is **NOT** a bimodule of $KHA_T(S)$.

Definition

We say that

- $\mathcal{M}$ is a right Hecke pattern for $\mathfrak{X}$ if it is open and

$$\underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{M},\mathcal{M},\mathfrak{X}}(S) \simeq \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\bullet,\mathcal{M},\mathfrak{X}}(S)$$

- $\mathcal{M}$ is a left Hecke pattern for $\mathfrak{X}$ if it is open and

$$\underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{M},\mathcal{M},\mathfrak{X}}(S) \simeq \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{M},\bullet,\mathfrak{X}}(S)$$

- $\mathcal{M}$ is a 2-sided Hecke pattern for $\mathfrak{X}$ if

$$\underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\bullet,\mathcal{M},\mathfrak{X}}(S) \simeq \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{M},\mathcal{M},\mathfrak{X}}(S) \simeq \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{M},\bullet,\mathfrak{X}}(S)$$

Proposition

$$\text{If } \mathcal{M} \text{ is } \begin{cases} \text{a left HP} \\ \text{a right HP} \\ \text{a 2-sided HP} \end{cases} \Longrightarrow \begin{cases} (L\mathcal{M}) \text{ holds.} \\ (R\mathcal{M}) \text{ holds.} \\ (L\mathcal{M}) \text{ and } (R\mathcal{M}) \text{ hold.} \end{cases}$$

Attention ⚠

We can consider Hecke patterns for $\mathfrak{X} \subset \underline{\mathbb{R}\mathrm{Coh}}_{\tilde{\tau}}(S)$ closed

$\Longrightarrow$ consider repr.s of the subalgebra of $\mathrm{KHA}_{\tilde{\tau}}$ generated by $K_0(\mathfrak{X})$.

3. Nakajima type operators

Fix a 2-sided Hecke pattern $\mathfrak{M}$ for $\mathfrak{X}$. Consider

$$\begin{tikzcd} & \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S) \arrow[dl, "ev_1^{\mathfrak{M}}"'] \arrow[d, "ev_3^{\mathfrak{X}}"] \arrow[dr, "ev_2^{\mathfrak{M}}"] & \\ \mathfrak{M} & \mathfrak{X} & \mathfrak{M} \end{tikzcd}$$

Definition

We define the Nakajima operators:

$$\boxed{e := (ev_2^{\mathfrak{M}})_* \circ (ev_1^{\mathfrak{M}} \times ev_3^{\mathfrak{X}})^* \circ \boxtimes \quad \text{and} \quad f := (ev_1^{\mathfrak{M}})_* \circ (ev_3^{\mathfrak{X}} \times ev_2^{\mathfrak{M}})^* \circ \boxtimes}$$

4. Examples of 2-sided HP for 0-dimensional sheaves

In this case, we can choose $\mathfrak{X} =$

- $\mathbb{R}\underline{\mathrm{Coh}}_0(S)$, or
- $\mathbb{R}\underline{\mathrm{Coh}}_0(S; \text{length}=1) \simeq \tilde{S} \times [pt/\mathbb{C}^*]$ a derived enh. of S

and we can choose $\mathcal{M} =$

- $\underline{\mathrm{Hilb}}(S)$ = Hilbert stack of pts of $S \simeq \mathrm{Hilb}(S) \times [pt/\mathbb{C}^*]$, or
- $\underline{M}^{st}_H(r, c_1)$

$\Longrightarrow$ Nakajima type operators e and f for $\tilde{S} \times [pt/\mathbb{C}^*]$ decompose, w.r.t. the $\mathbb{C}^*$-action, as

$$e = \bigoplus_{n \in \mathbb{Z}} e_n \quad \text{and} \quad f = \bigoplus_{n \in \mathbb{Z}} f_n$$

= Schiffmann-Vasserot and Neguț's ops e_n and f_n

5. Example of 2-sided HP for 1-dimensional sheaves

Fix an effective divisor $D \overset{i}{\hookrightarrow} S$, an ample divisor $H \subset S$, and $\alpha \in \mathbb{Q}$.

Let $\mathrm{Coh}^{(s)s}_{\alpha}(S) = \{ \mathcal{E} \in \mathrm{Coh}_{\leq 1}(S) : \mathcal{E}$ is H-(semi)stable of slope $\alpha \}$

slope $= \dfrac{\chi(-)}{H \cdot c_1(-)}$

Definition

A subcategory $\mathcal{X} \subset \mathrm{Coh}^{s}_{\alpha}(S)$ is admissible if

- $\mathcal{E} \in \mathcal{X}$ is scheme-theoretically supported on D
- $\mu_{H\text{-max}}(\mathcal{E} \otimes \mathcal{O}_S(D)) \leq \alpha \leq \mu_{H\text{-min}}(\mathcal{E} \otimes \mathcal{O}_S(-D)) \quad \forall \mathcal{E} \in \mathcal{X}$

We assume that the corresponding moduli stack $\mathfrak{X}$ is open and closed in $\mathbb{R}\underline{\mathrm{Coh}}^{ss}_{\alpha}(S)$.

Example

Let B be a smooth projective curve and let $S \xrightarrow{\pi} B$ be a smooth projective elliptic surface, which admits a section.

Let D be a singular fiber such that D_{red} = affine ADE configuration Q of rational (-2)-curves C_k.

Then, for any $d \in \mathbb{Z}$, $\{ i_* \mathcal{O}_{C_k}(d) \}_k \subset \mathrm{Coh}^{s}_{\alpha}(S)$ is admissible

α depending on d and $H \cdot C_k$

Going back to the general setting, fix
$\mathcal{M} \stackrel{\text{def}}{=}$ subcategory of $\mathrm{Coh}_{t.f.}(S)$ consisting of locally free sheaves $\mathcal{F}$ on S such that

$$\boxed{\mu_{H\text{-max}}\left(i_* i^* \mathcal{F} \otimes \mathcal{O}_S(D)\right) \leq \alpha \leq \mu_{H\text{-min}}\left(i_* i^* \mathcal{F}\right)}$$

Remark: $\mathfrak{M} \subseteq \mathbb{R}\underline{\mathrm{Coh}}_{t.f.}(S)$ is open.

Theorem
$\mathfrak{M}$ is a 2-sided Hecke pattern for $\mathfrak{X}$.

$\Longrightarrow$ $\exists$ Nakajima type operators for $\mathfrak{X}$ acting on the K-theory of $\mathfrak{M}$

Our result can be now reformulated as:

Theorem (Diaconescu-Porta-S.-Y. Zhao)
Let $\pi: S \longrightarrow B$ be a smooth projective elliptic surface which admits a section.
Let $D \subset S$ be a singular fiber such that D_{red} = affine ADE

configuration Q of (-2)-rational curves C_k (with $Q \neq A_1^{(1)}$).

Let $\mathcal{X} = \{ i_* \mathcal{O}_{C_k}(-1) \} \subset \mathrm{Coh}_0^s(S)$. Then, the Nakajima type operators for $\mathcal{X}$ give rise to an action of U_Q on $K_0(\mathcal{M})$.

Remark

- This provides a K-theoretical version of the paper "Langlands reciprocity for algebraic surfaces" by Ginzburg-Kapranov-Vasserot
- If instead of U_Q, one considers the Lie algebra $\mathfrak{g}_Q^{KM}$ Nakajima and Yoshioka constructed actions

$$\mathfrak{g}_Q^{KM} \curvearrowright H^*\left(\begin{array}{l}\text{moduli space of}\\ \text{Gieseker/Simpson stable}\\ \text{sheaves on } S\end{array}\right)$$

- Our framework works also when D = smooth projective curve of genus ≥ 1 with $D^2 < 0$

$\Longrightarrow$ current goal: compute the relations in this case.

6. 2-Sided Hecke patterns are rare for $\mathfrak{X} := \underline{\mathrm{RCoh}}_{\leq 1}(S)$

For example, set $\mathcal{M} := \underline{\mathrm{RCoh}}_{\mathrm{t.f.}}(S)$. Then

- a subsheaf of a torsion-free sheaf is torsion-free

$\Longrightarrow \underline{\mathrm{RCoh}}^{\mathrm{ext}}_{\bullet, \mathcal{M}, \mathfrak{X}}(S) \simeq \underline{\mathrm{RCoh}}^{\mathrm{ext}}_{\mathcal{M}, \mathcal{M}, \mathfrak{X}}(S)$

$\Longrightarrow \mathcal{M}$ is a right Hecke pattern for $\mathfrak{X}$

BUT

- an extension between a torsion and a torsion-free sheaves is not torsion-free:

$$0 \longrightarrow \underset{\mathrm{t.f.}}{E_1} \longrightarrow G \longrightarrow \underset{\mathrm{torsion}}{E_3} \longrightarrow 0 \quad \not\Longrightarrow \quad G = \mathrm{t.f.}$$

i.e., $\underline{\mathrm{RCoh}}^{\mathrm{ext}}_{\mathcal{M}, \bullet, \mathfrak{X}}(S) \not\simeq \underline{\mathrm{RCoh}}^{\mathrm{ext}}_{\mathcal{M}, \mathcal{M}, \mathfrak{X}}(S)$

i.e., $\mathcal{M}$ is not a left Hecke pattern for $\mathfrak{X}$

However, we are able to prove:

Theorem (Diaconescu-Porta-S.)
$K_0(\mathbb{R}\underline{\mathrm{Coh}}_{t.f.}(S))$ is a left and right module of $\mathrm{KHA}_{\leq 1}(S)$.

Remark
The Thm holds for any $\mathcal{M} \subseteq \mathbb{R}\underline{\mathrm{Coh}}_{t.f.}(S)$ which is either a left or right Hecke pattern for $\mathcal{X}$.

$\Longrightarrow$ This approach enlarges the family of possible moduli spaces and stacks for which we can define Nakajima type operators.

The proof follows from this observation: we can "rotate"

$$0 \longrightarrow \underset{\text{t.f.}}{E_1} \longrightarrow G \longrightarrow \underset{\text{torsion}}{E_3} \longrightarrow 0$$

in $D^b(Coh(S))$:

$$E_3 \longrightarrow E_1[1] \longrightarrow G$$

This triangle is a short exact sequence in the tilted heart:

$$D^b(Coh(S))^{\heartsuit_\tau} = \{ E \in D^b(Coh(S)) : \mathcal{H}^{-1}(E) = \text{t.f.},$$
$$\mathcal{H}^0(E) = \text{torsion}, \ \mathcal{H}^i(E) = 0 \ \forall i \neq -1, 0 \}$$

Attention ⚠ :

$(Coh_{t.f.}(S)[1], Coh_{\leq 1}(S))$ is a torsion pair of $D^b(Coh(S))^{\heartsuit_\tau}$

In particular,

$$\underset{\text{torsion}}{E_3} \longrightarrow \underset{\text{t.f.}[1]}{E_1[1]} \longrightarrow G \implies G = E_2[1] \text{ with } E_2 = \text{t.f.}$$

i.e., $\underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{M},\mathcal{M},\mathcal{X}}(S) \underset{\text{"rotation"}}{\simeq} \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{X},\mathcal{M}[1],\mathcal{M}[1]}(S;\heartsuit_\tau)$

$$\simeq \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{X},\mathcal{M}[1],\bullet}(S;\heartsuit_\tau)$$

i.e., $\mathcal{M}$ is a right Hecke pattern for $\mathcal{X}$ in the tilted heart.