Cohomological Hall algebras, moduli spaces, and quantum groups

Plan

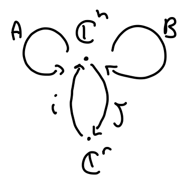
1. Motivation: moduli spaces vs. vertex algebras
2. Plethora of Cohomological Hall Algebras
3. COHA of the minimal resolution of a ADE singularity

1. moduli spaces vs. vertex algebras

Consider

$\mathcal{M}(r,n)$ = moduli space of framed sheaves ($\mathcal{E}$ torsion-free, $\phi : \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$ framing) on $\mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$ of $\mathrm{rk} = r \in \mathbb{Z}_{\geq 1}$, $c_1 = 0$, $c_2 = n \in \mathbb{Z}_{\geq 0}$

$$\simeq \left\{ (A,B,i,j) : [A,B] + ij = 0 \ ;\ \nexists\, 0 \subsetneq S \subsetneq \mathbb{C}^n \text{ s.t. } A(S) \subseteq S, B(S) \subseteq S, \mathrm{Im}(i) \subseteq S \right\} / GL_n(\mathbb{C})$$



this is the ADHM description of $\mathcal{M}(r,n)$

via conjugation: $(gAg^{-1}, gBg^{-1}, gi, jg^{-1})$

= smooth quasi-projective variety of dimension $2rn$

Rmk

- $\mathcal{M}(1,n) \simeq \mathrm{Hilb}^n(\mathbb{C}^2)$ = Hilbert scheme of n pts in $\mathbb{C}^2$
- $\mathcal{M}(r,n)$ = Nakajima quiver variety associated to the 1-loop quiver 

torus of $\mathbb{P}^2$

Fact: $\exists\, T := \underset{(t_1,t_2)}{(\mathbb{C}^*)^2} \times \underset{D}{(\mathbb{C}^*)^r} \curvearrowright \mathcal{M}(r,n)$:

D = diagonal matrix

- $(t_1,t_2,D)\cdot(\mathcal{E},\phi) = \left((F^{-1}_{t_1,t_2})^* \mathcal{E},\ \check{\phi} : (F^{-1}_{t_1,t_2})^* \mathcal{E} \longrightarrow (F^{-1}_{t_1,t_2})^* \mathcal{O}_{\ell_\infty}^{\oplus r} \longrightarrow \mathcal{O}_{\ell_\infty}^{\oplus r} \xrightarrow{\cdot D} \mathcal{O}_{\ell_\infty}^{\oplus r} \right)$
- $(t_1,t_2,D)\cdot(A,B,i,j) = (t_1 A, t_2 B, i D^{-1}, t_1 t_2 D j)$

The main player from the geometric side is the following vector space:

Def. $\mathbb{L}^{(r)}_n := H^*_T(\mathcal{M}(r,n))$ module over $H^*_T(pt) \simeq \mathbb{C}[\varepsilon_1,\varepsilon_2,a_1,\dots,a_r] =: R_r$

$\mathbb{L}^{(r)} := \bigoplus_{n\geq 0} \mathbb{L}^{(r)}_n$; $\mathbb{L}^{(r)}_K := \mathbb{L}^{(r)} \otimes_{R_r} K_r$ $\left(K_r := \mathrm{Frac}(R_r) = \text{field of fractions} \right)$

The main player from the algebraic side is:

<u>Recall</u>: $\mathcal{W}(sl(r))$ = $\mathbb{Z}$-graded vertex algebra generated by $\widetilde{W}_i(z) = \sum_{\ell \in \mathbb{Z}} \widetilde{W}_{i,\ell}\, z^{-\ell-i}$ for $i=2,\dots,r$

$$\mathcal{W}(gl(r)) := \mathcal{W}(sl(r)) \otimes \underbrace{\text{Heisenberg algebra}}_{\langle b_\ell, c \,:\, \ell \in \mathbb{Z}\setminus\{0\} \rangle / [b_\ell, c]=0,\ [b_\ell, b_{-k}] = \ell\delta_{\ell,k}\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)c}$$

<u>Examples</u>:

- $r=1$: $\mathcal{W}(gl(1))$ = Heisenberg algebra
- $r=2$: $\mathcal{W}(gl(2)) :=$ Virasoro algebra $\otimes$ Heisenberg algebra

<u>Thm</u> (Schiffmann-Vasserot, Maulik-Okounkov)

$\exists$ an action of $\mathcal{W}(gl(r))$ on $\mathbb{L}_K^{(r)}$ s.t. $\mathbb{L}_K^{(r)} \simeq$ Verma module with highest weight vector $[M(r,0)]$ (= fundamental class of $M(r,0)$), i.e., the module with basis

$$\left\{ \widetilde{W}_{i_1,-\ell_1} \cdots \widetilde{W}_{i_s,-\ell_s} [M(r,0)] \,:\, s \geq 0,\ \ell_i \geq 1 \right\}$$

<u>Rmk</u>

$r=1$: one recovers Nakajima and Grojnowski's result:

$\exists$ an action of the Heisenberg algebra on $\mathbb{L}_K^{(1)}$ s.t. $\mathbb{L}_K^{(1)} \simeq$ Fock space

Question: How did SV, MO prove the Theorem?

Attention ⚠: They are unable to construct the action directly, rather they use the action of another algebra:

Consider

- $F^c := \mathbb{C}(-\varepsilon_2/\varepsilon_1)[c_0, c_1, c_2, \dots]$ — infinitely many variables
- $\exists\ F^c \longrightarrow K_r,\ c_k \longmapsto p_k\left(\frac{a_1}{\varepsilon_1}, \dots, \frac{a_r}{\varepsilon_1}\right)$ - kth power sum
- $\forall\ M$ = module over F^c, $M_{K_r} := M \otimes_{F^c} K_r$

Thm (SV, MO)

1. $\exists$ a faithful representation of $Y(\hat{\mathfrak{gl}}(1))_{K_r}$ (= affine Yangian of $\hat{\mathfrak{gl}}(1)$) on $\mathbb{L}_K^{(r)}$ s.t. $\mathbb{L}_K^{(r)}$ is generated by $[M(r,0)]$.

2. $\exists$ an embedding $Y(\hat{\mathfrak{gl}}(1))_{K_r} \hookrightarrow U(\mathcal{W}(\mathfrak{gl}(r)))$ (universal enveloping algebra)

   as subalgebras of $\mathrm{End}(\mathbb{L}_K^{(r)})$ s.t. $\exists$ an equivalence of categories:

$$\left\{\text{admissible } U(\mathcal{W}(\mathfrak{gl}(r)))\text{-modules}\right\} \xrightarrow{\sim} \left\{\text{admissible } Y(\hat{\mathfrak{gl}}(1))_{K_r}\text{-modules}\right\}$$

Rmk

- $Y(\widehat{\mathfrak{gl}}(1)) = $ deformation of $U($central extension of $\widehat{\mathfrak{gl}}(1) \otimes \mathbb{C}[z])$

  $\widehat{\mathfrak{gl}}(1) \otimes \mathbb{C}[z] = \mathbb{C}[t^{\pm 1}, z]$

  deformation $= \exists$ a $\mathbb{N}$-filtration $F^\bullet$ s.t. $gr(F^\bullet) \simeq U(\text{——})$

- The two representations are different:

depend on r

$$\overbrace{Y(\widehat{\mathfrak{gl}}(1))}^{\text{always the same}}{}_{K_r} \xrightarrow[\text{faithful}]{} \mathbb{L}^{(r)}_K \xleftarrow[\text{Verma module}]{} \mathcal{W}(\mathfrak{gl}(r))$$

$Y(\widehat{\mathfrak{gl}}(1))_{K_r}[M(r,0)] \simeq \mathbb{L}^{(r)}_K$ is obtained geometrically via:

$$\begin{cases} \text{MO: theory of stable envelopes} + R\text{-matrix realization} \\ \text{SV: definition via explicit operators} \\ \text{SV: COHA of 1-loop quiver} \end{cases}$$

Attention ⚠ : This is a 2d COHA and <u>not</u> 1d/3d Kontsevich-Soibelman COHA

Def.

- $$\mathrm{COHA}_{1\text{-loop}} = \bigoplus_{k\geq 0} H_*^{(\mathbb{C}^*)^2\times GL_k(\mathbb{C})}(C_k) \quad \text{as a vector space}$$

where

$$\underbrace{(\mathbb{C}^*)^2}_{\text{scaling}}\times \underbrace{GL_k(\mathbb{C})}_{\text{conjugation}} \curvearrowright C_k = \{(A,B)\in \mathrm{Mat}(k,\mathbb{C}) : [A,B]=0\} \text{ - commuting variety}$$

- multiplication : $m = \bigoplus_{k_1,k_2} m_{k_1,k_2}$, where

$$m_{k_1,k_2} : H_*^{(\mathbb{C}^*)^2\times GL_{k_1}(\mathbb{C})}(C_{k_1}) \times H_*^{(\mathbb{C}^*)^2\times GL_{k_2}(\mathbb{C})}(C_{k_2}) \longrightarrow H_*^{(\mathbb{C}^*)^2\times GL_{k_1+k_2}(\mathbb{C})}(C_{k_1+k_2})$$

$m_{k_1,k_2} := q_* \circ \overset{\text{refined Gysin pullback}}{p^*}$ induced by the Hall convolution diagram:

$$\begin{tikzcd}
 & C_{k_1+k_2} \cap \text{parabolic of } \mathfrak{gl}(k_1+k_2,\mathbb{C}) \text{ corresponding to fixing } \mathbb{C}^{k_1}\subset\mathbb{C}^{k_1+k_2} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
C_{k_2}\times C_{k_1} & & C_{k_1+k_2}
\end{tikzcd}$$

The key result is:

Thm (SV)

1. $(\mathrm{COHA}_{1\text{-loop}})_{K_r} \simeq Y^+(\hat{\mathfrak{gl}}(1))_{K_r} =: Y^+(\mathfrak{g}_{1\text{-loop}})_{K_r}$

2. $(\mathrm{COHA}_{1\text{-loop}})_{K_r} \curvearrowright \mathbb{L}^{(r)}_K$

Question: Where is the 1-loop quiver $\circlearrowleft\bullet$ in the definition of $\mathrm{COHA}_{1\text{-loop}}$?

Answer:

- $$\mathrm{COHA}_{1\text{-loop}} = \bigoplus_{k\geq 0} H_*^{(\mathbb{C}^*)^2 \times GL_k(\mathbb{C})}(C_k)$$

$$\simeq \bigoplus_{k\geq 0} H_*^{(\mathbb{C}^*)^2}\left(\left[C_k / GL_k(\mathbb{C})\right]\right) \simeq \bigoplus_{k\geq 0} H_*^{(\mathbb{C}^*)^2}\left(\mathrm{Rep}(\Pi_{1\text{-loop}}; k)\right)$$

stack quotient

stack of repr.s of $\Pi_{1\text{-loop}}$ of dimension k

$\Downarrow$

$A \circlearrowleft \mathbb{C}^k \circlearrowright B \longmapsto \Pi_{1\text{-loop}}$ - module (= repr.)

$+$ $\qquad\qquad\qquad\qquad\qquad\qquad \Vert$

$[A,B]=0$ $\qquad$ preprojective algebra of $\circlearrowleft\bullet$ $= \mathbb{C}\langle x,y\rangle / [x,y]$

Set $\underline{\mathrm{Rep}}(\Pi_{1\text{-Loop}}) = \bigsqcup_{k\geq 0} \underline{\mathrm{Rep}}(\Pi_{1\text{-Loop}}, k)$

<u>Hall convolution diagram</u> becomes: stack of extensions of finite-dim. reprs

$$\begin{tikzcd}
& \underline{\mathrm{Rep}}^{ext}(\Pi_{1\text{-loop}}) \arrow[dl, bend right, "p"'] \arrow[dr, bend left, "q"] & \\
\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}) \times \underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}) \ni (E_2, E_1) & 0 \to E_1 \to E \to E_2 \to 0 \arrow[l, mapsto] \arrow[r, mapsto] & E \in \underline{\mathrm{Rep}}(\Pi_{1\text{-loop}})
\end{tikzcd}$$

$$m := q_* \circ p^* : H_*^{(\mathbb{C}^*)^2}(\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}))^{\times 2} \longrightarrow H_*^{(\mathbb{C}^*)^2}(\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}))$$

This way of interpreting $\mathrm{COHA}_{1\text{-loop}}$ shows that the same construction can be applied to other cases:

2. Plethora of COHAs

Q = quiver = $(I = \{\text{vertices}\}, \Omega = \{\text{arrows}\})$

Schiffmann-Vasserot, Yang-Zhao: 2d (equivariant) COHA of Q:

$$\mathrm{COHA}_Q^{(A)} := H_*^{(A)}(\underline{\mathrm{Rep}}(\Pi_Q)) + \text{Hall multiplication}$$

certain torus $\subset \mathbb{C}^* \times (\mathbb{C}^*)^{\Omega}$

Characterization of the "quantum nature":

Thm (SV): $COHA_Q^A \hookrightarrow Y^+(\mathfrak{g}_Q^{MO})$ = Yangian of the Maulik-Okounkov Lie algebra of Q

Conjecture: —"— $\simeq$ ——"—— (True for Q = 1-loop, ADE, affine ADE)

X = smooth projective curve/$\mathbb{C}$

We define 3 COHAs associated to X:

- Schiffmann, Minets for $rk=0$: Dolbeault COHA of X = $COHA_{Dol}(X)$ = COHA of Higgs sheaves ($\mathcal{E}$ coherent sheaf, $\phi: \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega_X^1$) on X

Porta–: de Rham COHA of X = $COHA_{dR}(X)$ = COHA of flat bundles ($\mathcal{F}$ vector bundle, $\nabla: \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega_X^1$ conn. $\nabla^2 = 0$) on X

Porta–, Davison: Betti COHA of X = $COHA_B(X)$ = COHA of $\pi_1(X) \longrightarrow GL_n(\mathbb{C})$ for some n

Rmk

$\exists$ COHA versions of

- Riemann-Hilbert correspondence: $\mathrm{COHA}_{dR}(X) \simeq \mathrm{COHA}_B(X)$
- non-abelian Hodge correspondence: $\exists\ F^\bullet \subset \mathrm{COHA}_{dR}(X)$ s.t. $gr_{F_\bullet} \simeq \mathrm{COHA}_{Dol}(X;0)$

S smooth quasi-projective surface$/\mathbb{C}$

Schiffmann-Vasserot: COHA of zero-dimensional sheaves on $\mathbb{C}^2 = \mathrm{COHA}_{1\text{-loop}}$

Kapranov-Vasserot, Yu Zhao for $rk=0$: COHA of properly supported coherent sheaves on S

Rmk

1. $H_*^{BM}(-) \rightsquigarrow K_0(-)$ = Grothendieck group of coherent sheaves

COHAs $\rightsquigarrow$ K-theoretical Hall algebras

Yangians $\rightsquigarrow$ quantum loop algebras

2. Porta- : $\exists$ categorified version: monoidal structure on $D^b(\mathrm{Coh}(\underline{\qquad}))$

Question: What is the "quantum nature" of the COHAs of curves or the COHA of a surface?

3. COHA of ADE sing.

based on joint project with Diaconescu, Porta, Schiffmann, and Vasserot

Consider

- $G \subset SL(2,\mathbb{C})$ finite group $\longleftrightarrow$ ADE quiver Q_{fin}
- $\pi : Y \longrightarrow X := \mathbb{C}^2/G$ resolution of singularities
- $C := \pi^{-1}(0) = C_1 \cup \cdots \cup C_e$, $C_i \simeq \mathbb{P}^1$, $(C_i \cdot C_j) = -$ Cartan matrix of Q_{fin}
- torus $A \subset GL(2,\mathbb{C})$ centralizing G $(A = \{1\}, \mathbb{C}^*, \text{or } \mathbb{C}^* \times \mathbb{C}^*)$

Set

$\underline{\mathrm{Coh}}_C(Y) :=$ moduli stack of coherent sheaves on Y set-theoretically supported on C

$$\mathrm{COHA}^A_{Y,C} := H^A_*(\underline{\mathrm{Coh}}_C(Y))$$

Thm (DP–SV)

1. $\mathrm{COHA}^A_{Y,C}$ depends only on the formal neighborhood $\hat{Y}_C$ of Y along C.

2.

- $\exists$ a surjective map, induced by the Hall multiplication:

$$\phi : H^A_*\left(\underline{\mathrm{Coh}}_{C_1}(Y) \times \cdots \times \underline{\mathrm{Coh}}_{C_r}(Y)\right) \longrightarrow \mathrm{COHA}^A_{Y,C}$$

- $\mathrm{Ker}(\phi)$ only depends on the classes supported at the intersections

$$\underline{\mathrm{Coh}}_{C_i \cap C_{i+1}}(Y)$$

Rmk: (2) provides a PBW-type Theorem.

Attention ⚠: We also provide a description in terms of the Yangian of the affine ADE quiver

Notation:

- Q = affine ADE quiver $= (I = \{\text{vertices}\}, \Omega = \{\text{arrows}\})$
- $\mathfrak{g}_Q$ = (derived) Kac-Moody algebra of Q
- $L\mathfrak{g}_Q$ = (central extension of) $\mathfrak{g}_Q[z] \oplus K$, $K := \bigoplus_{\substack{n\neq 0 \\ \ell > 0}} \mathbb{C} c_{n,\ell}$
- $\underline{d} \in \mathbb{Z}I \rightsquigarrow \underline{d} = \underline{d}_{fin} + n\delta \in \mathbb{Z}I_{fin} + \mathbb{Z}\delta$, δ = imaginary root

For any $\ell \in \mathbb{Z}$:

$$L\mathfrak{g}_\ell := \bigoplus_{\underline{d}} L\mathfrak{g}_{\underline{d}},$$

where $d \in \mathbb{Z}I$ such that $\begin{cases} \underline{d}_{fin} \in \Delta^-_{fin}, \\ \ell(\check{\rho}_0, \underline{d}_{fin}) \leq n < (\ell+1)(\check{\rho}_0, \underline{d}_{fin}). \end{cases}$

sum of the fundamental coweights of Q_{fin}

Thm (DP-SV)

1. $\mathrm{COHA}_{Y,C} \simeq \text{"}\lim_{\ell \leq 0}\text{"}\, U(Lm)/(L\mathfrak{g}_{<\ell} \cdot U(Lm))$ as algebras

where

$$Lm := \bigoplus_{\underline{d}_{fin} \in \Delta^-_{fin} \cup \{0\}} L\mathfrak{g}_{\underline{d}}, \quad L\mathfrak{g}_{<\ell} := \bigoplus_{k<\ell} L\mathfrak{g}_k$$

2. $\mathrm{COHA}^{A}_{Y,C} \simeq \hat{Y}_Q := \text{"}\lim_{\ell \leq 0}\text{"}\ \hat{Y}_{Q,\ell}$ as algebras

where

$\hat{Y}_{Q,\ell}$ is a certain quotient of the Yangian $Y^+(\mathfrak{g}_Q)$

Attention ⚠ :

1. This is the first example of an explicit characterization of a COHA of sheaves of dimension ≥ 0

2. $\mathrm{COHA}^{A}_{Y,C} \simeq$ new positive part of $\hat{Y}(\mathfrak{g}_Q)$ completion