Categorified Hall algebras and their representations

versus

categorified Nakajima operators

The first part of this talk is devoted to the explanation of the terms appearing in the title.

1. Categorified Hall algebras of surfaces

Let S = smooth proj. surface$/_{\mathbb{C}}$.

Let

$$\boxed{\underline{\mathbb{R}\mathrm{Coh}}(S) = \text{derived moduli stack of coherent sheaves on } S}$$

Consider the "convolution diagram":

$$\underline{\mathbb{R}\mathrm{Coh}}(S) \times \underline{\mathbb{R}\mathrm{Coh}}(S) \xleftarrow{q} \underbrace{\underline{\mathbb{R}\mathrm{Coh}}^{ext}(S)}_{= \text{stack of extensions}} \xrightarrow{p} \underline{\mathbb{R}\mathrm{Coh}}(S)$$

where:

$$p = ev_2 : 0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0 \longmapsto E_2$$

$$q = ev_3 \times ev_1 : \quad \text{———} \, \text{''} \, \text{———} \quad \longmapsto (E_3, E_1)$$

- The map p is proper representable.
- The map q is derived lci.

Theorem (Porta-S.)

suitable dg enhancement

$D^b_{coh}(\underline{\mathbb{R}Coh}(S))$ has the structure of an E_1-monoidal dg category, whose underlying tensor product is given by:

$$\boxed{\begin{gathered} m: D^b_{coh}(\underline{\mathbb{R}Coh}(S)) \times D^b_{coh}(\underline{\mathbb{R}Coh}(S)) \xrightarrow{\boxtimes} \\ D^b_{coh}(\underline{\mathbb{R}Coh}(S)\times\underline{\mathbb{R}Coh}(S)) \xrightarrow{p_* \circ q^*} D^b_{coh}(\underline{\mathbb{R}Coh}(S)) \end{gathered}}$$

This is the categorified Hall algebra of S.

In particular, $\underbrace{G_0(\underline{Coh}(S))}^{\text{K-theoretical HA}}$ and $\overbrace{H^{BM}_*(\underline{Coh}(S))}^{\text{COHA}}$ have the structures of unital associative algebras.

Notation: $H(-)$ denotes $H^{BM}_*(-)$, $G_0(-)$, $D^b_{coh}(-)$
$HA(-)$ —— " — COHA, KHA, CatHA

relevant in
- categ. of enu. inv.s
- Knot invariants

Remark

1. We have:
- a version of the Thm for $\underline{\mathbb{R}Coh}_{ps}$ (non-proper S)
- an equivariant version of the Thm w.r.t. T = torus $\curvearrowright S$

2. There are been other equivalent constructions of KHAs and COHAs of surfaces due to:
Yu Zhao (0-dim case), Kapranov-Vasserot, Minets ($S = T^*C$, 0-dim. case), S.-Schiffmann ($S = T^*C$)

3. Note that

- $D^b_{coh}(\underline{\mathbb{R}\mathrm{Coh}}(S)) \not\simeq D^b_{coh}(\underline{\mathrm{Coh}}(S))$, and
- $\nexists$ CatHA over

2. Representations

In the theory of HAs, representations are essential to give a description in terms of generators and relations of COHA(S), KHA(S), ...

Attention ⚠:

In general, COHA(S) and KHA(S) are too big.
First, it is better to restrict ourselves to "smaller" algebras

Consider

to shorten the notation

$$\boxed{\begin{array}{l} \mathfrak{X} = \mathbb{R}\underline{\mathrm{Coh}}_0(S) = \text{derived moduli stack of} \\ \qquad\qquad\qquad\qquad \text{0-dimensional sheaves on } S \\ \text{it is a Serre subcategory} \\ \Longrightarrow HA_0(S) \end{array}}$$

<u>Example:</u>

$$\begin{cases} COHA_0^{(\mathbb{C}^*)^2}(\mathbb{C}^2) \simeq Y^+(\hat{\mathfrak{gl}}(1)) & \text{- affine Yangian} \\ KHA_0^{(\mathbb{C}^*)^2}(\mathbb{C}^2) \simeq U^+(\hat{\hat{\mathfrak{gl}}}(1)) & \text{- elliptic Hall algebra} \end{cases}$$

The key notion to construct representations of these algebras is that of "Hecke patterns".

Consider a s.e.s.

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{T} \longrightarrow 0 \qquad (*)$$

with 0-dimensional sheaf $\mathcal{T}$, and

$\mathcal{E}, \mathcal{F} \in \mathrm{Coh}_{\geq 1}(S) = \{\mathcal{E} \in \mathrm{Coh}(S) : \mathcal{E}$ does not contain any 0-dimensional subsheaf$\}$

Def.

Let $\mathcal{M} \longrightarrow \underline{\mathrm{RCoh}}_{\geq 1}(S)$ be a derived stack together with a map (e.g., $\mathcal{M}$ is an open substack).

- $\mathcal{M}$ is a left Hecke pattern (LHP) for $\mathcal{X}$ if for any s.e.s. $\circledast$

$$\boxed{\mathcal{T} \in \mathcal{X}, \mathcal{F} \in \mathcal{M} \Longrightarrow \mathcal{E} \in \mathcal{M}}$$

- $\mathcal{M}$ is a right (RHP) ——— // ——— $\circledast$

$$\boxed{\mathcal{T} \in \mathcal{X}, \mathcal{E} \in \mathcal{M} \Longrightarrow \mathcal{F} \in \mathcal{M}}$$

- $\mathcal{M}$ is a 2-sided (2-SHP) ——— // ——— if it is both a left and right Hecke pattern.

Thm

Let $\mathcal{M}$ be a 2-SHP for $\mathcal{X}$. Then, the convolution diagrams

$$\mathcal{X} \times \mathcal{M} \xleftarrow{q} \underline{\mathrm{RCoh}}^{ext}_{\mathcal{M},\mathcal{M},\mathcal{X}}(S) \xrightarrow{p} \mathcal{M} \qquad (L)$$

$$\mathcal{M} \times \mathcal{X} \xleftarrow[ev_2 \times ev_3]{q' \,=} \underline{\mathrm{RCoh}}^{ext}_{\mathcal{M},\mathcal{M},\mathcal{X}}(S) \xrightarrow{p' = ev_1} \mathcal{M} \qquad (R)$$

induces left and right actions of

$$\boxed{\quad HA_0(S) \quad \text{on} \quad H(\mathcal{M}) \quad}$$

Remark

This is a consequence of my work with Diaconescu and Porta, in Borel-Moore homology a similar result has been obtained by Mellit-Minets-Schiffmann-Vasserot

Examples

- $\mathcal{M} = \underline{\mathrm{Hilb}}(S)$ = Hilbert stack of pts of S

$$\simeq \mathrm{Hilb}(S) \times pt/\mathbb{C}^*$$

- Fix H ample divisor, $r \geq 1$, $c_1 \in NS(S)$ with $\gcd(r, c_1 \cdot H) = 1$.
  $\mathcal{M} = \underline{\mathbb{R}\mathrm{Coh}}^{H-s}(S; r, c_1)$.

3. Categorified Nakajima operators

Nakajima operators are operators acting on the Borel-Moore homology of Hilbert schemes of pts on S.
In the above setting, consider:

$$\mathcal{X} = S \xhookrightarrow[i]{} \tilde{S} \xleftarrow[j]{} \tilde{S} \times pt/\mathbb{C}^* \simeq \underline{\mathbb{R}\mathrm{Coh}}_0(S;1)$$

$\tilde{S} \times pt = \operatorname{Spec} \operatorname{Sym}(\omega_S[1])$; 1 : length 1

$$\mathcal{M} = \mathrm{Hilb}(S)$$

Then, we have:

$$\begin{tikzcd}
\mathcal{X} \times \mathcal{M} & \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{M},\mathcal{M},\mathcal{X}}(S) \arrow[l, "q"'] \arrow[r, "p"] & \mathcal{M} \\
& \mathbb{P}(\mathcal{U}^\vee \otimes \omega_S[1]) \arrow[u, phantom, "\wr\wr"] \arrow[ul, "\tilde{q}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{M} \times \mathcal{X} & \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{M},\mathcal{M},\mathcal{X}}(S) \arrow[l, "q' = ev_2 \times ev_3"'] \arrow[r, "p' = ev_1"] & \mathcal{M} \\
& \mathbb{P}(\mathcal{U}) \arrow[u, phantom, "\wr\wr"] \arrow[ul, "\tilde{q}'"] &
\end{tikzcd}$$

Def. (Categorified Nakajima operators)

We define the functors:

$$\begin{cases} \mu^+_d := \tilde{q}'_!\left(p'^*(-) \otimes \mathcal{O}(d)\right) : \\ \mu^-_k := \tilde{q}_*\left(p^*(-) \otimes \mathcal{O}(-k)\right) : \end{cases} \quad D^b_{coh}(\mathcal{M}) \longrightarrow D^b_{coh}(\mathcal{M} \times S)$$

"simple" Nakajima operators

Remark

- Relation to Hall product: $\forall\, \mathcal{E} \in D^b\mathrm{Coh}(S)$:

$$\boxed{pr_{m,*}\left(\mu^-_k(-) \otimes pr_S^* \mathcal{E}\right) = m\left(-, \vartheta^* i_*(\mathcal{E}) \otimes \mathcal{O}(k)\right)}$$

$\Longrightarrow$ negative Nakajima operators are elements in $HA_0(S)$

- $\exists\ \mu^+_{d_1,\dots,d_n}$ and $\mu^-_{k_1,\dots,k_m}$ defined by Neguț ("iterated" Nakajima operators)

Thm (Neguț for G_0, Mellit-Minets-Schiffmann-Vasserot for H^{BM}_*)
Categorified Nakajima operators induce an action of

$$\boxed{\begin{array}{ll} \text{the elliptic Hall algebra of } S & \text{on } G_0(\mathrm{Hilb}(S)) \\ \text{the affine Yangian of } S & \text{on } H^{BM}_*(\mathrm{Hilb}(S)) \end{array}}$$

Moreover, $COHA_0(S)$ is generated by $H^{BM}_*(\underline{\mathbf{R}\mathrm{Coh}}_0(S;1))$
i.e., $COHA_0(S)$ is spherically generated.

Remark

- Negut has computed rels between categorified $\mu^+_\bullet$'s
- Yu Zhao ——————— " ——————————— $\mu^+_\bullet$'s and $\mu^-_\bullet$'s

This concludes this first part.

1. Categorified Hall algebras in general

Fix

- $\mathcal{C}$ = $\mathbb{C}$-linear stable ∞-category of finite type
(e.g. $\mathcal{C} = \mathrm{QCoh}(S)$)
(i.e. $\mathrm{Fun}(\mathcal{C}, -)$ commutes with filtered colimits)

- $\mathbb{R}\underline{\mathrm{Perf}}_{ps}(\mathcal{C})$ = Toën-Vaquié's derived moduli stack of pseudo-perfect objects of $\mathcal{C}$

$$\mathbb{R}\underline{\mathrm{Perf}}_{ps}(\mathcal{C})(\underset{\substack{\cap \\ \mathrm{dAff}_{\mathbb{C}}}}{A}) := \left\{ E \in \mathcal{C}_A := \mathcal{C} \otimes_{\mathbb{C}} \mathrm{Mod}_A : \forall G \in \mathcal{C}_A^{\omega}, \ \mathrm{Hom}_{\mathcal{C}}(G, E) \in \mathrm{Perf}(A) \right\}^{\simeq}$$

It is a locally Artin derived stack locally of finite type$/\mathbb{C}$

Remark

- $\mathcal{C} = \mathrm{QCoh}(S) \implies$ pseudo-coherent = properly supported

Now we introduce the stack of coherent objects.

Now, we fix:

- $\tau = (\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ t-structure

Fact: For any $A \in \mathrm{dAff}_{\mathbb{C}}$, $\exists$ a t-structure on $\tau_A = (\mathcal{C}_A^{\leq 0}, \mathcal{C}_A^{\geq 0})$ on

$$\mathcal{C}_A := \mathcal{C} \otimes_{\mathbb{C}} \mathrm{Mod}_A$$

induced by τ (w.r.t. the equivalence:

$$\mathcal{C}_A \simeq \underbrace{\mathrm{Fun}^R_{\mathbb{C}}(\mathrm{Mod}_A^{op}, \mathcal{C})}_{}$$ functors which are right adjoint

τ_A is induced via the forgetful morphism $\mathcal{C}_A \longrightarrow \mathcal{C}$ (evaluation at A))

Def. (flat objects)

An object $E \in \mathcal{C}_A$ is τ-flat if $\forall\, M \in \mathrm{Mod}_A^{\heartsuit}$, one gets $M \otimes E \in \mathcal{C}_A^{\heartsuit}$ (heart w.r.t. τ_A)

Example

$\mathcal{C} = \mathrm{QCoh}(S)$, $\tau = \tau_{std}$ = standard t-structure

For $A \in \mathrm{dAff}_{\mathbb{C}}$, $\tau_A = \tau_{std}$ on $\mathrm{QCoh}(S \times \mathrm{Spec} A)$

Then

$$E \in \mathcal{C}_A \text{ flat} \iff p^*(M) \otimes E \in \mathcal{C}_A^{\heartsuit} \quad \forall\, M \in \mathrm{Mod}_A^{\heartsuit}$$

$p : S \times \mathrm{Spec} A \longrightarrow \mathrm{Spec} A$

Attention ⚠: if A is classical, it is equivalent to the usual notion of flat families

$\exists\ \underline{\mathbb{R}\mathrm{Coh}}_{ps}(\mathcal{C},\tau) \subseteq \underline{\mathbb{R}\mathrm{Perf}}_{ps}(\mathcal{C})$
= derived moduli stack of (τ-) flat pseudo-perfect objects of $\mathcal{C}$

$$\underline{\mathbb{R}\mathrm{Coh}}_{ps}(\mathcal{C},\tau)(A) = \underline{\mathbb{R}\mathrm{Perf}}_{ps}(\mathcal{C}_A,\tau_A) \cap \mathrm{Coh}(\mathcal{C}_A,\tau_A)$$

Prop:

$\underline{\mathbb{R}\mathrm{Coh}}_{ps}(\mathcal{C},\tau)$ is Artin $\Leftrightarrow$ $\underline{\mathbb{R}\mathrm{Coh}}_{ps}(\mathcal{C},\tau) \subseteq \underline{\mathbb{R}\mathrm{Perf}}_{ps}(\mathcal{C})$ open

Def. We say that τ satisfies openness of flatness if • holds

Fix $\mathcal{C}$ and τ as above.

Consider the "convolution diagram" as before:

$$\underline{\mathbb{R}\mathrm{Coh}}_{ps}(\mathcal{C},\tau) \times \underline{\mathbb{R}\mathrm{Coh}}_{ps}(\mathcal{C},\tau) \xleftarrow{q_\tau} \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{ps}(\mathcal{C},\tau) \xrightarrow{p_\tau} \underline{\mathbb{R}\mathrm{Coh}}_{ps}(\mathcal{C},\tau)$$

Thm (Porta-S., Diaconescu-Porta-S.)

Assume that

1. $\mathcal{C}$ is of finite type,
2. τ satisfies openness of flatness,
3. Serre functor $S_{\mathcal{C}}$ s.t. $S_{\mathcal{C}}[-2]$ is t-exact, ($\Rightarrow q_\tau$ derived lci)
4. p_τ proper.

Then, $\exists$ a $\mathbb{E}_1$-monoidal structure on $D^b\mathrm{Coh}(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau))$ induced by $(p_\tau)_* q_\tau^*$.

In particular, $G_0(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau))$ and $H_*^{BM}(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau))$ have the structures of unital associative algebras.

Furthermore, if $\mathfrak{X} \subseteq \mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau)$ is an open substack s.t. $\forall$ field K, $\mathfrak{X}(K)$ is a Serre subcategory.
Then, $\exists$ induced $HA(\mathfrak{X})$.

Example

- $\mathcal{C} = \mathrm{Ind}(\mathrm{Perf}(K3))$ or $\mathcal{C} = \mathrm{Ind}(\text{Kuznetsov component})$ (of a smooth cubic 4fold $\subset \mathbb{P}^5$, Fano 3fold of $rk(\mathrm{Pic})=1$, GM varieties)
  $\tau = t$-structure associated to a stability condition σ
  $\mathfrak{X} = \mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau)$, $\mathbb{R}\underline{\mathrm{Coh}}^{\sigma\text{-ss}}_{ps}(\mathcal{C},\tau; v\in\Lambda)$

- $\mathcal{C} = QCoh(S)$, $\tau = \tau_{std}$

Fix a torsion pair $v = (\mathcal{T}or, \mathcal{F})$ of $Coh(S)$, i.e.,

- $Hom(T,F) = 0 \quad \forall\, T \in \mathcal{T}or,\ F \in \mathcal{F}$;
- $\forall\, E \quad \exists\ 0 \longrightarrow \underset{\mathcal{T}or}{\underset{\in}{T}} \longrightarrow E \longrightarrow \underset{\mathcal{F}}{\underset{\in}{F}} \longrightarrow 0$

$\Longrightarrow$ $\exists$ new t-structure τ_v such that its heart is:

$$\boxed{\begin{array}{l} \mathcal{C}_v^{\heartsuit} = \left\{ E \in \mathcal{C} : \mathcal{H}^{-1}_{\tau}(E) \in \mathcal{F},\ \mathcal{H}^{0}_{\tau}(E) \in \mathcal{T}or,\ \mathcal{H}^{i}_{\tau}(E) = 0 \quad \forall\, i \neq -1, 0 \right\} \\ \text{and } (\mathcal{F}[1], \mathcal{T}or) \text{ torsion pair of } \mathcal{C}_v^{\heartsuit} \end{array}}$$

Then

$$\exists\ \underline{\mathbb{R}Coh}_{\mathcal{T}or}(S),\ \underline{\mathbb{R}Coh}_{\mathcal{F}}(S) \subseteq \underline{\mathbb{R}Coh}(S),\ \subseteq \underline{\mathbb{R}Coh}_{ps}(\mathcal{C}, \tau_v)$$

Def. We say that v is open if • open.

Facts:

- Lieblich: τ_v satisfies openness of flatness
- DPS: τ_v is proper.

$\boxed{\Longrightarrow \exists\ HA_{\tau_v} \text{ associated to the tilted t-structure } \tau_v}$

Moreover,

$\boxed{\text{if } \mathcal{T}or \text{ is a Serre subcategory} \Longrightarrow \exists\ HA_{\mathcal{T}or} \text{ associated to } \mathcal{T}or}$

2. Representations in general

Now, we discuss how to construct representations.

Fix, $\mathcal{C}$, τ again.

Fix $v = (\mathcal{T}or, \mathcal{F})$ torsion pair of $\mathcal{C}^{\heartsuit}$.

Note that

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{T} \longrightarrow 0 \quad \text{in } \mathcal{C}^{\heartsuit}$$

if $\mathcal{T} \in \mathcal{T}or$, $\mathcal{E} \in \mathcal{F} \Longrightarrow \mathcal{E}' \in \mathcal{F}$

$$\boxed{\Longrightarrow \mathcal{F} \text{ left HP for } \mathcal{T}or \text{ w.r.t. } \tau}$$

but

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{T} \longrightarrow 0 \quad \text{in } \mathcal{C}^{\heartsuit}$$

if $\mathcal{T} \in \mathcal{T}or$, $\mathcal{E}' \in \mathcal{F}$ $\not\Longrightarrow$ $\mathcal{E} \in \mathcal{F}$

$\Longrightarrow \mathcal{F}$ not right HP for $\mathcal{T}or$ w.r.t. τ

On the other hand, if we "rotate":

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{E}'[1] \longrightarrow \mathcal{G} \longrightarrow 0 \quad \text{in } \mathcal{C}^{\heartsuit}_{v} = \text{tilted heart}$$

if $\mathcal{T} \in \mathcal{T}or$, $\mathcal{E}' \in \mathcal{F}$ $\Longrightarrow \mathcal{G} = \mathcal{E}[1] \in \mathcal{F}[1]$

because $\mathcal{F}[1]$ is torsion in $\mathcal{C}^{\heartsuit}_{v}$

This observation is essential to prove:

Thm (Diaconescu-Porta-S.)

Assume that:

1. $\mathscr{C}$ is of finite type,

2. τ and τ_v satisfy openness of flatness,

3. $\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{Tor}}(\mathscr{C},\tau)$ and $\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathscr{C},\tau)$ are open in both $\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{ps}}(\mathscr{C},\tau)$ and $\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{ps}}(\mathscr{C},\tau_v)$

4. Serre functor $S_{\mathscr{C}}$ s.t. $S_{\mathscr{C}}[-2]$ is t-exact w.r.t. τ and τ_v,

5. p_τ and p_{τ_v} are proper,

6. Tor is a Serre subcategory.

7. $\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{Tor}}(\mathscr{C},\tau)$ is closed in both $\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{ps}}(\mathscr{C},\tau)$ and $\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{ps}}(\mathscr{C},\tau_v)$

Then

- $H(\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathscr{C},\tau))$ is a left module of HA_{Tor}

induced by:

$$\begin{tikzcd}
 & \mathbb{R}\underline{\mathrm{Coh}}^{\mathrm{ext}}_{\mathcal{F},\mathcal{F},\mathrm{Tor}}(\mathscr{C},\tau) \arrow[dl, "q_\tau"'] \arrow[dr, "p_\tau"] & \\
\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{Tor}}(\mathscr{C},\tau)\times\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathscr{C},\tau) & & \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathscr{C},\tau)
\end{tikzcd}$$

- $H(\underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau))$ is a right module of $HA_{\mathcal{T}or}$ induced by

$$\underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau) \times \underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{T}or}(\mathcal{C},\tau) \xleftarrow{q_{\tau_v}} \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{T}or,\mathcal{F}[1],\mathcal{F}[1]}(\mathcal{C},\tau) \xrightarrow{p_{\tau_v}} \underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau)$$

Thus, the use of the tilted heart overcomes the lack of 2-sided Hecke patterns.

Example

- S = smooth projective surface$/\mathbb{C}$

$$\boxed{\begin{aligned} \mathcal{T}or &= \mathrm{Coh}_{\leq 1}(S) := \{\mathcal{F} \in \mathrm{Coh}(S) : \dim(\mathrm{supp}(\mathcal{F})) \leq 1\} \\ \mathcal{F} &= \mathrm{Coh}_{t.f.}(S) := \{\text{torsion free sheaves on } S\} \end{aligned}}$$

or

$$\boxed{\begin{aligned} \mathcal{T}or &= \mathrm{Coh}_0(S) \\ \mathcal{F} &= \mathrm{Coh}_{\geq 1}(S) \end{aligned}}$$

$\Longrightarrow$ We recover the examples discussed before, moreover we get:

$P(S)$:= moduli space of Pandharipande-Thomas stable pairs on S

$$\| $$

$(\mathcal{F}, s : \mathcal{O}_S \longrightarrow \mathcal{F})$ with $\mathcal{F}$ pure 1-dimen.
$\mathrm{CoKer}(s)$ 0-dimen.

Thm (DPS)

- $H(P(S))$ is a right and left module of $HA_0(S)$
- ——— " ——— a left module of $HA_{\leq 1}(S)$.

3. Categorified Nakajima operators in general

Fix $\mathcal{C} = QCoh(S)$, $\tau = \tau_{std}$

$\mathcal{M} \subseteq \underline{\mathbb{R}Coh}_{f.f.}(S)$ a 2-sided HP for $\mathcal{X} \subseteq \underline{\mathbb{R}Coh}_{tor}(S)$ open

Then

$$\boxed{\begin{tikzcd} \mathcal{X} \times \mathcal{M} & \underline{\mathbb{R}Coh}^{ext}_{\mathcal{M},\mathcal{M},\mathcal{X}}(S) \arrow[l, "q"'] \arrow[r, "p"] \arrow[d] & \mathcal{M} \\ & \mathbb{P}(\quad) \arrow[ul, "\tilde{q}"] & \end{tikzcd}}$$

$$\boxed{\begin{tikzcd} \mathcal{M} \times \mathcal{X} & \underline{\mathbb{R}Coh}^{ext}_{\mathcal{M},\mathcal{M},\mathcal{X}}(S) \arrow[l, "q"'] \arrow[r, "p' = ev_1"] \arrow[d] & \mathcal{M} \\ & \mathbb{P}(\quad) \arrow[ul, "\tilde{q}'"] & \end{tikzcd}}$$

$\Longrightarrow$ p, p' are derived lci and $\mathbb{C}^*$-equiv., $\tilde{q}, \tilde{q}'$ are proper

Def. (Categorified Nakajima operators)

We define the functors:

$$\boxed{\begin{cases} \mu^+_d := \tilde{q}'_!\left(p'^*(-) \otimes \mathcal{O}(d)\right) : \\ \mu^-_k := \tilde{q}_*\left(p^*(-) \otimes \mathcal{O}(-k)\right) : \end{cases} \quad D^b_{coh}(\mathcal{M}) \longrightarrow D^b_{coh}(\mathcal{M} \times \mathcal{X})}$$

At the moment, together with Yu Zhao, we are studying these operators for:

Example

- $\mathfrak{X} \subseteq \underline{\mathbb{R}\mathrm{Coh}}^{H\text{-}s}_{\mu}(S)$ parametrizing sheaves scheme-theoretically supported on a fixed effective divisor D

- $\mathfrak{M} \subseteq \underline{\mathbb{R}\mathrm{Coh}}_{l.f.}(S)$ parametrizing locally free sheaves $\mathcal{F}$ on S with $\mu_{max}(\mathcal{F}) \leq \mu$.

This is a setting motivated by an old paper by Ginzburg-Kapranov-Vasserot

Geometric idea behind the proof:

Consider:

$$(\ast)\qquad \begin{tikzcd}[column sep=tiny] & \mathbb{R}\underline{\mathrm{Coh}}^{ext}_{Tor,\mathcal{F}}(\mathcal{C},\tau) \arrow[dl, "q_\tau"'] \arrow[dr, "p_\tau"] & \\ \mathbb{R}\underline{\mathrm{Coh}}_{Tor}(\mathcal{C},\tau)\times\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau) & & \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau) \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{E} \arrow[r] \arrow[dl, "q_\tau"'] \arrow[dr, "p_\tau"] & T \arrow[r] & 0 \\ & (T,\mathcal{E}') & & \mathcal{E} & \end{tikzcd}$$

in the abelian category $\mathcal{C}^{\heartsuit}_{\tau}$

Now, q_τ is quasi-smooth

Fact: $\mathcal{E}\in\mathcal{F} \Longrightarrow \mathcal{E}'\in\mathcal{F}$

$\Longrightarrow$ The fiber of p_τ at $\mathcal{E}$ is the Quot scheme parametrizing its torsion quotients

$\Longrightarrow$ p_τ is proper $\Longrightarrow$ $(\ast)$ gives rise to the left action.

Now, consider:

$$\begin{tikzcd}
 & \mathbb{R}\underline{\mathrm{Coh}}^{\mathrm{ext}}_{\mathcal{F},\mathcal{T}or}(\mathcal{C},\tau) \arrow[dl, "q_\tau"'] \arrow[dr, "p_\tau"] & \\
\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau) \times \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{T}or}(\mathcal{C},\tau) & & \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau)
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] & \mathcal{E} \arrow[r] \arrow[dl, "q_\tau"'] \arrow[dr, "p_\tau"] & \mathcal{E}' \arrow[r] & 0 \\
 & (\mathcal{E}', T) & & \mathcal{E} &
\end{tikzcd}$$

in the abelian category $\mathcal{C}^{\heartsuit}_{\tau}$

Attention ⚠: $\mathcal{E} \in \mathcal{F} \not\Longrightarrow \mathcal{E}' \in \mathcal{F}$

Thus, the fiber of p_τ at $\mathcal{E}$ is not proper

Consider

(**)

$$\begin{tikzcd}[column sep=small]
& \mathbb{R}\underline{\mathrm{Coh}}^{ext}_{\mathcal{T}or,\mathcal{F}}(\mathcal{C},\tau_v) \arrow[dl, "q_{\tau_v}"'] \arrow[dr, "p_{\tau_v}"] & \\
\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau)\times \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{T}or}(\mathcal{C},\tau) \arrow[d, phantom, "\wr"] & & \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau) \arrow[d, phantom, "\wr"] \\
\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}[1]}(\mathcal{C},\tau_v)\quad \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{T}or}(\mathcal{C},\tau_v) & & \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}[1]}(\mathcal{C},\tau_v)
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] & \mathcal{E} \arrow[r] \arrow[dl, "q"'] \arrow[dr, "p"] & \mathcal{E}' \arrow[r] & 0 \\
& (\mathcal{E}',T) & & \mathcal{E} &
\end{tikzcd}$$

in the abelian category $\mathcal{C}^{\heartsuit}_{\tau_v}$

- $T \simeq \mathcal{H}^0(T) \in \mathcal{T}or \subset \mathcal{C}^{\heartsuit}_{\tau_v}$

- $\mathcal{E} \simeq \mathcal{H}^{-1}(\mathcal{E})[1] \in \mathcal{F} \subset \mathcal{C}^{\heartsuit}_{\tau_v} \Longrightarrow \mathcal{E}' \simeq \mathcal{H}^{-1}(\mathcal{E}')[1] \in \mathcal{F}$

$\Longrightarrow$ as before, p_{τ_v} is proper

$\Longrightarrow$ $(**)$ gives rise to the right action. $\square$