From Hilbert schemes of pts on a smooth surface to cohomological Hall algebras

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Plan

1. Elliptic Hall algebra and the K-theory of Hilbert schemes of pts
2. Cohomological Hall algebras
3. Representations of Cohomological Hall algebras

Elliptic Hall algebra and the K-theory of Hilbert schemes of pts

S = smooth (quasi-) projective surface$/\mathbb{C}$, $n \in \mathbb{N}$

$\boxed{\begin{array}{l} \mathrm{Hilb}^n(S) = \text{Hilbert scheme of } n\text{-pts on } S \\ \qquad\qquad = \text{moduli space parametrizing zero-dim. subschemes} \\ \qquad\qquad\ \ Z \subset S \text{ such that } \dim_{\mathbb{C}} H^0(Z, \mathcal{O}_Z) = n \end{array}}$

Remarks

- $\mathrm{Hilb}^n(S)$ is a smooth (quasi-) projective variety$/\mathbb{C}$ of dimension $2n$

- $Z \subset S$ subscheme $\longleftrightarrow$ $I_Z \subset \mathcal{O}_S$ ideal sheaf

$$\Longrightarrow \mathrm{Hilb}^n(S) \text{ "="} \left\{ I \subset \mathcal{O}_S : \dim \operatorname{supp}(\mathcal{O}_S/I) = 0, \dim H^0(\mathcal{O}_S/I) = n \right\}$$

$$\Longrightarrow \mathrm{Hilb}^n(S) \underset{\substack{\uparrow \\ S \text{ projective}}}{=} M_S^{st}(\underset{\substack{\uparrow \\ rk}}{1}, \underset{\substack{\uparrow \\ c_1}}{0}, \underset{\substack{\uparrow \\ ch_2}}{-n}) = \text{moduli space of Gieseker-stable sheaves on } S$$

- $\pi : \mathrm{Hilb}^n(S) \longrightarrow \mathrm{Sym}^n(S) := \overbrace{S \times \cdots \times S}^{n\text{-th copies}} \big/ \underbrace{\mathfrak{S}_n}_{\text{symmetric group of } n \text{ letters}}$

is a resolution of singularities, i.e., π is a proper morphism which is an iso over the smooth locus of $\mathrm{Sym}^n(S)$.

Examples

- $n=1$: $\mathrm{Sym}^1(S) = S \simeq \mathrm{Hilb}^1(S)$

- $n=2$: $Z \in \mathrm{Hilb}^2(S) \rightsquigarrow \begin{cases} Z = \{x,y\},\ x \neq y & \Longrightarrow \pi(Z) = x+y \\ Z_{red} = \{x\} & \Longrightarrow \pi(Z) = x+x = 2x \end{cases}$

$$\Longrightarrow \mathrm{Hilb}^2(S) \simeq \mathrm{Blow}_{\underbrace{\Delta}_{\text{diagonal}}}(S \times S)/\mathfrak{S}_2 \xrightarrow{\pi} \mathrm{Sym}^2(S)$$

Notation: set $\mathrm{Hilb}(S) := \bigsqcup_{n \geq 0} \mathrm{Hilb}^n(S)$

Assume that $\exists\ T = \text{torus} \curvearrowright S$

- if S is projective, T could be trivial

- if S is quasi-projective, the fixed locus S^T is proper

Set

- $G_0^T(-)$ = Grothendieck group of T-equivariant coherent sheaves

- $R := G_0^T(pt)$, $K := \mathrm{Frac}(R)$, $M_K := M \otimes_R K$ for any R-mod. M

Thm. (Schiffmann-Vasserot for $S = \mathbb{C}^2$, Neguț for arbitrary S)

$\exists$ an associative algebra, generated by

$$f_{\pm 1, \ell},\, e_{0, \pm s} \in \mathrm{End}\left(G_0^T(\mathrm{Hilb}(S))_K \right) \quad \left(\text{with } \ell \in \mathbb{Z},\ \underset{s \neq 0}{s \in \mathbb{N}}\right)$$

which is isomorphic to the elliptic Hall algebra $\mathcal{E}$.

Construction of $f_{\pm 1, \ell}, e_{0,\pm s} \in \mathrm{End}\left(G_0^T(\mathrm{Hilb}(S))_K\right)$

1. Hecke correspondences

for $k>0$: $\mathrm{Hilb}^{n,n+k}(S) := \left\{ 0 \to J \to I \to Q_x \to 0 : \begin{array}{l} \bullet\ (I,J) \in \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^{n+k}(S) \\ \bullet\ \mathrm{supp}(Q_x) = \{x\} \end{array} \right\}$

reduced closed subscheme

$$\downarrow$$

$$\mathrm{Hilb}^n(S) \times \mathrm{Hilb}^{n+k}(S) \times S$$

for $k>0$: $\mathrm{Hilb}^{n+k,n}(S) \hookrightarrow \mathrm{Hilb}^{n+k}(S) \times \mathrm{Hilb}^n(S) \times S$

for $k=0$: $\mathrm{Hilb}^{n,n}(S) = \text{diagonal} \hookrightarrow \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^n(S)$

Attention ⚠ : $\mathrm{Hilb}^{n,n'}(S)$ is smooth $\Leftrightarrow |n-n'| \le 1$

2. Tautological bundles on Hecke correspondences

$\tau_{n,n+1}$:= line bundle on $\mathrm{Hilb}^{n,n+1}(S)$ having fiber

$$\tau_{n,n+1}\Big|_{(I,J)} \simeq I/J$$

$\tau_{n+1,n}$:= —— " —— on $\mathrm{Hilb}^{n+1,n}(S)$ —— " ——

$\tau_{n,n}$:= vector bundle on $\mathrm{Hilb}^{n,n}(S)$ of rank n having fiber

$$\tau_{n,n}\Big|_{(I,I)} \simeq \mathcal{O}_S/I$$

3. Nakajima type operators:

$$f_{-1,\ell} = \prod_n \underbrace{\mathbb{R}(pr_{n+1})_*\left(\left[\tau_{n,n+1}\right]^{\otimes \ell} \overset{\mathbb{L}}{\otimes} pr_n^*(-)\right)}_{: G_0^T(\mathrm{Hilb}^n(S))_K \longrightarrow G_0^T(\mathrm{Hilb}^{n+1}(S))_K} \quad \text{for } \ell \in \mathbb{Z}$$

$f_{1,\ell}$ = as above after $n \longleftrightarrow n+1$ for $\ell \in \mathbb{Z}$

$e_{0,s} = \prod_n \mathbb{R}(pr_n)_* \left([\Lambda^s \tau_{n,n}] \overset{\mathbb{L}}{\otimes} pr_n^*(-) \right)$ for $s \in \mathbb{N}, s \neq 0$

$e_{0,-s}$ = as above after $\tau_{n,n} \longleftrightarrow \tau_{n,n}^\vee$ for $s \in \mathbb{N}, s \neq 0$

$$f_{\pm 1,\ell},\ e_{0,\pm s} \in \mathrm{End}\left(G_0^T(\mathrm{Hilb}(S))_K \right)$$

The elliptic Hall algebra

$\mathcal{E}$ = associative algebra over $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]^{\mathrm{Sym}}$ generated by $e_k, f_k, h_s^\pm$ with $k \in \mathbb{Z}, s \in \mathbb{N}$ subject to:

- Lie theoretic relations
- quadratic and cubic relations depending on (a suitable normalization of) $\zeta(z)$:

zeta function of an elliptic curve over a finite field with $q_1 q_2$ elements $\longleftrightarrow$ $\zeta(z) = \dfrac{(1-q_1 z)(1-q_2 z)}{(1-z)(1-q_1 q_2 z)}$

Attention ⚠ : later, I will give a "geometric" definition of $\mathcal{E}$.

Let us state again the result of Schiffmann-Vasserot and Neguţ :

Thm.
The assignment

$$q_1+q_2 \longmapsto [\Omega^1_S] \quad , \quad q_1 q_2 \longmapsto [\omega_S]$$

induces an homomorphism $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]^{Sym} \longrightarrow G_0^T(S)_K$.
Then, $\exists$ an injective homomorphism

$$\mathcal{E} \longrightarrow End\left(G_0^T(Hilb(S))_K \right)$$

of $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]^{Sym}$-algebras.

Remarks

- Neguţ and Yu Zhao have investigated the categorification of the above result.

- Schiffmann-Vasserot proved a cohomological version of the above result for $S=\mathbb{C}^2 \circlearrowleft T=\mathbb{C}^*\times\mathbb{C}^*$, replacing

$$\mathcal{E} \longleftrightarrow \text{the affine Yangian of } \mathfrak{gl}(1)$$

$\Longrightarrow$ Nakajima-Grojnowski's action of the Heisenberg algebra

Advantages of Nakajima type operators:

- It allows to compute "easily" relations between the generators

- One realize geometric repr.s of the whole elliptic Hall algebra (its categorification, etc) via

$$M_S(r,c_1) := \bigsqcup_{ch_2} M_S^{st}(r,c_1,ch_2) \qquad (\text{here } S \text{ is projective})$$

Problems:

- It is useful to realize repr.s of one (!) algebra

▸ The elliptic Hall algebra does NOT contain all possible operators acting on $G_0(-)$, $D^b_{coh}(-)$, etc, of

$$\mathcal{M}_S(r) := \bigsqcup_{c_1} \bigsqcup_{ch_2} \mathcal{M}_S^{st}(r, c_1, ch_2) \qquad \text{(here } S \text{ is projective)}$$

In particular, $\mathcal{E}$ does NOT contain operators changing the first Chern class c_1, i.e., which e.g. depend on

$$\left\{ 0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow i_* \mathcal{L}_C \longrightarrow 0 \right\}$$

where $C \overset{i}{\hookrightarrow} S$ is a smooth proj. curve and $\mathcal{L}_C$ line bundle on C.

Solution: define an algebra bigger than $\mathcal{E}$ and construct geom. repr.s of it

via the theory of Cohomological Hall algebras

Cohomological Hall algebras

S = smooth quasi-projective surface$/\mathbb{C}$.

$(\mathbb{R})\underline{\mathrm{Coh}}_{ps}(S)$ = (derived) moduli stack of properly supported coherent sheaves on S

Consider the Hall convolution diagram

derived stack of short exact sequences

$$\begin{tikzcd}[column sep=small] & \mathbb{R}\underline{\mathrm{Coh}}^{ext}_{ps}(S) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathbb{R}\underline{\mathrm{Coh}}_{ps}(S) \times \mathbb{R}\underline{\mathrm{Coh}}_{ps}(S) & & \mathbb{R}\underline{\mathrm{Coh}}_{ps}(S) \end{tikzcd}$$

$$q: 0 \to \mathcal{E}_2 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_1 \longrightarrow 0 \longmapsto (\mathcal{E}_1, \mathcal{E}_2)$$
$$p: \qquad \text{''} \qquad \longmapsto \mathcal{E}$$

Facts:

- q is derived l.c.i. (i.e., $\mathbb{L}_p$ is perfect and in Tor-amplitude $[-1,1]$)
- p is representable by proper schemes (which are Quot schemes)

Thm (Porta–S.)

1. $D^b_{coh}(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(S))$ has a monoidal structure induced by $q_* \circ p^*$.

2. It induces to an associative algebra structure $\mathrm{KHA}(S)$ on $G_0(\underline{\mathrm{Coh}}(S))$

———— 〃 ———— $\mathrm{COHA}(S)$ on $H^{BM}_*(\underline{\mathrm{Coh}}(S))$

Remarks

- $D^b_{coh}(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(S)) \neq D^b_{coh}(\underline{\mathrm{Coh}}_{ps}(S))$

  $\Longrightarrow \nexists$ Hall-monoidal structure on $D^b_{coh}(\underline{\mathrm{Coh}}_{ps}(S))$

- $D^b_{coh}(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(S))^{\heartsuit} \simeq D^b_{coh}(\underline{\mathrm{Coh}}_{ps}(S))^{\heartsuit}$

  $\Longrightarrow G_0(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(S)) \simeq G_0(\underline{\mathrm{Coh}}_{ps}(S))$

- The above Theorem holds also for $\mathbb{R}\underline{\mathrm{Coh}}_{\bullet}(S) \hookrightarrow \mathbb{R}\underline{\mathrm{Coh}}_{ps}(S)$, where

$$\tau_{\leq 1} := \{ \mathcal{F} \in \mathrm{Coh}_{ps}(S) : \dim(\mathrm{supp}(\mathcal{F})) \leq 1 \}$$

$$\tau_{0} := \{ \mathcal{F} \in \mathrm{Coh}_{ps}(S) : \dim(\mathrm{supp}(\mathcal{F})) = 0 \}$$

- (2) recovers known constructions by:

- Kapranov and Vasserot via perfect obstruction theory
- Schiffmann and Vasserot for $\mathbb{C}^2$ via the "Lagrangian formalism"
- S.-Schiffmann for $S = T^*(\text{curve})$ ——— " ———

Thm (Schiffmann-Vasserot)

1. $KHA^T(\mathbb{C}^2)_K = \left(G_0^T(\underline{Coh}_0(\mathbb{C}^2))_K \, , \text{Hall product}\right)$

$\simeq \mathcal{E}^+$ = positive part of the elliptic Hall algebra

2. $\exists$ an action $KHA^T(\mathbb{C}^2)_K$ on $G_0^T(Hilb(\mathbb{C}^2))_K$ such that

$$\begin{tikzcd}
KHA^T(\mathbb{C}^2)_K \arrow[d, "\wr"'] \arrow[drr] & & \\
\mathcal{E}^+ \arrow[rr] \arrow[u, phantom, "\circlearrowright", shift left=6ex] & & End\left(G_0^T(Hilb(\mathbb{C}^2))_K\right)
\end{tikzcd}$$

Similar results hold in BM homology.

Advantages of COHAs:

- We can realize more algebras than simply $\mathcal{E}^+$, as the COHA associated with sheaves of dim. ≤ 1

- The construction of the algebra is "intrinsic"

Problem: COHAs only realize halves of the whole algebras we are interested in

$\Longrightarrow$ this approach is NOT completely complementary to the one with explicit operators, but:

Solution: "doubling" COHAs

Representations of COHAs (and "doubling" Hall algebras)

S = smooth projective surface$/\mathbb{C}$.

$$\mathbb{R}\,\underline{\mathcal{C}oh}(S)$$

$$\rightsquigarrow$$

$$D^b_{coh}(\mathbb{R}\,\underline{\mathcal{C}oh}(S)),\quad G_0(\underline{\mathcal{C}oh}(S)),\quad H^{BM}_*(\underline{\mathcal{C}oh}(S))$$

algebras

Torsion sheaves $\longleftrightarrow$ algebra | module $\longleftrightarrow$ Torsion-free sheaves

$\mathbb{R}\,\underline{\mathcal{C}oh}_0(S)$, $\mathbb{R}\,\underline{\mathcal{C}oh}_{\leq 1}(S)$ $\rightsquigarrow$ | $\mathbb{R}\,\underline{\mathcal{C}oh}_{t.f.}(S)$ $\rightsquigarrow$

$D^b_{coh}(\mathbb{R}\,\underline{\mathcal{C}oh}_\bullet(S))$ algebra $\longleftrightarrow$ module $D^b_{coh}(\mathbb{R}\,\underline{\mathcal{C}oh}_{t.f.}(S))$

$G_0(\underline{\mathcal{C}oh}_\bullet(S))$ algebra $\longleftrightarrow$ module $G_0(\underline{\mathcal{C}oh}_{t.f.}(S))$

$H^{BM}_*(\underline{\mathcal{C}oh}_\bullet(S))$ algebra $\longleftrightarrow$ module $H^{BM}_*(\underline{\mathcal{C}oh}_{t.f.}(S))$

Attention: we shall recover the constructions via Nakajima type ops after restricting to $\mathcal{M}_S^{st}(r, c_1, ch_2) \subset \underline{\mathrm{Coh}}_{t.f.}(S; r, c_1, ch_2)$

More precisely, we obtain:

Thm (Diaconescu-Porta-S.)

- $D^b_{coh}(\mathbb{R}\,\underline{\mathrm{Coh}}_{t.f.}(S))$ is a left and right categorical module over $D^b_{coh}(\mathbb{R}\,\underline{\mathrm{Coh}}_{\leq 1}(S))$

In particular,

- $G_0(\underline{\mathrm{Coh}}_{t.f.}(S))$ is a left and right module over $\mathrm{KHA}_{\leq 1}(S)$,

$$H_*^{BM}(\underline{\mathrm{Coh}}_{t.f.}(S)) \text{ ---------- // ---------- } \mathrm{COHA}_{\leq 1}(S)$$

Moreover, the same result holds after replacing

$$\underline{\mathrm{Coh}}_{t.f.}(S) \longleftrightarrow \underline{\mathrm{Coh}}_{t.f.}(S; r)$$

fixed rank r

Remark: when $r=1$, we can replace $\underline{Coh}_{t.f.}(S;1)$ with

$M(S;1) :=$ moduli space of rank-one torsion-free sheaves on S

Geometric idea behind the proof:

Consider:

$$(*) \quad \begin{tikzcd}
 & \mathbb{R}\underline{Coh}^{ext}_{\leq 1, t.f.}(S) \arrow[dl, "q"'] \arrow[dr, "p"] & \\
\mathbb{R}\underline{Coh}_{\leq 1}(S) \times \mathbb{R}\underline{Coh}_{t.f.}(S) & & \mathbb{R}\underline{Coh}_{t.f.}(S)
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{E} \arrow[r] \arrow[dl, "q"'] \arrow[dr, "p"] & T \arrow[r] & 0 \\
 & (T, \mathcal{E}') & & \mathcal{E} &
\end{tikzcd}$$

in the abelian category $Coh(S)$

Now, q is derived l.c.i.

Fact: $\mathcal{E}$ is torsion-free $\Longrightarrow$ $\mathcal{E}'$ is torsion free

$\Longrightarrow$ The fiber of p at $\mathcal{E}$ is the Quot scheme parametrizing its torsion quotients

$\Longrightarrow$ p is proper.

$\Longrightarrow$ $(*)$ gives rise to the left action.

Now, consider:

$$\begin{tikzcd}
 & \mathbb{R}\underline{\mathrm{Coh}}^{\mathrm{ext}}_{\leq 1, \mathrm{t.f.}}(S) \arrow[dl, "q"'] \arrow[dr, "p"] & \\
\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{t.f.}}(S) \times \mathbb{R}\underline{\mathrm{Coh}}_{\leq 1}(S) & & \mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{t.f.}}(S)
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] & \mathcal{E} \arrow[r] \arrow[dl, "q"'] \arrow[dr, "p"] & \mathcal{E}' \arrow[r] & 0 \\
& (\mathcal{E}', T) & & \mathcal{E} &
\end{tikzcd}$$

in the abelian category $\mathrm{Coh}(S)$

Attention ⚠: $\mathcal{E}$ is torsion-free $\centernot\Longrightarrow$ $\mathcal{E}'$ is torsion free

Thus, the fiber of p at $\mathcal{E}$ is not proper

Solution: we consider the abelian category

$$\mathrm{Coh}^{\#}(S) := \left\{ E \in D^b_{coh}(S) : \mathcal{H}^i(E) = 0,\ \mathcal{H}^{-1}(E) \text{ is torsion-free},\ \mathcal{H}^0(E) \text{ is torsion} \right\}$$

(heart of the tilted t-structure induced by the torsion pair (torsion sheaves, torsion-free sheaves))

$\Longrightarrow$ $\mathbb{R}\underline{\mathrm{Coh}}^{\#}(S)$ = derived moduli stack of objects $\in \mathrm{Coh}^{\#}(S)$

Facts:

- $\mathbb{R}\underline{\mathrm{Coh}}_{t.f.}(S) \xrightarrow[{[1]}]{\sim} \mathbb{R}\underline{\mathrm{Coh}}^{\#}_{tor}(S)$
- $\mathbb{R}\underline{\mathrm{Coh}}_{\leq 1}(S) \xrightarrow{\sim} \mathbb{R}\underline{\mathrm{Coh}}^{\#}_{t.f.}(S)$

We have

$(**)^{\#}$

$$\begin{tikzcd}
 & \mathbb{R}\underline{\mathrm{Coh}}^{\#,\,ext}_{\leq 1,\,t.f.}(S) \arrow[dl, "q"'] \arrow[dr, "p"] & \\
\mathbb{R}\underline{\mathrm{Coh}}_{t.f.}(S) \times \mathbb{R}\underline{\mathrm{Coh}}_{\leq 1}(S) & & \mathbb{R}\underline{\mathrm{Coh}}_{t.f.}(S)
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] & \mathcal{E} \arrow[r] \arrow[dl, "q"'] \arrow[dr, "p"] & \mathcal{E}' \arrow[r] & 0 \\
 & (\mathcal{E}', T) & & \mathcal{E} &
\end{tikzcd}$$

in the abelian category $\mathrm{Coh}^{\#}(S)$

Now

- $T \simeq \mathcal{H}^0(T)$ torsion in $\mathrm{Coh}^{\#}(S)$

- $\mathcal{E} \simeq \mathcal{H}^{-1}(\mathcal{E})[1]$ torsion-free $\Longrightarrow$ $\mathcal{E}' \simeq \mathcal{H}^{-1}(\mathcal{E}')[1]$ torsion-free

$\Longrightarrow$ as before, p is proper

$\Longrightarrow$ $(**)^{\#}$ gives rise to the right action. $\square$

<u>Def.</u> (Algebras of "Hecke modifications along curves")

The Yangian of $\underline{\mathrm{Coh}}_{\leq 1}(S)$ is the subalgebra of $\mathrm{End}\left(H_*^{BM}\left(\mathcal{M}(S;1)\right)\right)$ generated by the images of

$$\text{left action} \quad a_\ell : \mathrm{COHA}_{\leq 1}(S) \longrightarrow \mathrm{End}\left(H_*^{BM}\left(\mathcal{M}(S;1)\right)\right)$$

$$\text{right action} \quad a_r : \mathrm{COHA}_{\leq 1}(S) \longrightarrow \mathrm{End}\left(H_*^{BM}\left(\mathcal{M}(S;1)\right)\right)$$

Similarly, we define: (categorified) quantum loop algebra of $\mathrm{Coh}_{\leq 1}(S)$

Remark

- Similarly, we may define

  Algebras of "Hecke modifications at points" $\longleftrightarrow$ $\underline{\mathrm{Coh}}_0(S)$

- The theorem above holds after considering:
  - $\underline{\mathrm{Coh}}_0(S)$ (algebra)
  - $\mathrm{Hilb}(S)$ instead of $\mathcal{M}(S,1)$ (module)

$\Longrightarrow$ we recover Neguț's construction of the action of the elliptic Hall algebra.

Now, consider

$P(S) :=$ moduli space of Pandharipande-Thomas stable pairs on S

$$\| $$

$(\mathcal{F}, s: \mathcal{O}_S \longrightarrow \mathcal{F})$ with $\mathcal{F}$ pure 1-dimen.

$\mathrm{Coker}(s)$ 0-dimen.

Thm (Diaconescu-Porta-S.)

- $D^b_{coh}(\mathbb{R}P(S))$ is a left and right categorical module

  over $D^b_{coh}(\mathbb{R}\,\underline{Coh}_0(S))$

In particular,

- $G_0(P(S))$ is a left and right module over $KHA_0(S)$,

- $H^{BM}_*(P(S))$ ——〃—— $COHA_0(S)$

Remark

We can replace $S \rightsquigarrow T^*X$, X = smooth projective curve$/\mathbb{C}$

$\Longrightarrow$ the above Theorem holds.

In particular,

PT stable pair on T^*X = cyclic Higgs bundle $(\underbrace{\mathcal{E}, \phi}_{\text{Higgs bundle}}, \tau : \mathcal{O}_X \longrightarrow \mathcal{E})$ on X

$\nexists$ saturated Higgs subbundle $(\mathcal{E}', \phi')$
s.t. $\mathrm{Im}(\tau) \subseteq \mathcal{E}'$.

The previous theorem is a consequence of the following more general framework.

Thm (Diaconescu-Porta-S.)

Assume that

1. $\mathcal{C}$ is a "nice" triangulated category (e.g. for which Toën-Vaquié's moduli of objects is an Artin derived stack)
2. τ is a t-structure which satisfies openness of flatness
3. $\exists$ a Serre functor $S_{\mathcal{C}}$ such that $S_{\mathcal{C}}[-2]$ is t-exact
4. the Quot functor for $(\mathcal{C},\tau)$ is represented by a proper algebraic space

Then

- $D^b_{coh}(\mathbb{R}\underline{Coh}_{ps}(\mathcal{C},\tau))$ has a monoidal structure induced by $(q_\tau)_* \circ p_\tau^*$.

- It descends to an associative algebra structure $KHA(\mathcal{C},\tau)$ on $G_0(\underline{Coh}(\mathcal{C},\tau))$

  ——— // ——— $COHA(\mathcal{C},\tau)$ on $H_*^{BM}(\underline{Coh}(\mathcal{C},\tau))$

Remark

$(1)+(2) \Longrightarrow \mathbb{R}\underline{Coh}(\mathcal{C},\tau)$ is Artin

$(3) \Longrightarrow p_\tau$ is derived l.c.i. $\Longrightarrow \exists\ p_\tau^*$

$(4) \Longrightarrow q_\tau$ is proper $\Longrightarrow \exists\ (q_\tau)_*$

Consider:

1. $\mathcal{C}$ = "nice" triangulated category (as before)

2. τ = t-structure which satisfies openness of flatness

3. $\nu = (\mathcal{T}or, \mathcal{F})$ = torsion pair in $\mathcal{C}^\heartsuit$, i.e.,
   - $\mathrm{Hom}(\mathcal{T}or, \mathcal{F}) = 0$
   - $\forall E \in \mathcal{C}^\heartsuit$ $\exists$ $0 \longrightarrow \underset{\mathcal{T}or}{T} \longrightarrow E \longrightarrow \underset{\mathcal{F}}{F} \longrightarrow 0$ with $T \in \mathcal{T}or$, $F \in \mathcal{F}$

   $\Longrightarrow$ τ_ν = tilted t-structure on $\mathcal{C}$, whose heart is:

$$\mathcal{C}_\nu^\heartsuit = \left\{ E \in \mathcal{C} : \mathcal{H}_\tau^{-1}(E) \in \mathcal{F},\ \mathcal{H}_\tau^{0}(E) \in \mathcal{T}or,\ \mathcal{H}_\tau^{i}(E) = 0\ \forall i \neq 0, -1 \right\}$$

4. $\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{T}or}(\mathcal{C},\tau)$, $\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau)$ are open in $\mathbb{R}\underline{\mathrm{Coh}}(\mathcal{C},\tau)$

Facts:

(1) + (2) $\Longrightarrow$ $\mathbb{R}\underline{\mathrm{Coh}}(\mathcal{C},\tau)$ is Artin

Lieblich shows: (4) $\Longrightarrow$ $\mathbb{R}\underline{\mathrm{Coh}}(\mathcal{C},\tau_\nu)$ is Artin

Thm (Diaconescu-Porta-S.)

Assume that

1. p_τ is derived l.c.i., q_τ is proper
2. p_{τ_v} —— " ——, q_{τ_v} —— " ——
3. $\mathcal{T}or$ is a Serre subcategory

Then

$$D^b_{coh}\left(\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{T}or}(\mathcal{C},\tau)\right)$$

has a monoidal structure induced from the one on

$$D^b_{coh}\left(\mathbb{R}\underline{\mathrm{Coh}}(\mathcal{C},\tau)\right) \quad \text{or} \quad D^b_{coh}\left(\mathbb{R}\underline{\mathrm{Coh}}(\mathcal{C},\tau_v)\right)$$

equivalently

Assume furthermore that

4. $\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{T}or}(S,\tau)$ is closed in both $\mathbb{R}\underline{\mathrm{Coh}}(\mathcal{C},\tau)$ and $\mathbb{R}\underline{\mathrm{Coh}}(\mathcal{C},\tau_v)$

Then

- $D^b_{coh}(\mathbb{R}\underline{Coh}_F(\mathcal{C},\tau))$ is a left (resp. right) categorical module of $D^b_{coh}(\mathbb{R}\underline{Coh}_{Tor}(\mathcal{C},\tau))$ induced by the monoidal structure of $D^b_{coh}(\mathbb{R}\underline{Coh}(\mathcal{C},\tau))$ (resp. $D^b_{coh}(\mathbb{R}\underline{Coh}(\mathcal{C},\tau_v))$).

Similar statements hold for $G_0(-)$ and $H^{BM}_*(-)$.

Remark

The first result can be applied to $\mathcal{C}$ = noncommutative K3 surface, i.e., a category with the same properties of $D^b_{coh}(K3)$ (e.g. $\exists$ Serre functor $\simeq$ shift by 2).

A famous example of noncommutative K3 surfaces is

$$\mathcal{C} = Ku(X) = \text{Kuznetsov component}$$

$$\text{of } X = \begin{cases} \text{Fano 3folds of Picard rank one} \\ \text{cubic 4folds} \\ \text{Gushel-Mukai 4folds or 6folds} \end{cases}$$

To apply the second result, one needs "nice" torsion pairs of $\mathcal{C}^{\heartsuit}$:

under investigation.