Cohomological Hall algebras of 1-dimensional sheaves and Yangians

(joint work with D.E. Diaconescu, M. Porta, O. Schiffmann, and É. Vasserot: arXiv:2502.19445)

Plan:

1. Overview of 2d Cohomological Hall algebras

2. Nilpotent COHA of a surface and affine Yangians

1. Overview of 2d Cohomological Hall algebras

▸ Quivers

$Q = \text{quiver} = \left(I = \{\text{vertices}\}, \Omega = \{\text{edges}\} \right)$

$\rightsquigarrow Q^{db} = \text{double quiver} = \left(I, \Omega \sqcup \Omega^{op} =: \Omega^{db} \right)$

$\Omega^{op} = \left\{ e^* : j \longrightarrow i \;\middle|\; e : i \longrightarrow j \in \Omega \right\}$

$\rightsquigarrow \mathbb{C}Q^{db} = \text{path algebra of } Q^{db}$

$\rightsquigarrow \Pi_Q =$ preprojective algebra of $Q = \mathbb{C}Q^{db} \Big/ \sum_{e \in \Omega} [e, e^*]$

(preprojective rels)

Example

$Q : \circlearrowleft_{e}$ = one-loop quiver $\Longrightarrow \Pi_{1\text{-loop}} \simeq \mathbb{C}\langle e, e^* \rangle \big/ [e, e^*] \simeq \mathbb{C}[e, e^*]$

Denote:

$$\boxed{\underline{\mathrm{Rep}}(\Pi_Q) = \text{moduli stack of finite-dimensional representations of } \Pi_Q}$$

- $\underline{\mathrm{Rep}}(\Pi_Q) = \coprod_{\underline{d} \in \mathbb{N}^I} \underline{\mathrm{Rep}}(\Pi_Q; \underline{d})$

dimension vector

- $\underline{\mathrm{Rep}}(\Pi_Q; \underline{d})$ = quotient stack $C_{\underline{d}} / GL(\mathbb{C}; \underline{d})$

where

- $GL(\mathbb{C}; \underline{d}) := \prod_{i \in I} GL(\mathbb{C}; d_i)$
- $C_{\underline{d}} = \left\{ (A_e)_{e \in \Omega^{db}} \in \bigoplus_{e \in \Omega^{db}} \mathrm{Hom}\left(\mathbb{C}^{d_{s(e)}}, \mathbb{C}^{d_{t(e)}}\right) : \sum_{e \in \Omega} [A_e, A_{e^*}] = 0 \right\}$

Example: Q = 1-loop quiver

- $\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}; d) = C_d / GL(\mathbb{C}; d)$
- $C_d = \left\{ (A_1, A_2) \in \mathrm{Mat}(\mathbb{C}, d)^{\times 2} : [A_1, A_2] = 0 \right\}$ = commuting variety

- $\exists$ torus action given as:

$$(\mathbb{C}^*)^{\Omega} \times \mathbb{C}^* \curvearrowright \underline{\mathrm{Rep}}(\Pi_Q; \underline{d})$$

$$\underset{\displaystyle (t_e, t)}{\mathbin{\rotatebox[origin=c]{90}{$\in$}}} \cdot \underset{\displaystyle (A_e, A_e^* := A_{e^*})_{e \in \Omega}}{\mathbin{\rotatebox[origin=c]{90}{$\in$}}} = (t_e A_e, t_e^{-1} t A_{e^*})_{e \in \Omega}$$

Example: Q = 1-loop quiver

- $(\mathbb{C}^*)^\Omega \times \mathbb{C}^* = (\mathbb{C}^*)^{\times 2}$

- $(t_1, t) \cdot (A_1, A_2) = (t_1 A_1, t_1^{-1} t A_2)$

Schiffmann-Vasserot, Yang-Zhao: fix $T \subseteq (\mathbb{C}^*)^\Omega \times \mathbb{C}^*$ subtorus

$$\boxed{\exists\ \mathrm{COHA}_Q^T = T\text{-equivariant COHA associated to finite-dim. reprs of } \Pi_Q}$$

= unital associative algebra structure on

$$H_*^T(\underline{\mathrm{Rep}}(\Pi_Q)) \simeq \bigoplus_{\underline{d} \in \mathbb{Z} I} H_*^{GL(\mathbb{C};\underline{d}) \times T}(C_{\underline{d}})$$

with multiplication $p_* \circ q^!$ induced by:

$$\underline{\mathrm{Rep}}(\Pi_Q) \times \underline{\mathrm{Rep}}(\Pi_Q) \xleftarrow{q} \underbrace{\underline{\mathrm{Rep}}^{ext}(\Pi_Q)}_{=\text{stack of extensions}} \xrightarrow{p} \underline{\mathrm{Rep}}(\Pi_Q)$$

where:

$p: 0 \to E_2 \to E \to E_1 \to 0 \mapsto E$ (proper)

$q:$ ——— " ——— $\mapsto (E_2, E_1)$

Also, Schiffmann-Vasserot:

$\exists$ $\mathrm{COHA}_Q^{(T),nil}$ = (T-equivariant) COHA associated to the moduli stack Λ_Q of strongly semi-nilpotent reprs of Π_Q

Attention ⚠:

strongly semi-nilpotent = nilpotent (i.e. both A_e and A_{e^*} nil.) if Q is without edge-loops

Set

$$\boxed{\mathrm{HA}_Q^T := \mathrm{COHA}_Q^{T,nil}}$$

Now, let's recall the main result relating COHAs of quivers and Yangians:

Theorem (Schiffmann-Vasserot, Botta-Davison)

Let Q be an arbitrary quiver and $T = T_{max} = (\mathbb{C}^*)^{\Omega} \times \mathbb{C}^*$.

$\exists$ an isomorphism of $H_T^*(pt)$-algebras:

$$\Psi : \mathrm{HA}_Q^T \xrightarrow{\sim} \mathbb{Y}_{Q,T}^{MO,-}$$

Here, $Y^{MO,-}_{Q,T}$ = negative part of Maulik-Okounkov Yangian $Y^{MO}_{Q,T}$ of Q

w.r.t. triangular dec.

Remark

Maulik-Okounkov: definition of $Y^{MO}_{Q,T}$ via R-matrix

= filtered deformation of $U(\mathfrak{g}^{MO}_{Q}[t])$

where $\mathfrak{g}^{MO}_{Q}$ = $\mathbb{Z}$-graded Lie algebra.

Remark

- McBreen: Q = finite ADE

$$\Longrightarrow \begin{cases} \mathfrak{g}^{MO}_{ADE} = \text{simple Lie algebra of type ADE} \\ Y^{MO}_{ADE,\mathbb{C}^*} = \text{Drinfeld's Yangian} \end{cases}$$

- $\left.\begin{array}{l}\text{Schiffmann-Vasserot} \\ \text{Maulik-Okounkov}\end{array}\right\}$: $\mathfrak{g}^{MO}_{1\text{-loop}} = W_{1+\infty}$

= u.c.e. of the Lie algebra of regular diff. operators on the circle

- DPSSV: Q = affine ADE, Q_{fin} = finite ADE, $(\mathbb{C}^*)^{\times 2} \subset T_{max}$

$$\Longrightarrow \begin{cases} \mathfrak{g}_Q^{MO}[t] = \text{universal central extension of } \mathfrak{g}_{Q_{fin}}[s^{\pm 1}, t] =: \mathfrak{g}_{ell} \\ \mathcal{Y}^{MO}_{Q,(\mathbb{C}^*)^{\times 2}} = \text{ass. algebra given by gen.s and rel.s} \end{cases}$$

Attention

when Q is without edge-loops : $\mathfrak{g}_Q^{MO}[0]$ = Kac-Moody Lie algebra of Q

Surfaces

S = smooth quasi-projective surface $/\mathbb{C}$
T = (possibly trivial) torus $\curvearrowright S$

$\boxed{\underline{\mathrm{Coh}}_{ps}(S) = \text{moduli stack of properly supported coherent sheaves on } S}$

Remark

We can also define:

- $\underline{\mathrm{Coh}}_0(S) \subset \underline{\mathrm{Coh}}_{ps}(S)$ corresponding to 0-dim. sheaves
- $\underline{\mathrm{Coh}}_{\leq 1}(S) \subset \underline{\mathrm{Coh}}_{ps}(S)$ corresponding to sheaves of dim ≤ 1

Attention ⚠

$\exists$ a derived enhancement

$$\mathbb{R}\mathrm{Coh}_{ps}(S) \times \mathbb{R}\mathrm{Coh}_{ps}(S) \xleftarrow{\mathbb{R}q} \mathbb{R}\mathrm{Coh}^{ext}_{ps}(S) \xrightarrow{\mathbb{R}p} \mathbb{R}\mathrm{Coh}_{ps}(S)$$

such that $\mathbb{R}q$ is quasi-smooth $\Longrightarrow \exists (\mathbb{R}q)^!$

Kapranov-Vasserot, Yu Zhao (in dim=0), DPSSV:

$\boxed{\exists\ \mathrm{COHA}^{(T)}_S = (T\text{-equivariant})\ \mathrm{COHA}\text{ associated to properly supported sheaves on } S}$

= unital associative algebra structure on

$$H^{(T)}_*(\mathbb{R}\mathrm{Coh}_{ps}(S)) = H^{(T)}_*(\mathrm{Coh}_{ps}(S))$$

with multiplication $(\mathbb{R}p)_* \circ (\mathbb{R}q)^!$.

Remark

- $\exists\ \mathrm{COHA}^{(T)}_{S,0\text{-dim}}$ associated to $\mathrm{Coh}_0(S)$

- $\exists\ \mathrm{COHA}^{(T)}_{S,\leq 1}$ associated to $\underline{\mathrm{Coh}}_{\leq 1}(S)$

Example

$$S = \mathbb{C}^2 : \quad \underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}) \xrightarrow{\sim} \underline{\mathrm{Coh}}_0(\mathbb{C}^2)$$

$$\begin{array}{c} \cup \\ A_1 \circlearrowright \mathbb{C}^d \circlearrowleft A_2 \end{array} \longmapsto \mathbb{C}^d = \mathbb{C}[A_1, A_2]\text{-module}$$

$$\Longrightarrow \mathrm{COHA}^{(T)}_{\mathbb{C}^2, 0\text{-dim}} \simeq \mathrm{COHA}^{(T)}_{1\text{-loop}}$$

In dim $= 0$, we have a complete characterization:

Theorem (Mellit-Minets-Schiffmann-Vasserot)
$\mathrm{COHA}^{(T)}_{S,0\text{-dim}}$ can be described explicitly by generators and relations.
In particular, if $\omega_S \simeq \mathcal{O}_S$:

$$\mathrm{COHA}_{S,0\text{-dim}} \simeq U(W_{1+\infty}(S)[t])$$

where $W_{1+\infty}(S)$ is a Lie algebra assocaated with $H^*(S)$.

The questions I would like to address today are:

Question 1: is $\mathrm{COHA}^T_{S,\leq 1}$ related to Yangians?

Question 2: can we describe $\mathrm{COHA}^T_{S,\leq 1}$ by generators and relations?

2. COHA of a surface and affine Yangians

We saw that Yangians are related to COHAs of nilpotent representations.
First, we introduce a "nilpotent" version of COHA_S.

- S = smooth quasi-projective surface$/\mathbb{C}$
- $C \subset S$ reduced closed subscheme

Consider

$\mathrm{Coh}(S, C)$ = moduli stack of coherent sheaves on S set-theoretically supported on C

sheaf analog of nilpotency

Example: X = smooth projective curve $/\mathbb{C}$

$\underline{\mathrm{Coh}}(T^*X, X) \simeq$ moduli stack of Higgs sheaves $(\mathcal{F}, \phi: \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega^1_X)$ on X such that ϕ is nilpotent

zero section

Theorem 1 (DPSSV)

1. $\exists$ an associative algebra structure on $H^{BM}_*(\underline{\mathrm{Coh}}(S, C))$

$$\Longrightarrow \mathrm{COHA}_{S,C} =: \mathrm{HA}_{S,C}$$

If T = torus $\curvearrowright S$, C T-invariant $\Longrightarrow \exists\ \mathrm{COHA}^T_{S,C} =: \mathrm{HA}^T_{S,C}$

2. $\mathrm{HA}^{(T)}_{S,C}$ depends "locally" on C, i.e., given (S_1, C_1) and (S_2, C_2) s.t. the formal completions $\widehat{(S_1)}_{C_1} \simeq \widehat{(S_2)}_{C_2}$, we have:

$$\mathrm{HA}^{(T)}_{S_1, C_1} \simeq \mathrm{HA}^{(T)}_{S_2, C_2}$$

The first relation between $\mathrm{HA}^T_{S,C}$ and Yangians is when

S = minimal resolution of ADE singularity

- $G \subset SL(2,\mathbb{C})$ finite group

$\updownarrow$

ADE quiver $Q_{fin} \subset$ affine ADE quiver Q

- $\pi : S \longrightarrow \mathbb{C}^2/G$ Kleinian resolution of singularities

$C_{red} := \pi^{-1}(0)_{red} = C_1 \cup \dots \cup C_e$; $C_i \simeq \mathbb{P}^1$; $(C_i \cdot C_j) = -$ Cartan matrix of Q_{fin}

- torus $T \subset GL(2,\mathbb{C})$ centralizing G (T = trivial or $\mathbb{C}^*$ or $\mathbb{C}^* \times \mathbb{C}^*$)

Example

$G = \mathbb{Z}_2 \Longrightarrow Q_{fin} = \bullet = A_1$, $Q = \bullet \rightleftarrows \bullet = A_1^{(1)}$

$\Longrightarrow \mathbb{C}^* \times \mathbb{C}^* \curvearrowright S = T^*\mathbb{P}^1 \supset C = \mathbb{P}^1$ = zero section

Recall

$\mathfrak{g}_{ell}$:= universal central extension of $\mathfrak{g}_{Q_{fin}}[s^{\pm 1}, t]$

$$\simeq \mathfrak{g}_{Q_{fin}}[s^{\pm 1}, t] \oplus K$$

$$K = \bigoplus_{\ell \in \mathbb{N}} \mathbb{Q} c_\ell \oplus \bigoplus_{\substack{k \in \mathbb{Z}_{\neq 0} \\ \ell \in \mathbb{N}_{\geq 1}}} \mathbb{Q} c_{k,\ell}$$

central elements

Example

$$G = \mathbb{Z}_2 \Longrightarrow \mathfrak{g}_{ell} \simeq \mathfrak{sl}(2)[s^{\pm 1}, t] \oplus K$$

Define

$$\mathfrak{g}_{ell}^+ := \mathfrak{n}^-_{Q_{fin}}[s^{\pm 1}, t] \oplus s^{-1}\mathfrak{h}_{Q_{fin}}[s^{-1}, t] \oplus \bigoplus_{k<0} \mathbb{Q} c_{k,\ell}$$

Theorem 2 (DPSSV)

completion

- $\exists$ a canonical algebra isomorphism $HA_{S,C} \simeq \hat{U}(\mathfrak{g}_{ell}^+)$
- $\exists$ a canonical algebra isomorphism $HA_{S,C}^T \simeq \mathbb{Y}_\infty^+$

where $\mathbb{Y}_\infty^+$ is a filtered deformation of $\hat{U}(\mathfrak{g}_{ell}^+)$

Question: how do we prove this theorem?

Recall the derived McKay correspondence:

$$\tau : D^b(\mathrm{Coh}(S)) \xrightarrow{\sim} D^b(\mathrm{Mod}(\Pi_Q))$$

τ is not t-exact w.r.t. the standard t-structures

$$\Longrightarrow \underline{\mathrm{Coh}}(Y, C) \not\simeq \Lambda_Q = \text{stack of nilpotent repr.s of } \Pi_Q$$

$$\Longrightarrow HA^T_{Y,C} \not\simeq HA^T_Q$$

<u>Attention</u> ⚠ : we will "interpolate" between the 2 hearts by using:

- braid group actions on bounded derived cat.s
- Bridgeland stability conditions

Recall that

- extended affine braid group $B_{ex} \simeq \left(B_{fin} \cup \{L_{\check{\lambda}} : \check{\lambda} \in \check{X}_{fin}\}\right) / \text{rels}$ (finite coweight lattice)
- $\check{X}_{fin} \xrightarrow{\sim} \mathrm{Pic}(S)$, $\check{\lambda} \longmapsto \mathcal{L}_{\check{\lambda}}$

Lemma

$\exists$ a group homomorphism

abelian category

$$\rho : B_{ex} \longrightarrow \mathrm{Aut}(D^b(\mathrm{Coh}_C(S)))$$

such that $\rho(L_{\check\lambda}) = (\mathcal{L}_{\check\lambda} \otimes -) =: L_{\check\lambda}$, $\forall\, \check\lambda \in \check X_{fin}$

$\Longrightarrow$ by the McKay equivalence, $L_{\check\lambda} \in \mathrm{Aut}(D^b(\mathrm{nilp}(\Pi_Q)))$

Now fix a coweight $\check\Theta = \sum_{i\in I} \check\Theta_i \check\omega_i \in \check X_{aff}$ s.t.

$$\boxed{\Theta_i > 0 \ \ \forall i \neq 0 \ \text{ and } \ \check\Theta_0 = -\sum_{i=1}^{e} \check\Theta_i}$$

$\Longrightarrow \Theta_{fin} := \sum_{i\neq 0} \check\Theta_i \check\omega_i \in \check X_{fin}$

Consider the stability function on $\mathrm{nilp}(\Pi_Q)$:

$$\sum_i \check\omega_i$$

$$\begin{aligned} Z_{\check\Theta} : K_0(\mathrm{nilp}(\Pi_Q)) \simeq \mathbb{Z}I &\longrightarrow \mathbb{C} \\ \underline{d} &\longmapsto -(\check\Theta, \underline{d}) + (\check\rho, \underline{d}) \end{aligned}$$

Attention: This is **only** a reformulation of King's (semi)-stability for quiver repr.s.

$\Longrightarrow$ $\exists$ associated $(Z_{\check{\theta}}, \mathcal{P}_{\check{\theta}})$ = Bridgeland's stability condition on $D^b(\mathrm{nilp}(\Pi_Q))$

Here, $\mathcal{P}_{\check{\theta}}$ = slicing = family of full additive subcategories

$$\mathcal{P}_{\check{\theta}}(\phi) \subset D^b(\mathrm{nilp}(\Pi_Q)) \quad \forall \phi \in \mathbb{R}$$

such that

- $\mathcal{P}_{\check{\theta}}(\phi+1) = \mathcal{P}_{\check{\theta}}(\phi)[1] \quad \forall \phi \in \mathbb{R}$;

- $\forall\, \phi_1 > \phi_2$ and $F_j \in \mathcal{P}_{\check{\theta}}(\phi_j)$, $\mathrm{Hom}(F_1, F_2) = 0$;

- $\forall\, 0 \neq E \in D^b(\mathrm{nilp}(\Pi_Q))$, $\exists\, \phi_1 > \cdots > \phi_n$ and triangles

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[dl] \\
F_1 \in \mathcal{P}_{\check{\theta}}(\phi_1) \arrow[u, dashed] & & & F_n \in \mathcal{P}_{\check{\theta}}(\phi_n) \arrow[u, dashed] &
\end{tikzcd}$$

(Harder-Narasimhan type filtration)

Set $\mathcal{P}_{\check{\theta}}(I) := \langle \mathcal{P}_{\check{\theta}}(\phi) : \phi \in I \rangle \quad \forall$ interval $I \subset \mathbb{R}$

Lemma

1. $\forall k \in \mathbb{Z}$, $L_{-2k\check{\theta}_{fin}} : \mathrm{nilp}(\Pi_Q) \xrightarrow{\sim} \mathcal{P}_{\check{\theta}}((\nu_{-k}, \nu_{-k}+1]))$

with $\nu_\ell := \frac{1}{\pi} \arctan(2h\ell)$ — Coxeter number

2. $\mathcal{P}_{\check{\theta}}((-\frac{1}{2}, \frac{1}{2}]) \simeq \mathrm{Coh}_C(S)$

Remark

- $\nu_k \xrightarrow{k\to+\infty} \frac{1}{2}$

- we have a "sequence" of t-structures $\{\tau_k\}_{k\in\mathbb{N}}$ all equivalent to $\tau_0 :=$ standard t-structure s.t.

$$\boxed{\tau_k \xrightarrow{k\longrightarrow+\infty} \tau_\infty = \text{t-structure with heart } \mathrm{Coh}_C(S)}$$

For $\ell, k \in \mathbb{N}$, $k \geq \ell$, set

$\Lambda_Q^{\ell,k} :=$ (derived) moduli stack of objects in $\mathcal{P}_{\check{\theta}}((\nu_{-\ell}, \nu_{-k}+1]))$

Attention: $L_{2k\check{\theta}_{fin}} : \Lambda_Q^{\ell,k} \xrightarrow{\sim} \Lambda_Q^{\ell-k,0} \simeq \Lambda_Q^{>\nu_{-\ell+k}} =$ HN stratum

Lemma

1. The vector space

$$\boxed{HA_\infty^{(T)} := \lim_{\ell} \operatorname{colim}_{k \geq \ell} H_*^{(T)}\left(\Lambda_Q^{\ell,k}\right)}$$

has the structure of an unital associative algebra with multiplication induced from that of $HA_Q^{(T)}$.

2. $\exists$ an algebra isomorphism $HA_{S,C}^{(T)} \simeq HA_\infty^{(T)}$

Finally, a careful analysis of the compatibility between the action of B_{ex} on Yangians and on HA_Q^T yields:

Lemma

$\exists$ an algebra isomorphism $HA_\infty^T \xrightarrow{\sim} \mathbb{Y}_\infty^+$, where

$$\mathbb{Y}_\infty^+ := \lim_{\ell} T_{2\ell\check{\theta}_{fin}}\left(\mathbb{Y}_Q^-\right) \Big/ T_{2\ell\check{\theta}_{fin}}(J)$$

where $J := \sum_{\mu_{\check{\theta}}(\underline{d})>0} \mathbb{Y}_Q^- \cdot \mathbb{Y}_{Q,-\underline{d}}^-$.

The proof of Thm 2 follows from the above lemmas.

$\square$

<u>Conjecture</u> Y_∞^+ is a new half of $\widehat{Y}_{\mathbb{Q}}^{MO}$.