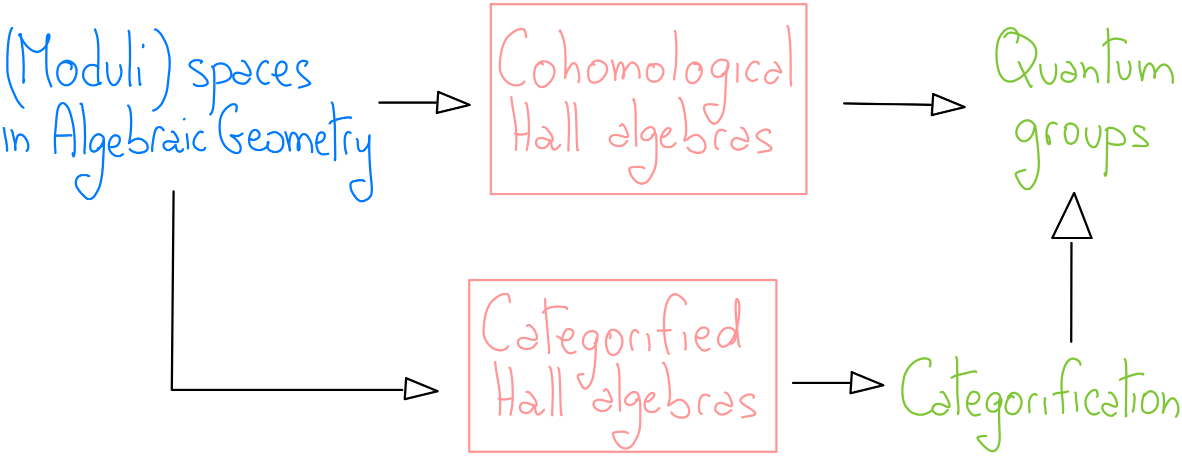
Cohomological Hall algebras, quantum groups, and their categorification

The goal of the talk is to explain:



1. Motivating example

Set $R := \mathbb{C}[\varepsilon_1, \varepsilon_2]$. Consider the following R-module

$$\bigoplus_{n\geq 0} R[t_1, \ldots, t_n]^{\mathfrak{S}_n}$$

endowed with the shuffle multiplication

$$(P * Q)(t_1, \ldots, t_{n+m}) = \frac{1}{n!m!} \mathrm{Sym}_{n+m}\Big(\prod_{\substack{1\leq i\leq n \\ n<j\leq n+m}} \zeta(t_i - t_j)$$

$$\cdot P(t_1, \ldots, t_n) \cdot Q(t_{n+1}, \ldots, t_{n+m})\Big)$$

where

$$\zeta(t) = \frac{(\varepsilon_1 - t)(\varepsilon_2 - t)(\varepsilon_1 + \varepsilon_2 + t)}{t}$$

Set Shuffle := the subalgebra generated by t_1^k for $k \geq 0$.

Fact:

Shuffle is an example of cohomological Hall algebras.

Let us introduce the geometrical framework.

Consider the commuting variety:

$$\mathrm{Comm}_n := \{(A,B) \in \mathrm{Mat}(n,\mathbb{C}) : [A,B] = 0\}$$

$\exists$ actions

- $GL(n,\mathbb{C}) \curvearrowright \mathrm{Comm}_n \quad : g\cdot(A,B) = (gAg^{-1}, gBg^{-1})$
- $T = \mathbb{C}^* \times \mathbb{C}^* \curvearrowright \mathrm{Comm}_n \quad : (t_1,t_2)\cdot(A,B) = (t_1A, t_2B)$

The cohomological Hall algebra (of the 1-loop quiver) is

- as vector space:

$$COHA^T(n) := \underbrace{H_c^{T\times GL(n)}(\mathrm{Comm}_n)^\vee}_{\text{dual of equivariant compactly supported cohomology}} \text{ and } COHA^T := \bigoplus_n COHA^T(n)$$

Rmk: $COHA^T(n) \hookrightarrow H^*_{T\times GL(n)}(pt) \simeq R[t_1,\dots,t_n]^{\mathfrak{S}_n}$

- The Hall multiplication $m = \bigoplus_{n_1,n_2} m_{n_1,n_2}$ is

$$m_{n_1,n_2} := (p_{n_1,n_2})_* \circ q^*_{n_1,n_2} : COHA^T_{n_1} \otimes COHA^T_{n_2} \longrightarrow COHA^T_{n_1+n_2}$$

$$\mathrm{Comm}_{n_1} \times \mathrm{Comm}_{n_2} \xleftarrow{q_{n_1,n_2}} \widetilde{\mathrm{Comm}}_{n_1,n_2} \xrightarrow{p_{n_1,n_2}} \mathrm{Comm}_{n_1+n_2}$$

$$\|$$

$$\mathrm{Comm}_{n_1+n_2} \cap \left(\text{parabolic of } \mathfrak{gl}(n_1+n_2,\mathbb{C}) \text{ fixing } \mathbb{C}^{n_1} \subset \mathbb{C}^{n_1+n_2}\right)^{\times 2}$$

Attention ⚠:

$$(A,B) \in \widetilde{\mathrm{Comm}}_{n_1,n_2} \Longrightarrow A = \left(\begin{array}{c|c} A_1 & * \\ \hline 0 & A_2 \end{array}\right),\ B = \left(\begin{array}{c|c} B_1 & * \\ \hline 0 & B_2 \end{array}\right)$$

such that $[A_1,B_1]=0$, $[A_2,B_2]=0$

$$\Longrightarrow p_{n_1,n_2} : (A,B) \longmapsto (A,B) \text{ and } q_{n_1,n_2} : (A,B) \longmapsto \big((A_1,B_1),(A_2,B_2)\big)$$

Thm (Schiffmann-Vasserot)

- m endows COHA^T with the structure of a unital associative algebra over $R \simeq \mathbb{C}[\varepsilon_1,\varepsilon_2]$.

- Shuffle $\simeq \mathrm{COHA}^T \simeq Y^+(\hat{\mathfrak{gl}}(1))$

$$= \text{a positive part of } Y(\hat{\mathfrak{gl}}(1))$$

Attention ⚠:

- $Y(\widehat{\mathfrak{gl}}(1)) = Y^-(\widehat{\mathfrak{gl}}(1)) \otimes Y^0(\widehat{\mathfrak{gl}}(1)) \otimes Y^+(\widehat{\mathfrak{gl}}(1))$

  = affine Yangian of $\mathfrak{gl}(1)$ = deformation of $U(\mathfrak{g}^{MO}_{1\text{-loop}}[z])$

- $Y(\widehat{\mathfrak{gl}}(1))$ is a (topological) Hopf algebra defined via

  – an R-matrix (geometrically def. by Maulik and Okounkov)

  – generators and relations (by Schiffmann-Vasserot)

- COHA^T realizes only an half of the Yangian

  $\Longrightarrow$ to get the full Yangian one has to "double" it

Attention ⚠:

$Y(\widehat{\mathfrak{gl}}(1))$ is an example of a family of Yangians defined through geometric techniques by Maulik-Okounkov:

$$Q = (I, E) = \text{quiver} \rhd \begin{cases} Y_Q^{MO} = \mathbb{Z} \times \mathbb{Z}^I\text{-graded Hopf algebra} \\ \qquad = \text{deformation of } U(\mathfrak{g}_Q^{MO}[z]) \\ \mathfrak{g}_Q^{MO} = \mathbb{Z} \times \mathbb{Z}^I\text{-graded Lie algebra} \\ \mathfrak{g}_Q^{MO}[0] = \mathfrak{g}_Q^{KM} = \text{Kac-Moody Lie algebra of } Q \end{cases}$$

Going back to Comm_n, note that

$$(A, B) \in \mathrm{Comm}_n \rightsquigarrow A \circlearrowleft \mathbb{C}^n \circlearrowright B \quad + [A, B] = 0$$

$$\rightsquigarrow \text{n-dim. left module of } \Pi_{1\text{-loop}}$$

$\Longrightarrow \Pi_{1\text{-loop}}$ = preprojective algebra of 1-loop quiver $(= \overset{x}{\circlearrowleft})$

$= \mathbb{C}\langle x, y \rangle / [x, y] \simeq$ algebra of polynomials in x, y

$\Longrightarrow \mathrm{Comm}_n / GL(n) \simeq$ space $\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}, n)$ of n-dimensional left modules of $\Pi_{1\text{-loop}}$

$\widetilde{\mathrm{Comm}_{n_1,n_2}}/P \simeq$ space $\underline{\mathrm{Rep}}^{ext}(\Pi_{1\text{-loop}}, n_1, n_2)$ of extensions

$$0 \longrightarrow M_{n_1} \longrightarrow M_{n_1+n_2} \longrightarrow M_{n_2} \longrightarrow 0$$

of left modules of $\Pi_{1\text{-loop}}$

$$\Longrightarrow \mathrm{COHA}^T = \mathrm{COHA}^T_{1\text{-loop}} = H^T_c\Big(\underbrace{\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}})}_{\coprod_n \underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}, n)}\Big)^\vee + \text{Hall mult.}$$

By replacing $1\text{-loop} = \circlearrowleft$ with arbitrary quivers Q:

<u>Thm</u> (Schiffmann-Vasserot)

1. $\exists$ a Hall multiplication on $\mathrm{COHA}^{T_Q}_Q := H^{T_Q}_c(\underline{\mathrm{Rep}}(\Pi_Q))^\vee$ such that $\mathrm{COHA}^{T_Q}_Q$ is a unital associative algebra over $H^*_{T_Q}(pt)$

2. $\mathrm{COHA}^{T_Q}_Q \simeq Y^{MO,+}_Q$

<u>Attention</u> ⚠:

- (2) $\Longrightarrow$ $\exists$ geometrically defined generators of $Y^{MO,+}_Q$

$\Longrightarrow Y_Q^{MO}$ has a presentation by generators and relations for certain quivers, e.g., $Q = \circlearrowleft$, ADE, affine ADE

- We can replace:

$H_c^T(-)^\vee \rightsquigarrow G_0^T(-)$ = Grothendieck group of coherent sheaves

$COHA_Q^{T_Q} \rightsquigarrow KHA_Q^{T_Q}$ = K-theoretical Hall algebra

$Y_Q^{MQ} \rightsquigarrow$ quantum loop algebra of Q = quantum enveloping algebra of $L\mathfrak{g}_Q^{MO}$

Goal: Replace $\mathrm{Rep}(\Pi_Q)$ with another (moduli) space

$\Downarrow$

define new families of Lie algebras, and their Yangians and quantum loop algebras.

Thm (S-Schiffmann, Kapranov-Vasserot)
Let S = smooth quasi-projective complex surface.
Assume that $\exists$ (possibly trivial) torus $T \curvearrowright S$.

Let $\underline{Coh}_{ps}(S)$ be the "space" (derived moduli stack) of properly supported coherent sheaves on S.

Then $\exists$ a Hall multiplication on

$$\mathrm{COHA}_S^T := H_c^T(\underline{Coh}_{ps}(S))^\vee$$

such that it is a unital associative algebra.

A similar result holds also in K-theory.

These algebras have been described in some cases:

<u>Thm</u> (Diaconescu-Porta-S-Schiffmann-Vasserot)
Let $S \longrightarrow \mathbb{C}^2/G$ be resolution of ADE singularity.
Then

$$\mathrm{COHA}_S^T \simeq \lim_{\ell \leq 0} Y_{(\ell)} \quad \text{where } Y_{\widehat{ADE}}^{MO,+} \twoheadrightarrow Y_{(\ell)}$$

where the transition maps in are induced by the action of a fixed $T \in B_{ex}$ = extended affine braid group of type ADE

Conjecture

COHA_S^T is a new half of $Y_{\widehat{ADE}}^{\widehat{MO}}$ — completion

For this choice of the surface S, we get $Y_{\widehat{ADE}}^{MO}$

For other choices of the surface, we may obtain new Yangians, e.g.:

Conjecture

- $\exists$ a deformation $Y(\widehat{\mathfrak{gl}}(1))$ of $U(\mathfrak{g}^{MO}_{1\text{-loop}}[z])$ such that

$$\mathrm{COHA}^{\mathbb{C}^*}_{T^*(\text{Elliptic curve})} \simeq Y^+(\widehat{\mathfrak{gl}}(1))$$

Let us finish by mentioning the categorification:

Thm (Porta-S, Diaconescu-Porta-S.)
Let $\mathcal{X} = \mathrm{Rep}(\Pi_Q)$ or $\mathrm{Coh}_{ps}(S)$ (or another "nice" space).
Let $D^b\mathrm{Coh}(\mathcal{X})$ be (a dg enhancement of) the bounded derived category of coherent sheaves on $\mathcal{X}$.

Then, the functor

$$p_* \circ q^* : D^b\mathrm{Coh}(\mathcal{X}) \otimes D^b\mathrm{Coh}(\mathcal{X}) \longrightarrow D^b\mathrm{Coh}(\mathcal{X})$$

endows $\mathrm{CatHA}_{\mathcal{X}} := D^b\mathrm{Coh}(\mathcal{X})$ with the structure of a ($\mathbb{E}_1$-) monoidal category.

Moreover, by passing to $K(D^b\mathrm{Coh}(\mathcal{X})) \simeq G_0(\mathcal{X})$ one recovers the K-theoretical Hall algebra of $\mathcal{X}$.

Rmk
The categorified Hall algebras of $\mathrm{Rep}(\Pi_Q)$ or $\mathrm{Coh}_{ps}(T^*X)$ or $\mathrm{Coh}_{ps}(\text{K3 surface})$ can be described explicitly (due to Pădurariu and Toda)