Cohomological Hall algebras,

their representations, and Nakajima operators

1. Cohomological Hall algebras of surfaces

S = smooth projective surface$/\mathbb{C}$.

$$\boxed{\underline{\mathrm{Coh}}(S) = \text{moduli stack of coherent sheaves on } S}$$

Attention ⚠ : we work within the framework of Derived Algebraic Geometry

$\Longrightarrow$ $\underline{\mathbb{R}\mathrm{Coh}}(S)$ = derived enhancement of $\underline{\mathrm{Coh}}(S)$

Construction of COHA of S:

Consider the "convolution diagram":

$$\underline{\mathbb{R}\mathrm{Coh}}(S) \times \underline{\mathbb{R}\mathrm{Coh}}(S) \xleftarrow{ev_1 \times ev_3} \underline{\mathbb{R}\mathrm{Coh}}^{ext}(S) \xrightarrow{ev_2} \underline{\mathbb{R}\mathrm{Coh}}(S)$$

= stack of extensions

where:

$$ev_2 : \quad 0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0 \longmapsto E_2$$

$$ev_1 \times ev_3 : \quad \text{————— // —————} \longmapsto (E_1, E_3)$$

- ev_2 is proper representable
- $ev_1 \times ev_3$ is derived lci (i.e., the cotangent complex $\mathbb{L}_{ev_1 \times ev_3}$ is perfect of amplitude $[-1,1]$)

Theorem

- Kapranov-Vasserot (Yu Zhao for 0-dim. sheaves):

$H_*^{BM}(\underline{Coh}(S))$ = Borel-Moore homology of $\underline{Coh}(S)$

(resp. $G_0(\underline{Coh}(S))$ = Grothendieck group of coh. sheaves on $\underline{Coh}(S)$)

has the structure of an associative algebra, whose product is given by:

$$\boxed{\begin{aligned} m: H_*^{BM}(\underline{Coh}(S)) \times H_*^{BM}(\underline{Coh}(S)) &\xrightarrow{\boxtimes} \\ H_*^{BM}(\underline{Coh}(S) \times \underline{Coh}(S)) &\xrightarrow{(ev_2)_* \circ (ev_1 \times ev_3)^!} H_*^{BM}(\underline{Coh}(S)) \end{aligned}}$$

(and similarly for $G_0(\underline{Coh}(S))$.

$\Longrightarrow$ COHA of S (resp. K-theoretical HA of S)

▶ Porta-S. : $D^b_{coh}(\underline{\mathbb{R}Coh}(S))$ has the structure of a $(\mathbb{E}_1\text{-})$ monoidal dg-category, whose tensor product is given by:

$$\boxed{D^b_{coh}(\underline{\mathbb{R}Coh}(S)) \times D^b_{coh}(\underline{\mathbb{R}Coh}(S)) \xrightarrow{m} D^b_{coh}(\underline{\mathbb{R}Coh}(S))}$$

where $m = \left((ev_2)_* \circ (ev_1 \times ev_3)^*\right) \circ \boxtimes$

$\Longrightarrow$ Categorified HA of S

Remark

▶ the Thm holds also for
- S only quasi-proj.
- $\underline{Coh}_{ps}(S)$ = moduli stack of properly supported sheaves on S

▶ $\exists$ an equivariant version of the Thm w.r.t.

$$T = \text{torus} \curvearrowright S \rightsquigarrow T \curvearrowright \underline{Coh}_{ps}(S)$$

▶ Note that $D^b_{coh}(\underline{\mathbb{R}\mathrm{Coh}}(S)) \not\simeq D^b_{coh}(\underline{\mathrm{Coh}}(S))$

Moreover, $\nexists$ CatHA over $D^b_{coh}(\underline{\mathrm{Coh}}(S))$

$\Longrightarrow$ "categorification" requires DAG

Notation: in the following, some results hold for

$$H^{BM}_*(-),\ G_0(-),\ D^b_{coh}(-) \rightsquigarrow H(-)$$

Similarly, $HA(-)$ denotes $COHA(-)$, $KHA(-)$, $CatHA(-)$

Attention ⚠: One of the main goals in the theory of COHAs is to obtain an explicit characterization of the algebra by GENERATORS and RELATIONS.

This has been achieved for "smaller" COHAs.
Note that if

▶ $\mathcal{T} \subset \mathrm{Coh}_{ps}(S)$ is a Serre subcategory, and

▶ the corresponding moduli stack

$$\underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{T}}(S)$$

is open in $\underline{\mathbb{R}\mathrm{Coh}}_{ps}(S)$. Then

$$\exists \quad \mathrm{COHA}_{\mathcal{T}}(S),\ \mathrm{KHA}_{\mathcal{T}}(S),\ \mathrm{catHA}_{\mathcal{T}}(S)$$

For example,

▶ $\mathcal{T} = \mathrm{Coh}_0(S) = \{$ 0-dimensional sheaves on $S\} \subset \mathrm{Coh}_{ps}(S)$

▶ $\underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{T}}(S) = \underline{\mathbb{R}\mathrm{Coh}}_0(S)$ = derived moduli stack of 0-dimensional sheaves on S

$$\Longrightarrow \mathrm{COHA}_0(S),\ \mathrm{KHA}_0(S),\ \mathrm{CatHA}_0(S)$$

In this case, we have a full understanding of the algebraic structure of these algebras.

Indeed, the first result concerns $S = \mathbb{C}^2 \circlearrowleft \mathbb{C}^* \times \mathbb{C}^*$:

Theorem (Schiffmann-Vasserot)

$\mathbb{C}^* \times \mathbb{C}^* \curvearrowright \mathbb{C}^2$:

$$\begin{cases} \mathrm{COHA}_0^{\mathbb{C}^*\times\mathbb{C}^*, sph}(\mathbb{C}^2) \simeq Y^+_{\varepsilon_1,\varepsilon_2}(\hat{\mathfrak{gl}}(1)) \\ \mathrm{COHA}_0^{\mathbb{C}^*\times\mathbb{C}^*}(\mathbb{C}^2) \simeq \mathrm{COHA}_0^{\mathbb{C}^*\times\mathbb{C}^*, sph}(\mathbb{C}^2) \end{cases}$$

= (pos. part of affine Yangian of $\mathfrak{gl}(1)$)

given by generators and relations

$$\begin{cases} \mathrm{KHA}_0^{\mathbb{C}^*\times\mathbb{C}^*, sph}(\mathbb{C}^2) \simeq U^+_{q,t}(\hat{\mathfrak{gl}}(1)) \simeq \mathcal{E}^+ \\ \mathrm{KHA}_0^{\mathbb{C}^*\times\mathbb{C}^*}(\mathbb{C}^2) \simeq \mathrm{KHA}_0^{\mathbb{C}^*\times\mathbb{C}^*, sph}(\mathbb{C}^2) \end{cases}$$

= (pos. part of elliptic Hall algebra)

Here, $\mathrm{HA}_0^{sph}(S)$ is the subalgebra generated by $H(\mathbb{R}\underline{\mathrm{Coh}}_0(S;1))$

length 1

For an arbitrary smooth surface S (with pure cohomology), we have:

Thm (Mellit-Minets-Schiffmann-Vasserot for H^{BM}_*, Negut for G_0)

Consider $S \supset T$ (it could be trivial)

$$\begin{cases} \mathrm{COHA}_0^{T, sph}(S) \simeq \text{(pos. part of) Yangian of } S \end{cases}$$

$\left(COHA_0^T(S) \simeq COHA_0^{T,sph}(S) \right.$

- $KHA_0^{sph}(S) \simeq$ (pos. part of) elliptic Hall algebra of S

Important

The proof follows by working at the level of representations of $HA_0(S)$ and using Nakajima type operators to compute the relations.

2. Representations

Let us explain the main ideas to construct representations.

Fix a torsion pair $v=(\mathcal{T},\mathcal{F})$ of $Coh(S)$, i.e., projective

- $Hom(T,F)=0 \quad \forall\, T\in\mathcal{T},\ F\in\mathcal{F}$;
- $\forall\, E\ \exists\ 0 \to \underset{\in \mathcal{T}}{T} \to E \to \underset{\in \mathcal{F}}{F} \to 0$

such that the corresponding moduli stacks

$$\mathbb{R}\underline{Coh}_{\mathcal{T}}(S) \text{ and } \mathbb{R}\underline{Coh}_{\mathcal{F}}(S)$$

are open in $\mathbb{R}\underline{Coh}(S)$.

Fix $\mathcal{X} \subset \mathcal{T}$ and $\mathcal{M} \subset \mathcal{F}$ such that the corresponding moduli stacks $\mathfrak{X} \subseteq \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{T}}(S)$ and $\mathfrak{M} \subseteq \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(S)$ are open.

Consider the induced diagram:

$$\begin{tikzcd}
\mathfrak{X} \times \mathfrak{X} \arrow[d] & \mathbb{R}\underline{\mathrm{Coh}}^{\mathrm{ext}}_{\mathfrak{X},\mathfrak{X},\mathfrak{X}}(S) \arrow[l, "ev_1^{\mathfrak{X}} \times ev_3^{\mathfrak{X}}"'] \arrow[r, "ev_2^{\mathfrak{X}}"] \arrow[d] & \mathfrak{X} \arrow[d] \\
\mathbb{R}\underline{\mathrm{Coh}}(S) \times \mathbb{R}\underline{\mathrm{Coh}}(S) & \mathbb{R}\underline{\mathrm{Coh}}^{\mathrm{ext}}(S) \arrow[l, "ev_1 \times ev_3"] \arrow[r, "ev_2"'] & \mathbb{R}\underline{\mathrm{Coh}}(S)
\end{tikzcd}$$

Our hope is that

$$\boxed{\begin{array}{c} \exists \ (ev_2^{\mathfrak{X}})_* \circ (ev_1^{\mathfrak{X}} \times ev_3^{\mathfrak{X}})^! \circ \boxtimes \\ \text{such that } H(\mathfrak{X}) \text{ is an associative algebra} \end{array}} \quad (A)$$

Example: $\mathcal{T} = \mathrm{Coh}_0(S)$, $\mathfrak{X} = \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{T}}(S) = \mathbb{R}\underline{\mathrm{Coh}}_0(S)$

Since $\mathcal{T}$ is Serre, the squares are cartesian $\Longrightarrow$ (A) holds

Assume that (A) holds with $\mathfrak{X}$ closed in $\underline{\mathbb{R}\mathrm{Coh}}(S)$.
We want to construct left and right representations of $HA(\mathfrak{X})$ via $H(\mathfrak{M})$.

Consider the induced diagram:

$$\begin{tikzcd}
\mathfrak{M} \times \mathfrak{X} \arrow[d] & \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S) \arrow[l, "ev_1^{\mathfrak{M}} \times ev_3^{\mathfrak{X}}"'] \arrow[r, "ev_2^{\mathfrak{M}}"] \arrow[d] & \mathfrak{M} \arrow[d] \\
\underline{\mathbb{R}\mathrm{Coh}}(S) \times \underline{\mathbb{R}\mathrm{Coh}}(S) & \underline{\mathbb{R}\mathrm{Coh}}^{ext}(S) \arrow[l, "ev_1 \times ev_3"] \arrow[r, "ev_2"'] & \underline{\mathbb{R}\mathrm{Coh}}(S)
\end{tikzcd}$$

Our hope is that

$$\exists\ (ev_2^{\mathfrak{M}})_* \circ (ev_1^{\mathfrak{M}} \times ev_3^{\mathfrak{X}})^! \circ \boxtimes$$

such that $H(\mathfrak{M})$ is a right $HA(\mathfrak{X})$-module (RM)

$$\exists\ (ev_1^{\mathfrak{M}})_* \circ (ev_3^{\mathfrak{X}} \times ev_2^{\mathfrak{M}})^! \circ \boxtimes$$

such that $H(\mathfrak{M})$ is a left $HA(\mathfrak{X})$-module (LM)

Attention: $H(\mathcal{M})$ is NOT a bimodule of $HA(\mathcal{X})$.

Important: (RM) and (LM) holds if the categories $\mathcal{X}$ and $\mathcal{M}$ satisfy "right/left Hecke pattern property".

Consider a s.e.s.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & E_3 & \longrightarrow & 0 \\ & & \cap & & \cap & & \cap & & \\ & & \mathcal{F} & & \mathcal{F} & & \mathcal{T} & & \end{array} \qquad (*)$$

Def.

- $\mathcal{M}$ is a right HP for $\mathcal{X}$ if for any s.e.s. $(*)$

$$\boxed{E_3 \in \mathcal{X}, E_2 \in \mathcal{M} \Longrightarrow E_1 \in \mathcal{M}}$$

- $\mathcal{M}$ is a left ——— // ——— $(*)$

$$\boxed{E_3 \in \mathcal{X}, E_1 \in \mathcal{M} \Longrightarrow E_2 \in \mathcal{M}}$$

- $\mathcal{M}$ is a 2-sided ——— // ——— if it is both a left and right Hecke pattern.

Examples of 2-sided HP for $\mathcal{X} = \underline{\mathbb{R}\mathrm{Coh}}_0(S)$

- $\mathcal{M} = \underline{\mathrm{Hilb}}(S)$ = Hilbert stack of pts of S

$$\simeq \mathrm{Hilb}(S) \times pt/\mathbb{C}^*$$

- Fix H ample divisor, $r \geq 1$, $c_1 \in NS(S)$ with $\gcd(r, c_1 \cdot H) = 1$.
  $\mathcal{M} = \underline{\mathbb{R}\mathrm{Coh}}^{H\text{-}s}(S; r, c_1)$.

In order to determine the algebraic structure of $HA_0(S)$

- 1st step: to work at the level of the left/right module $H(\underline{\mathrm{Hilb}}(S))$.

- 2nd Step: to consider Nakajima type operators.

3. Nakajima operators

- $\mathcal{X} = S \times pt/\mathbb{C}^* \overset{cl}{=} \underline{\mathbb{R}\mathrm{Coh}}_0(S; 1) \overset{i}{\hookrightarrow} \underline{\mathbb{R}\mathrm{Coh}}_0(S; 1) \simeq \tilde{S} \times pt/\mathbb{C}^*$

  length 1

- $\mathcal{M} = \mathrm{Hilb}(S)$, $\mathcal{U}$ = universal sheaf on $\mathcal{M} \times S$

We have:

$$
\begin{tikzcd}
\mathcal{M} \times \mathcal{X} & \underset{S^1}{\underline{\mathbb{R}\mathrm{Coh}}}^{ext}_{\mathcal{M},\mathcal{M},\mathcal{X}}(S) \arrow[l, "ev_1^{\mathcal{M}} \times ev_3^{\mathcal{X}}"'] \arrow[r, "ev_2^{\mathcal{X}}"] & \mathcal{M} \\
 & \mathbb{P}(\mathcal{U}^{\vee} \otimes \omega_S[1]) \arrow[ul] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{X} \times \mathcal{M} & \underset{S^1}{\underline{\mathbb{R}\mathrm{Coh}}}^{ext}_{\mathcal{M},\mathcal{M},\mathcal{X}}(S) \arrow[l, "ev_3^{\mathcal{X}} \times ev_2^{\mathcal{M}}"'] \arrow[r, "ev_1^{\mathcal{M}}"] & \mathcal{M} \\
 & \mathbb{P}(\mathcal{U}) \arrow[ul] &
\end{tikzcd}
$$

Def. (Schiffmann-Vasserot, Neguț)

We define the Nakajima operators:

$$e_d := (ev_2^{\mathcal{M}})_* \left((ev_1^{\mathcal{M}} \times ev_3^{\mathcal{X}})^! \left((-) \boxtimes ((-) \otimes \chi_d) \right) \right)$$

$$f_d := (ev_1^{\mathcal{M}})_* \left((ev_3^{\mathcal{X}} \times ev_1^{\mathcal{M}})^! \left(((-) \otimes \chi_d) \boxtimes (-) \right) \right)$$

as operators

$$H(\mathcal{M}) \otimes H(S) \longrightarrow H(\mathcal{M})$$
$$H(S) \otimes H(\mathcal{M}) \longrightarrow H(\mathcal{M})$$

respectively.

Important: because of the appearance of $\mathbb{P}(-)$, one can explicitly compute relations.

4. What can we say beyond $HA_0(S)$?

Fix

$$\begin{cases} \mathcal{T} = \mathrm{Coh}_{\leq 1}(S) := \{\mathcal{F} \in \mathrm{Coh}(S) : \dim(\mathrm{supp}(\mathcal{F})) \leq 1\} \\ \mathcal{F} = \mathrm{Coh}_{t.f.}(S) := \{\text{torsion free sheaves on } S\} \end{cases}$$

Note that

- $\mathcal{T}$ is a Serre subcategory,
- $\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{T}}(S) := \mathbb{R}\underline{\mathrm{Coh}}_{\leq 1}(S)$ and $\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(S) := \mathbb{R}\underline{\mathrm{Coh}}_{t.f.}(S)$ are open in $\mathbb{R}\underline{\mathrm{Coh}}(S)$,
- $\mathbb{R}\underline{\mathrm{Coh}}_{\leq 1}(S)$ is also closed in $\mathbb{R}\underline{\mathrm{Coh}}(S)$.

$\Rightarrow \exists\, HA_{\leq 1}(S)$ = associative algebra structure on $H(\mathbb{R}\underline{\mathrm{Coh}}_{\leq 1}(S))$

Goal: construct representations of $HA_{\leq 1}(S)$.

Attention: 2-sided HPs for $Coh_{\leq 1}(S)$ are rare (do not exist?)

Note that

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0 \quad \text{in } Coh(S)$$

if $E_3 \in \mathcal{T}, E_2 \in \mathcal{F} \Longrightarrow E_1 \in \mathcal{F}$

$$\boxed{\Longrightarrow \mathcal{F} \text{ right HP for } \mathcal{T} \text{ in } Coh(S)}$$

but

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0 \quad \text{in } Coh(S)$$

if $E_3 \in \mathcal{T}, E_1 \in \mathcal{F} \;\not\Longrightarrow\; E_2 \in \mathcal{F}$

$\Longrightarrow \mathcal{F}$ not left HP for $\mathcal{T}$ in $Coh(S)$

On the other hand, we can "rotate" in $D^b(Coh(S))$

$$E_3 \longrightarrow E_1[1] \longrightarrow G$$

This triangle is a short exact sequence in the tilted heart:

$$D^b(\mathrm{Coh}(S))^{\heartsuit_\tau} = \{ E \in D^b(\mathrm{Coh}(S)) : \mathcal{H}^{-1}(E) \in \mathcal{F}, \mathcal{H}^0(E) \in \mathcal{T}, \mathcal{H}^i(E) = 0 \ \forall i \neq -1, 0 \}$$

and $(\mathcal{F}[1], \mathcal{T})$ torsion pair of $D^b(\mathrm{Coh}(S))^{\heartsuit_\tau}$

Moreover,

because $\mathcal{F}[1]$ is torsion

$$\text{if } E_3 \in \mathcal{T},\ E_1 \in \mathcal{F} \implies G = E_2[1] \in \mathcal{F}[1]$$

i.e., $\mathcal{F}$ is a right HP for $\mathcal{T}$ in the tilted heart

Attention ⚠:

the use of the tilted heart "compensate" the lack of 2-sided Hecke patterns.

This observation is essential to prove:

Thm (Diaconescu-Porta-S.)

$H(\mathbb{R}\underline{\mathrm{Coh}}_{t.f.}(S))$ is a left and right module over $HA_{\leq 1}(S)$.

By using this framework, we also proved:

Thm (Diaconescu-Porta-S.)

- $H(P(S))$ is a right and left module of $HA_0(S)$
- ——— " ——— a right module of $HA_{\leq 1}(S)$.

where

$P(S) :=$ moduli space of Pandharipande-Thomas stable pairs on S

$$\| $$

$(\mathcal{F}, s : \mathcal{O}_S \longrightarrow \mathcal{F})$ with $\mathcal{F}$ pure 1-dimen.
$\mathrm{Coker}(s)$ 0-dimen.

Remark

We are able to prove the above theorem for any "reasonable" tuple

$$\boxed{(\mathcal{C} = \text{triangulated category}, \tau = t\text{-structure}, (\mathcal{T}, \mathcal{F}) = \text{torsion pair in } \mathcal{C}^{\heartsuit})}$$

Question: What about Nakajima type operators in this case?

Fix an effective divisor D in S, an ample divisor H in S, and $\alpha \in \mathbb{Q}$.

Let
$$\mathrm{Coh}_{\alpha}^{(s)s}(S) = \left\{ E \in \mathrm{Coh}_{\leq 1}(S) : E \text{ is } H\text{-(semi)semistable of fixed slope } \alpha \right\}$$

$$\mu(-) = \frac{\chi(-)}{H \cdot ch_1(-)}$$

Definition

A subcategory $\mathcal{X} \subset \mathrm{Coh}_{\alpha}^{s}(S)$ is admissible if

- any sheaf $\mathcal{E} \in \mathcal{X}$ is scheme-theoretically supported on D
- $\mu_{H\text{-max}}(\mathcal{E} \otimes \mathcal{O}_S(D)) \leq \alpha \leq \mu_{H\text{-min}}(\mathcal{E} \otimes \mathcal{O}_S(-D))$

We assume that the corresponding moduli stack $\mathfrak{X}$ is open and closed in $\mathbb{R}\underline{\mathrm{Coh}}_{\alpha}^{ss}(S)$.

Let $i : D \hookrightarrow S$ be the inclusion.

Let $\mathcal{M} \subset \mathrm{Coh}_{t.f.}(S)$ be the subcategory of locally free sheaves $\mathcal{F}$ on S s.t.

$$\boxed{\mu_{H\text{-max}}(\iota_* \iota^* \mathcal{F} \otimes \mathcal{O}_S(D)) \leq \alpha \leq \mu_{H\text{-min}}(\iota_* \iota^* \mathcal{F})}$$

Lemma

$\mathcal{M}$ is a 2-sided HP for $\mathcal{X}$.

We have:

$$\mathfrak{M} \times \mathfrak{X} \xleftarrow{ev_1^{\mathfrak{M}} \times ev_3^{\mathfrak{X}}} \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S) \xrightarrow{ev_2^{\mathfrak{X}}} \mathfrak{M}$$

$$\mathfrak{X} \times \mathfrak{M} \xleftarrow{ev_3^{\mathfrak{X}} \times ev_2^{\mathfrak{M}}} \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S) \xrightarrow{ev_1^{\mathfrak{M}}} \mathfrak{M}$$

Def.

We define the Nakajima operators:

$$e := (ev_2^{\mathfrak{M}})_* \left((ev_1^{\mathfrak{M}} \times ev_3^{\mathfrak{X}})^! \circ \boxtimes \right. \qquad f := (ev_1^{\mathfrak{M}})_* \left((ev_3^{\mathfrak{X}} \times ev_1^{\mathfrak{M}})^! \circ \boxtimes \right.$$

Furthermore, $\exists$ $B\mathbb{G}_m$-action on $\mathfrak{X}$. Thus, we have also

$$e_d, f_d \quad \text{for } d \in \mathbb{Z}$$

Remark

By combining techniques from Hall algebras together with

Neguţ-Yu Zhao's approach, we are able to compute relations.

In particular, we obtain:

Thm (Diaconescu-Porta-S.-Yu Zhao)

Let $\pi\colon S \longrightarrow B$ be a relatively minimal smooth projective elliptic surface, which admits a section.

Assume that π admits a unique singular fiber D such that D_{red} is an affine ADE configuration of (-2)-rational curves, with at least 3 irreducible components E_i.

Assume that $\exists$ a (possibly trivial) torus T acting on S.

Let $\mathcal{X} \subset Coh_0^s(S)$ be the subcategory consisting of sheaves scheme-theoretically supported to a single irreducible component of D_{red} (i.e., of the form $\mathcal{O}_{E_i}(-1)$).

Then the action of the Nakajima operators associated to $\mathcal{X}$ gives rise to an action of

- the affine Yangian of type ADE on $H_*^{BM}(\mathcal{M})$,
- the quantum toroidal algebra of type ADE on $G_0(\mathcal{M})$.

Attention:

This provides a cohomological and K-theoretical version of an old construction of Ginzburg-Kapranov-Vasserot concerning operators arising from

$$\boxed{\text{Hecke modifications associated to } \mathcal{O}_{E_i}(-1)}$$

Remark

Our framework works also when D = smooth projective curve of genus ≥ 1 and $D^2 < 0$.

Goal: compute relations in this case!