Cohomological Hall algebras of 1-dimensional sheaves and Yangians over the Bridgeland's space of stability conditions

(based on

- joint work with D.-E. Diaconescu, M. Porta, O. Schiffmann, and E. Vasserot, arXiv: 2603.03386;

- joint work with O. Schiffmann and P. Shimpi, arXiv: 2511.08576)

1. (Non)standard halves of affine Lie algebras

- $Q_{fin} = (I_{fin}, \Omega_{fin})$ = finite ADE quiver (e.g. $Q = A_{N-1}$)
- $\mathfrak{g}_{fin}$ = simple Lie algebra assoc. to Q_{fin} (e.g. $\mathfrak{g}_{fin} = \mathfrak{sl}(N)$)
- Δ_{fin} = root system of $Q_{fin} = \Delta^+_{fin} \cup \Delta^-_{fin}$ (stand. deco.)
- W_{fin} = Weyl group of Q_{fin}

Standard fact: any set of positive roots of Δ_{fin} is $W_{fin} \times \{\pm 1\}$-conjugate to Δ^+_{fin}

Now, let

- $\mathfrak{g}_{aff}$:= universal central extension of $\mathfrak{g}_{fin}$ = Lie algebra of the affine quiver Q ($\overset{v.s.}{=} \mathfrak{g}_{fin} \otimes \mathbb{C}[s^{\pm 1}] \oplus \mathbb{C}c$)
- Δ = root system of $Q = \Delta^+ \cup \Delta^-$ (standard deco.)
- W_{aff} = Weyl group of Q

Attention ⚠: a set of positive roots of Δ is not necessarily $W_{aff} \times \{\pm 1\}$-conjugate to Δ^+

The sets of positive roots of Δ were classified by Jakobsen-Kac. They defined a recipe:

$$\phi \subseteq J \subseteq I_{fin} \longmapsto \Delta_J^{\pm} = \text{set of pos./neg. roots of } \Delta$$

Jacobsen-Kac: any set of positive roots of Δ is $W_{aff} \times \{\pm 1\}$-conj. to one of the Δ_J^{+}'s.

Example ($Q_{fin} = A_1 = \cdot$)

- $J = I_{fin}$:

$$\Delta_J^{+} = \mathbb{N}_{\geq 1}\delta \cup \left(\Delta_{fin}^{+} \oplus \mathbb{Z}\delta\right) = \text{'nonstandard' half}$$

- $J = \phi$:

$$\Delta_J^{+} = \Delta_{fin}^{+} \cup \left(\Delta_{fin} \oplus \mathbb{N}_{\geq 1}\delta\right) \cup \mathbb{N}_{\geq 1}\delta = \text{'standard' half}$$

Goal of the talk:

Give a 'geometric' interpretation of Jacobsen-Kac's classification in terms of COHAs and Yangians

2. The affine Yangian

Define the elliptic Lie algebra $\mathfrak{g}_{ell}$ as:

$$\boxed{\mathfrak{g}_{ell} := \text{universal central extension of the Lie algebra } \mathfrak{g}_{fin} \otimes \mathbb{Q}[s^{\pm 1}, t]}$$

As v.s.:

$$\mathfrak{g}_{ell} = \left(\mathfrak{g}_{fin} \otimes \mathbb{Q}[s^{\pm 1}, t]\right) \oplus \left(\bigoplus_{\ell \in \mathbb{N}} \mathbb{Q} c_\ell \oplus \bigoplus_{\substack{\ell \in \mathbb{N}, \ell \geq 1 \\ k \in \mathbb{Z}, k \neq 0}} \mathbb{Q} c_{k,\ell} \right)$$

central

Let $R := \mathbb{Q}[\varepsilon_1, \varepsilon_2]$.

The affine Yangian = associative algebra $\mathbb{Y}/R$ s.t.:

$$\boxed{\mathbb{Y} = \text{a 'deformation' of } U(\mathfrak{g}_{ell})}$$

i.e., $\mathbb{Y} \otimes_R \mathbb{Q} \simeq U(\mathfrak{g}_{ell})$ and $\exists$ a filtration $F^\bullet$ of $\mathbb{Y}$ s.t.

$$gr_{F^\bullet} \mathbb{Y} \simeq U(\mathfrak{g}_{ell}) \otimes R$$

Attention ⚠:

- $\mathcal{Y}$ is a Hopf algebra, but in this talk we will only focus on the algebra structure.
- In the following, we say that A = 'deformation' of B if the above properties hold for A, B.

Returning back to $\mathcal{Y}$, $\exists$ a triangular decomposition

$$\mathcal{Y}^{+} \otimes \mathcal{Y}^{0} \otimes \mathcal{Y}^{-} \xrightarrow{\sim} \mathcal{Y}$$

In addition, $\mathcal{Y}^{-}$ is a 'deformation' of $U(\mathfrak{n}_{\mathrm{ell}})$, where

- $\mathfrak{n}_{\mathrm{ell}} := \mathfrak{n}[t] \oplus \bigoplus_{k<0, \ell \geq 1} \mathbb{C}_{k,\ell}$
- $\mathfrak{n} :=$ 'standard' negative half of $\mathfrak{g}_{\mathrm{aff}}$

$= \text{negative half of } \mathfrak{g}_{\mathrm{aff}} \text{ w.r.t. } \Delta^{-} \subset \Delta$

Now, by Jakobsen-Kac: $\phi \subseteq J \subseteq I_{\mathrm{fin}} \rightsquigarrow \Delta^{-}_{J} \rightsquigarrow$

'nonstandard' halves of $\mathfrak{g}_{aff}$ and $\mathfrak{g}_{ell}$:

$$\begin{cases} \mathfrak{n}^J := \bigoplus_{\beta \in \bar{\Delta}_J} (\mathfrak{g}_{aff})_\beta \\ \mathfrak{n}^J_{ell} := \mathfrak{n}^J[t] \oplus \bigoplus_{\substack{k<0 \\ \ell \geq 1}} c_{k,\ell} \end{cases}$$

β-graded piece

Goals:

1. construct a 'deformation' of $U(\mathfrak{n}^J_{ell})$ in a 'geometric' way
2. organize these 'deformations' into a 'family' parametrized by J

More precisely, we shall consider

$$\boxed{\text{Stab} = \text{Bridgeland's space of stability conditions}}$$

(more precise later on)

completion

Theorem (S.-Schiffmann-Shimpi)

$\forall x \in \mathrm{Stab}$, $\exists\ \mathbb{Y}_x$ = 'deformation' of $\hat{U}^x(\mathfrak{n}^x_{ell})$, where

$$\mathfrak{n}^x_{ell} = b_x \cdot \mathfrak{n}^J_{ell}$$

for some $\emptyset \subseteq J \subseteq I_{fin}$ and for an element $b_x \in W$.

Conjecture

$\mathbb{Y}_x$ realizes a half of a completion $\hat{\mathbb{Y}}^x$ of $\mathbb{Y}$

Attention ⚠: The proof is geometric and based on the theory of cohomological Hall algebras.

3. COHAs of surfaces: Construction

$$\boxed{\text{Moduli stack} \longrightarrow \text{COHA} \longrightarrow \text{Associative algebra}}$$

S = smooth quasi-projective surface $/\mathbb{C}$

T = (possibly trivial) torus $\curvearrowright S$

$C \subset S$ = T-invariant closed reduced subscheme

Consider

$\mathrm{Coh}_C(S)$ = moduli stack of coherent sheaves on S set-theoretically supported on C

Example

$S = T^*X$, X = smooth projective curve $/\mathbb{C}$

$C = X \subset T^*X$ zero section

$\Longrightarrow$ $\underline{Coh}_X(T^*X)$ = moduli stack of Higgs sheaves $(\mathcal{F}, \phi: \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega^1_X)$ on X, such that ϕ is nilpotent

S-Schiffmann: $S = T^*X$; Diaconescu-Porta-S-Schiffmann-Vasserot:

1. $$\boxed{\exists\ HA^{(T)}_{S,C} = (T\text{-equivariant})\ \text{COHA associated to } \underline{Coh}_C(S)}$$

= unital associative algebra structure on

$$\underbrace{H^{(T)}_*}_{}(\mathbb{R}\underline{Coh}_C(S)) = \underbrace{H^{(T)}_*}_{}(\underline{Coh}_C(S))$$

(equivariant Borel-Moore homology)

where the multiplication $(\mathbb{R}p)_* \circ (\mathbb{R}q)^!$ is induced by

$$\mathbb{R}\underline{Coh}_C(S) \times \mathbb{R}\underline{Coh}_C(S) \xleftarrow{\mathbb{R}q} \underbrace{\mathbb{R}\underline{Coh}^{ext}_C(S)}_{} \xrightarrow{\mathbb{R}p} \mathbb{R}\underline{Coh}_C(S)$$

(stack of extensions)

Here,

$$p: \; 0 \to E_2 \to E \to E_1 \to 0 \;\longmapsto\; E$$

$$q: \; \text{—————— '' ——————} \;\longmapsto\; (E_2, E_1)$$

2. $HA^{(T)}_{S,C}$ depends "locally" on C, i.e., given (S_1, C_1) and (S_2, C_2) such that the formal completions $(\widehat{S_1})_{C_1} \simeq (\widehat{S_2})_{C_2}$

we have:

$$HA^{(T)}_{S_1,C_1} \simeq HA^{(T)}_{S_2,C_2}$$

4. COHA of minimal resolution of ADE singularity

- $G \subset SL(2,\mathbb{C})$ finite subgroup
  $\updownarrow$
  Q_{fin} = finite ADE quiver, Q = affine ADE quiver
- $\pi : S \longrightarrow \mathbb{C}^2/G$ Kleinian resolution of singularities
- $C := \pi^{-1}(0)$. Then, $C_{red} = C_1 \cup \cdots \cup C_e$, $C_i \simeq \mathbb{P}^1$

$(C_i \cdot C_j) = -$ Cartan matrix of Q_{fin}

- torus $T \subset GL(2,\mathbb{C})$ centralizing G
  ($T =$ trivial, $\mathbb{C}^*$, or $\mathbb{C}^* \times \mathbb{C}^*$)

Example: $G = \mathbb{Z}_2 \Longrightarrow Q_{fin} = A_1 = \cdot$, $Q = A_1^{(1)}$: 

$\Longrightarrow S = T^*\mathbb{P}^1 \supset C = \mathbb{P}^1 =$ zero section

Recall the derived McKay correspondence:

$$\tau : D^b(\mathrm{Coh}_C(S)) \xrightarrow{\sim} D^b(\mathrm{nilp}(\Pi_Q))$$

where

- $\Pi_Q =$ preprojective algebra of Q

  $:= \mathrm{End}($tilting bundle on S inducing $\tau)$

- $\mathrm{nilp}(\Pi_Q) =$ nilpotent finite-dim. right Π_Q-modules $=: \Pi_Q$-repr.s

Important:
The derived McKay correspondence is not t-exact w.r.t. the standard t-structures, i.e.,

$$
\begin{array}{ccc}
\mathrm{Coh}_c(S) & \cancel{\xrightarrow{\sim}} & \mathrm{nilp}(\Pi_Q) \\
& \Downarrow & \\
\underline{\mathrm{Coh}}_c(S) & \cancel{\simeq} & \Lambda_Q = \text{stack of nilpotent repr.s of } \Pi_Q \\
& \Downarrow & \\
\mathbb{HA}^{T}_{S,C} & \cancel{\simeq} & \mathbf{COHA}^{T,\mathrm{nil}}_Q \simeq \mathbb{Y}^-
\end{array}
$$

DPSSV

where $\mathbf{COHA}^{T,\mathrm{nil}}_Q$ = Schiffmann-Vasserot's nilpotent 2d COHA of the quiver Q

Attention ⚠: The relation between $\mathrm{Coh}_c(S)$ and $\mathrm{nilp}(\Pi_Q)$ is more subtle:

$$\boxed{\mathrm{Coh}_c(S) = \text{"limit" of } \mathrm{nilp}(\Pi_Q)}$$

More precisely,

- hearts of bounded t-structures on $\mathcal{C} = D^b(\mathrm{nilp}(\Pi_Q))$ form a partial ordered set:

$$\boxed{H_1 := \mathcal{C}_1^{\geq 0} \cap \mathcal{C}_1^{\leq 0} \leq H_2 := \mathcal{C}_2^{\geq 0} \cap \mathcal{C}_2^{\leq 0} \iff \mathcal{C}_1^{\geq 0} \subseteq \mathcal{C}_2^{\geq 0}}$$

- $\check{\theta} \in \check{X}_{fin}$ strictly dominant (i.e. $\check{\theta}(\alpha_i) > 0, \forall i$) $\underset{\text{via McKay}}{\rightsquigarrow}$ $\mathcal{L}_{\check{\theta}}$ π-ample Set

$$\tilde{L}_{\check{\theta}} := \tau \circ (\mathcal{L}_{\check{\theta}} \otimes -) \circ \tau^{-1} : D^b(\mathrm{nilp}(\Pi_Q)) \longrightarrow D^b(\mathrm{nilp}(\Pi_Q))$$

- Shimpi: $\boxed{\inf_{n \geq 0} \tilde{L}_{\check{\theta}}^{-n}(\mathrm{nilp}(\Pi_Q)) = \mathrm{Coh}_C(S)}$

By translating Shimpi's result in the theory of COHAs, we obtain:

<u>Theorem</u> (Diaconescu-Porta-S-Schiffmann-Vasserot)

- $\exists$ a canonical algebra isomorphism

$$\mathbb{HA}_{S,C} \simeq \hat{U}(\mathfrak{n}_{ell}^{I_{fin}})$$

(completion)

- $\exists$ a canonical algebra isomorphism

$$\mathbb{HA}^{T}_{S,C} \simeq \lim_{\ell} T^{2\ell}_{\check{\theta}}(\mathbb{Y}^{-}) \big/ T^{2\ell}_{\check{\theta}}(J)$$

extended affine braid group op.

where $J := \sum_{(\check{\theta}, \underline{d}) \leq 0} \mathbb{Y}^{-}_{-\underline{d}} \cdot \mathbb{Y}^{-}$.

In particular, $\mathbb{HA}^{T}_{S,C}$ is a 'deformation' of $\hat{U}(\mathfrak{n}^{I_{\mathrm{fin}}}_{\mathrm{ell}})$

Goal: $\forall\, \phi \neq J \subsetneq I_{\mathrm{fin}}$, define a 'deformation' of $\hat{U}(\mathfrak{n}^{J}_{\mathrm{ell}})$

Fix $\phi \neq J \subsetneq I_{\mathrm{fin}}$ and let $\check{\theta} \in \check{X}_{\mathrm{fin}}$ be s.t.:

$$\boxed{\check{\theta}(\alpha_i) = 0 \quad \forall\, i \in I_f \setminus J \quad \text{and} \quad \check{\theta}(\alpha_i) > 0 \quad \forall\, i \in J}$$

Shimpi: $\inf_{n \geq 0} \mathcal{I}^{-n}_{\check{\theta}}(\mathrm{nilp}(\Pi_Q)) = \mathcal{P}_c(S/S_J)$

where $\mathcal{P}_c(S/S_J)$ is defined as follows:

- contraction of C_i, for $i \in I_{fin} \setminus J$:

$$\begin{tikzcd} S \arrow[rr, "\pi_J"] \arrow[dr, "\pi"'] & & S_J \arrow[dl] \\ & \mathbb{C}^2/G & \end{tikzcd}$$

- $P_C(S/S_J) \subset D^b(Coh_C(S))$ = Van der Bergh's abelian category of perverse coherent sheaves (w.r.t. π_J), which are set-theoretically supported on C

- $P_C(S/S_J) \simeq Coh_{C_J}(\tilde{S}_J)$, where $\tilde{S}_J$ = canonical stack of S_J

$\Longrightarrow \exists\ HA_J^{(T)} := COHA^{(T)}$ of $P_C(S/S_J) \simeq Coh_{C_J}(\tilde{S}_J)$

<u>Theorem</u> (S-Schiffmann-Shimpi)

- $\exists$ a canonical algebra isomorphism

$$HA_J \simeq \hat{U}(n^J_{ell})$$

- HA_J^T is a 'deformation' of $\hat{U}(n^J_{ell})$

Now, let

- $\mathrm{Stab}(S) = \mathrm{Stab}(D^b(\mathrm{Coh}_c(S)))$ = space of stability conds

- $\mathrm{Stab}^\circ(S) \subset \mathrm{Stab}(S)$ be its distinguished connected component (containing all stab. conditions with heart $\mathrm{Coh}_c(S)$)

Lemma

$\forall\, x \in (Z, P) \in \mathrm{Stab}^\circ(S)$ and $\forall\, I \subset \mathbb{R}$ interval of lenght 1 $\exists\, \emptyset \subseteq J \subseteq_{\mathrm{fin}} I$ and an element b_x in the affine braid group s.t.

$$\mathcal{P}(I) \simeq b_x \cdot \mathcal{P}_c(S/S_J) \quad \text{(up to shifts)}$$

Therefore, we get:

Theorem (S-Schiffmann-Shimpi)

$\forall\, x = (\mathcal{P}, Z) \in \mathrm{Stab}^\circ(S)\ \exists\ HA_x^{(T)}$ associated to $\mathcal{P}(0,1]$:

$$\mathbb{Y}_x := HA_x^T = \text{'deformation' of } \widehat{U}^x(n_{ell}^x)$$

$$n_{ell}^x := b_x \cdot n_{ell}^J$$

Conjecture

$\mathbb{Y}_x$ realizes a half of a completion $\widehat{\mathbb{Y}}^x$ of $\mathbb{Y}$

Proof of the Thm for $J = I_{fin}$

We will use:

- Bridgeland stability conditions
- braid group actions on bounded derived cat's

Now, fix a strictly dominant coweight $\check{\theta} \in \check{X}_{fin} \hookrightarrow \check{X}$.

$\check{\theta}$ defines a King's stability condition on $\mathrm{nilp}(\Pi_Q)$:

$$Z_{\check{\theta}} : K_0(\mathrm{nilp}(\Pi_Q)) \simeq \mathbb{Z}I \longrightarrow \mathbb{C}$$
$$\underline{d} \longmapsto -(\check{\theta}, \underline{d}) + (\check{\rho}, \underline{d})i$$

$\check{\rho}(\alpha_i) = 1$

$$\Longrightarrow \boxed{\exists\ (Z_{\check{\theta}}, P_{\check{\theta}}) = \text{Bridgeland's stability condition on } D^b(\mathrm{nilp}(\Pi_Q))}$$

Here, $\mathcal{P}_{\check{\Theta}}$ = slicing = family of full additive subcats

$$\mathcal{P}_{\check{\Theta}}(\phi) \subset D^b(\mathrm{nilp}(\Pi_Q)) \quad \forall \phi \in \mathbb{R}$$

satisfying certain conditions.

Set $\mathcal{P}_{\check{\Theta}}(I) := \langle \mathcal{P}_{\check{\Theta}}(\phi) : \phi \in I \rangle$ $\forall$ interval $I \subset \mathbb{R}$

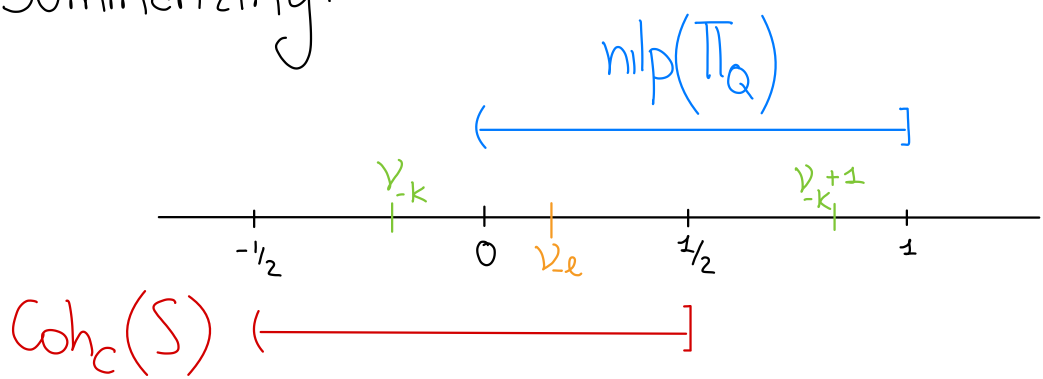
Lemma

1. $\forall k \in \mathbb{Z}$, $\tilde{L}^{-2k}_{\check{\Theta}} : \mathrm{nilp}(\Pi_Q) \xrightarrow{\sim} \mathcal{P}_{\check{\Theta}}\big((\nu_{-k}, \nu_{-k}+1]\big)$

with $\nu_\ell := \frac{1}{\pi}\arctan(2h\ell)$ Coxeter number

2. $\mathcal{P}_{\check{\Theta}}\big((-\frac{1}{2}, \frac{1}{2}]\big) \simeq \mathrm{Coh}_c(S)$

Summarizing:



$P_{\check\Theta}\big(\big(\nu_{-k}, \nu_{-k}+1\big]\big)$

$P_{\check\Theta}\big(\big(\nu_{-\ell}, \nu_{-k}+1\big]\big)$

For $\ell, k \in \mathbb{N}$, $k \geq \ell$, set

$$\boxed{\Lambda_Q^{\ell,k} := \text{derived moduli stack of objects in } P_{\check\Theta}\big(\big(\nu_{-\ell}, \nu_{-k}+1\big]\big)}$$

Attention:

1. Let $\Lambda_Q^{>\nu_{-\ell+k}} \subset \Lambda_Q$ = usual Harder-Narasimhan stratum
Then

$$\tilde{L}_{\check\Theta}^{2k} : \Lambda_Q^{\ell,k} \xrightarrow{\sim} \Lambda_Q^{\ell-k,0} \simeq \Lambda_Q^{>\nu_{-\ell+k}}$$

2. $\tilde{L}_{\check\Theta}^{2k}$ is a "braid group operator".

Recall that

- the extended affine braid group

$$\boxed{B_{ex} \simeq \big(B_{fin} \cup \{L_{\check\lambda} : \lambda \in \check{X}_{fin}\}\big) / \text{rel.s}}$$

Lemma

$\exists$ a group homomorphism

$$\rho : B_{ex} \longrightarrow \mathrm{Aut}(D^b(\mathrm{nilp}(\Pi_Q)))$$

such that $\rho(L_{\check{\lambda}}) = \tilde{L}_{\check{\lambda}}$ $\forall\, \lambda \in \check{X}_{fin}$

We have:

Lemma

1. The vector space

$$\boxed{HA^{(T)}_{\check{\Theta}} := \lim_{\ell} \operatorname*{colim}_{K \geq \ell} H^{(T)}_{*}(\Lambda^{\ell,K}_{Q})}$$

has the structure of an unital associative algebra with multiplication induced from that of $COHA^{(T),nil}_{Q}$

2. $\exists$ an algebra isomorphism $HA^{(T)}_{S,C} \simeq HA^{(T)}_{\check{\Theta}}$

Corollary

$HA^{(T)}_{\check{\Theta}}$ does not depend on the specific choice of strictly dominant finite coweight.

Now, we relate $HA^{T}_{\check{\Theta}} := \lim\limits_{\ell} \operatorname*{colim}\limits_{k\geq\ell} H^{T}_{*}(\Lambda^{\ell,k}_{Q})$ to Yangians.
Recall that

$$\mathrm{COHA}^{T,nil}_{Q} \simeq Y^{-}$$

(as proved by Schiffmann-Vasserot).

Then, as a vector space

$$\operatorname*{colim}_{k\geq\ell} H^{T}_{*}(\Lambda^{\ell,k}_{Q}) \simeq T^{2\ell}_{\check{\Theta}}(Y^{-}) \big/ T^{2\ell}_{\check{\Theta}}(J)$$

braid op.

where $J := \sum\limits_{(\check{\Theta},\underline{d})\leq 0} Y^{-}_{-\underline{d}} \cdot Y^{-}$.

Finally, we have:

Lemma

1. $\exists$ a unital associative algebra structure on:

$$Y_{S,C} := \lim_{\ell} T^{2\ell}_{\check{\Theta}}(Y^{-}) \big/ T^{2\ell}_{\check{\Theta}}(J)$$

2. $\exists$ an algebra isomorphism $HA^{T}_{\check{\Theta}} \xrightarrow{\sim} Y_{S,C}$. $\square$