# Cohomological Hall algebras of 1-dimensional sheaves and Yangians

(joint work with D.E. Diaconescu, M. Porta, O. Schiffmann, and É. Vasserot, arXiv:2502.19445, and work in progress with P. Shimpi and O. Schiffmann)

## Plan:

1. Overview of 2d Cohomological Hall algebras

2. Nilpotent COHA of a surface and affine Yangians

# 1. Overview of 2d Cohomological Hall algebras

## ▶ Quivers

$Q$ = quiver $= (I = \{\text{vertices}\}, \Omega = \{\text{edges}\})$

$\rightsquigarrow Q^{db}$ = double quiver $= (I, \underbrace{\Omega \sqcup \Omega^{op}}_{} =: \Omega^{db})$

$\Omega^{op} = \{e^* : j \longrightarrow i \mid e : i \longrightarrow j \in \Omega\}$

$\rightsquigarrow \mathbb{C}Q^{db}$ = path algebra of $Q^{db}$

$\rightsquigarrow \Pi_Q$ = preprojective algebra of $Q = \mathbb{C}Q^{db} / \underbrace{\sum_{e \in \Omega} [e, e^*]}_{\text{(preprojective rels)}}$

Denote:

$$\boxed{\underline{\mathrm{Rep}}(\Pi_Q) = \text{moduli stack of finite-dimensional representations of } \Pi_Q}$$

Rmk: $\underline{\mathrm{Rep}}(\Pi_Q) \simeq T^* \underline{\mathrm{Rep}}(Q)$

moduli stack of f.d. reprs of $Q$

▸ $\exists$ torus action: $(\mathbb{C}^*)^{\Omega} \times \mathbb{C}^* \curvearrowright \underline{\mathrm{Rep}}(\Pi_Q)$

<u>Schiffmann-Vasserot</u>, <u>Yang-Zhao</u>: fix $T \subseteq (\mathbb{C}^*)^{\Omega} \times \mathbb{C}^*$ subtorus

$\boxed{\exists\ \mathrm{COHA}_Q^T = T\text{-equivariant COHA associated to finite-dim. reprs of } \Pi_Q}$

= unital associative algebra structure on the

T-equiv. Borel-Moore homology of $H_*^T(\underline{\mathrm{Rep}}(\Pi_Q))$

with multiplication $p_* \circ q^!$ induced by:

$$\underline{\mathrm{Rep}}(\Pi_Q) \times \underline{\mathrm{Rep}}(\Pi_Q) \xleftarrow{q} \underbrace{\underline{\mathrm{Rep}}^{ext}(\Pi_Q)}_{=\text{stack of extensions}} \xrightarrow{p} \underline{\mathrm{Rep}}(\Pi_Q)$$

where:

$p: 0 \to E_2 \to E \to E_1 \to 0 \longmapsto E$

$q: \quad \text{---------} \,/\!/\, \text{---------} \quad \longmapsto (E_2, E_1)$

(proper)

Also, Schiffmann-Vasserot:

$\exists$ $COHA_Q^{(T),nil}$ = (T-equivariant) COHA associated to the moduli stack $\Lambda_Q$ of strongly semi-nilpotent reprs of $\Pi_Q$

Attention ⚠: if $Q$ is without 1-loops and cycles:
strongly semi-nilpotent = nilpotent as repr. of $Q^{db}$

Set

$$\boxed{HA_Q^T := COHA_Q^{T,nil}}$$

A central question in the theory of COHAs is to elucidate their "quantum nature".

In the quiver case, the relevant quantum group is the Maulik-Okounkov Yangian $\mathbb{Y}_{Q,T}^{MO}$ of the quiver $Q$:

$$\boxed{\mathbb{Y}_{Q,T}^{MO} = \text{filtered deformation of } U(\mathfrak{g}_{Q,T}^{MO}[t])}$$

where

$$\begin{cases} \mathfrak{g}^{MO}_{Q,T} \simeq \mathfrak{g}^{MO}_{Q} \otimes H^*_T(pt) & \text{(Davison-Botta)} \\ \mathfrak{g}^{MO}_{Q} = \mathbb{Z}I \times \mathbb{Z}\text{-graded Lie algebra} \end{cases}$$

## Remark

- McBreen: $Q$ = finite ADE

$$\Longrightarrow \begin{cases} \mathfrak{g}^{MO}_{ADE} = \text{u.c.e. of simple Lie algebra of type ADE} \\ \text{Drinfeld's Yangian} \subset \mathbb{Y}^{MO}_{ADE,\mathbb{C}^*} \end{cases}$$

- Schiffmann-Vasserot (SV), Maulik-Okounkov, Neguț:

$$\mathfrak{g}^{MO}_{1\text{-loop}} = W_{1+\infty} = \text{u.c.e. of the Lie algebra of regular diff. operators on } \mathbb{C}^*$$

- S. Jindal in type A, DPSSV:

$Q$ = affine ADE, $Q_{fin}$ = finite ADE, $(\mathbb{C}^*)^{\times 2} \subset T_{max}$

$$\Longrightarrow \begin{cases} \mathfrak{g}_Q^{MO}[t] = \text{Universal central extension of } \mathfrak{g}_{Q_{fin}}[s^{\pm 1}, t] =: \mathfrak{g}_{ell} \\ \mathbb{Y}^{MO}_{Q,(\mathbb{C}^*)^{\times 2}} \text{ has a presentation by generators and relations} \end{cases}$$

Now, let's recall the main result relating COHAs of quivers and Yangians:

**Theorem** (Negut for 1-loop quiver, Botta-Davison, SV)
Let $Q$ be an arbitrary quiver and $T = T_{max} = (\mathbb{C}^*)^{\Omega} \times \mathbb{C}^*$.
$\exists$ an isomorphism of $H_T^*(pt)$-algebras:

$$\Psi : HA_Q^T \xrightarrow{\sim} \mathbb{Y}^{MO,-}_{Q,T}$$

where $\mathbb{Y}^{MO,-}_{Q,T}$ = negative part of $\mathbb{Y}^{MO}_{Q,T}$

w.r.t. triangular dec.

## ▸ Surfaces

$S$ = smooth quasi-projective surface $/\mathbb{C}$
$T$ = (possibly trivial) torus $\curvearrowright S$

$$\boxed{\underline{\mathrm{Coh}}_{ps}(S) = \text{moduli stack of properly supported coherent sheaves on } S}$$

## Remark

We can also define:

- $\underline{\mathrm{Coh}}_0(S) \subset \underline{\mathrm{Coh}}_{ps}(S)$ corresponding to 0-dim. sheaves
- $\underline{\mathrm{Coh}}_{\leq 1}(S) \subset \underline{\mathrm{Coh}}_{ps}(S)$ corresponding to sheaves of dim $\leq 1$

## Attention ⚠

$\exists$ a derived enhancement

$$\mathbb{R}\underline{\mathrm{Coh}}_{ps}(S) \times \mathbb{R}\underline{\mathrm{Coh}}_{ps}(S) \xleftarrow{\mathbb{R}q} \mathbb{R}\underline{\mathrm{Coh}}^{ext}_{ps}(S) \xrightarrow{\mathbb{R}p} \mathbb{R}\underline{\mathrm{Coh}}_{ps}(S)$$

such that $\mathbb{R}q$ is quasi-smooth $\Longrightarrow \exists\, (\mathbb{R}q)^!$

Kapranov-Vasserot, Yu Zhao (in dim=0), DPSSV:

$$\boxed{\exists\ \mathrm{COHA}^{(T)}_S = (T\text{-equivariant})\ \mathrm{COHA} \text{ associated to properly supported sheaves on } S}$$

= unital associative algebra structure on

$$H_*^{(T)}\left(\underline{\mathbb{R}\mathrm{Coh}}_{ps}(S)\right) = H_*^{(T)}\left(\underline{\mathrm{Coh}}_{ps}(S)\right)$$

with multiplication $(\mathbb{R}p)_* \circ (\mathbb{R}q)^!$.

## Remark

- $\exists$ $\mathrm{COHA}^{(T)}_{S,0\text{-dim}}$ associated to $\underline{\mathrm{Coh}}_0(S)$
- $\exists$ $\mathrm{COHA}^{(T)}_{S,\leq 1}$ associated to $\underline{\mathrm{Coh}}_{\leq 1}(S)$

## Example

$S = \mathbb{C}^2$ : $\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}) \xrightarrow{\sim} \underline{\mathrm{Coh}}_0(\mathbb{C}^2)$

$$\begin{array}{c} \rotatebox{90}{$\in$} \\ A_1 \circlearrowright \mathbb{C}^d \circlearrowleft A_2 \longmapsto \mathbb{C}^d = \mathbb{C}[A_1, A_2]\text{-module} \end{array}$$

$$\Longrightarrow \mathrm{COHA}^{(T)}_{\mathbb{C}^2, 0\text{-dim}} \simeq \mathrm{COHA}^{(T)}_{1\text{-loop}}$$

In dim = 0, we have a complete characterization:

**Theorem** (Neguţ, Mellit-Minets-Schiffmann-Vasserot)
$COHA^{(T)}_{S,0\text{-dim}}$ can be described explicitly by generators and relations.
In particular, if $\omega_S \simeq \mathcal{O}_S$:

$$COHA_{S,0\text{-dim}} \simeq U^+(W_{1+\infty}(S))$$

where $W_{1+\infty}(S)$ is a Lie algebra associated with $H^*(S)$.

The questions I would like to address today are:

**Question 1:** is $COHA^T_{S,\leq 1}$ related to Yangians?

**Question 2:** can we describe $COHA^T_{S,\leq 1}$ by generators and relations?

## 2. COHA of a surface and affine Yangians

We saw that Yangians are related to COHAs of nilpotent representations.
First, we introduce a "nilpotent" version of $COHA_S$.

- $S$ = smooth quasi-projective surface$/\mathbb{C}$
- $C \subset S$ reduced closed subscheme

Consider

$$\boxed{\underline{\mathrm{Coh}}_C(S) = \text{moduli stack of coherent sheaves on } S \text{ set-theoretically supported on } C}$$

sheaf analog of nilpotency

Example: $X$ = smooth projective curve$/\mathbb{C}$

$\underline{\mathrm{Coh}}_X(T^*X) \simeq$ moduli stack of Higgs sheaves $(\mathcal{F}, \phi : \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega^1_X)$ on $X$ such that $\phi$ is nilpotent

zero section

Theorem 1 (DPSSV)

1. $\exists$ an associative algebra structure on $H^{BM}_*(\underline{\mathrm{Coh}}_C(S))$

$$\Longrightarrow \mathrm{COHA}_{S,C} =: \mathrm{HA}_{S,C}$$

If $T$ = torus $\curvearrowright S$, $C$ $T$-invariant $\Longrightarrow \exists\ \mathrm{COHA}^T_{S,C} =: \mathrm{HA}^T_{S,C}$

2. $HA^{(T)}_{S,C}$ depends "locally" on $C$, i.e., given $(S_1, C_1)$ and $(S_2, C_2)$ s.t. the formal completions $\widehat{(S_1)}_{C_1} \simeq \widehat{(S_2)}_{C_2}$, we have:

$$HA^{(T)}_{S_1,C_1} \simeq HA^{(T)}_{S_2,C_2}$$

The first relation between $HA^{T}_{S,C}$ and Yangians is when

$$\boxed{S = \text{minimal resolution of ADE singularity}}$$

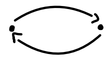
- $G \subset SL(2,\mathbb{C})$ finite group
  $\updownarrow$
  ADE quiver $Q_{fin}$ $\subset$ affine ADE quiver $Q$

- $\pi : S \longrightarrow \mathbb{C}^2/G$ Kleinian resolution of singularities

$C_{red} := \pi^{-1}(0)_{red} = C_1 \cup \ldots \cup C_e$ ; $C_i \simeq \mathbb{P}^1$; $(C_i \cdot C_j) = -$ Cartan matrix of $Q_{fin}$

- torus $T \subset GL(2,\mathbb{C})$ centralizing $G$ ($T$ = trivial or $\mathbb{C}^*$ or $\mathbb{C}^* \times \mathbb{C}^*$)

## Example

$G = \mathbb{Z}_2 \implies Q_{fin} = \bullet = A_1$ , $Q =$  $= A_1^{(1)}$

$\implies \mathbb{C}^* \times \mathbb{C}^* \curvearrowright S = T^*\mathbb{P}^1 \supset C = \mathbb{P}^1$ = zero section

Recall the derived McKay correspondence:

$$\tau : D^b(\mathrm{Coh}(S)) \xrightarrow{\sim} D^b(\mathrm{Mod}(\Pi_Q))$$

$$\Downarrow$$

$$\begin{cases} K_0(\mathrm{Coh}_C(S)) \xrightarrow{\sim} K_0(\mathrm{nilp}(\Pi_Q)) \simeq \text{root lattice } Y \text{ of } Q \\ \mathrm{Pic}(S) \xrightarrow{\sim} \text{coweight lattice } \check{X}_{fin} \text{ of } Q_{fin} \end{cases}$$

Recall

$\mathfrak{g}_{ell}$ := universal central extension of $\mathfrak{g}_{Q_{fin}}[s^{\pm 1}, t]$

$$\simeq \mathfrak{g}_{Q_{fin}}[s^{\pm 1}, t] \oplus K \qquad \Big( K = \bigoplus_{\ell \in \mathbb{N}} \mathbb{Q} c_\ell \oplus \bigoplus_{\substack{k \in \mathbb{Z}_{\neq 0} \\ \ell \in \mathbb{N}_{\geq 1}}} \mathbb{Q} c_{k,\ell} \Big)$$

central elements

is graded w.r.t. $Y \times \mathbb{Z}\delta_t$.

Define

$\mathfrak{g}_{ell}^{+}$ := Lie subalgebra of $\mathfrak{g}_{ell}$ associated to the monoid $K_0(\mathrm{Coh}_C(S))^{+} \subset Y$ of effective classes

$$= n^{-}_{Q_{fin}}[s^{\pm 1}, t] \oplus s^{-1} h_{Q_{fin}}[s^{-1}, t] \oplus \bigoplus_{k<0} \mathbb{Q}\, c_{k,\ell}$$

## Theorem 2 (DPSSV)

- $\exists$ a canonical algebra isomorphism $HA_{S,C} \simeq \hat{U}(\mathfrak{g}_{ell}^{+})$ completion
- $\exists$ a canonical algebra isomorphism $HA^{T}_{S,C} \simeq \mathbb{Y}^{+}_{S,C}$

where $\mathbb{Y}^{+}_{S,C}$ is a filtered deformation of $\hat{U}(\mathfrak{g}^{+}_{ell})$

## Remark

- we defined a set of generators given by fundamental classes of stacks of 0-dim. sheaves on $C$ and of line bundles on $C_i$'s.
- we described these generators in terms of Yangian generators.

Question: how do we prove this theorem?

Note that the derived McKay correspondence:

$$\tau : D^b(\mathrm{Coh}_c(S)) \xrightarrow{\sim} D^b(\mathrm{nilp}(\Pi_Q))$$

$\tau$ is not t-exact w.r.t. the standard t-structures

$\Longrightarrow$ $\underline{\mathrm{Coh}}_c(S) \cancel{\simeq} \Lambda_Q$ = stack of nilpotent repr.s of $\Pi_Q$

$\Longrightarrow$ $HA^T_{S,c} \cancel{\simeq} HA^T_Q$

The relation between $\mathrm{Coh}_c(S)$ and $\mathrm{nilp}(\Pi_Q)$ is more subtle:

$$\boxed{\mathrm{Coh}_c(S) = \text{"limit" of } \mathrm{nilp}(\Pi_Q)}$$

More precisely,

- hearts of bounded t-structures on $\mathcal{C} = D^b(\mathrm{nilp}(\Pi_Q))$ form a partial ordered set:

$$\boxed{H_1 := \mathcal{C}_1^{\geq 0} \cap \mathcal{C}_1^{\leq 0} \leq H_2 := \mathcal{C}_2^{\geq 0} \cap \mathcal{C}_2^{\leq 0} \iff \mathcal{C}_1^{\geq 0} \subseteq \mathcal{C}_2^{\geq 0}}$$

- Shimpi: $\check{\theta} \in \check{X}_f$ strictly dominant $\rightsquigarrow \mathcal{L}_{\check{\theta}} \in \mathrm{Pic}(S)$. Define

$$L_{\check{\theta}} := \tau \circ (\mathcal{L}_{\check{\theta}}^{-1} \otimes -) \circ \tau^{-1} : D^b(\mathrm{nilp}(\Pi_Q)) \longrightarrow D^b(\mathrm{nilp}(\Pi_Q))$$

Then

$$\boxed{\inf_{n \geq 0} L_{\check{\theta}}^n(\mathrm{nilp}(\Pi_Q)) = \mathrm{Coh}_c(S)}$$

Question: How do we make sense of this from the viewpoint of COHAs?

We shall use:

- braid group actions on bounded derived cat.s
- Bridgeland stability conditions

Lemma

Let $B_{ex}$ = the extended affine braid group. Then $\exists$ a group homomorphism

$$\rho : B_{ex} \longrightarrow \mathrm{Aut}(D^b(\mathrm{Coh}_c(S)))$$

such that $\rho(L_{\check\lambda}) = (\mathcal{L}_{\check\lambda} \otimes -)$ $\forall\, \check\lambda \in \check X_{fin}$ (w.r.t. the descri

$B_{ex} \simeq \left(B_{fin} \cup \{L_{\check\lambda} : \check\lambda \in \check X_{fin}\}\right) / _{rels}$.

Now, fix a strictly dominant coweight $\check\theta \in \check X_f \hookrightarrow \check X$.
$\check\theta$ defines a King's stability condition on $\mathrm{nilp}(\Pi_Q)$.

$\Longrightarrow$ $\exists$ associated Bridgeland's stability condition

$$(Z_{\check\theta}, \mathcal{P}_{\check\theta}) \quad \text{on } D^b(\mathrm{nilp}(\Pi_Q))$$

Lemma

1. $\forall\, k \in \mathbb{Z}$, $L^{-2k}_{\check\theta} : \mathrm{nilp}(\Pi_Q) \xrightarrow{\sim} \mathcal{P}_{\check\theta}\big((\nu_{-k}, \nu_{-k}+1]\big)$

with $\nu_\ell := \frac{1}{\pi} \arctan(2h\ell)$ Coxeter number

2. $\mathcal{P}_{\check\theta}\big((-\frac{1}{2}, \frac{1}{2}]\big) \simeq \mathrm{Coh}_C(S)$

For $\ell, k \in \mathbb{N}$, $k \geq \ell$, set

$\Lambda_Q^{\ell,k}$ := (derived) moduli stack of objects in $\mathcal{P}_{\check\theta}\big((\nu_{-\ell}, \nu_{-k}+1]\big)$

Attention: $L^{2k}_{\check{\theta}} : \Lambda^{\ell,k}_{Q} \xrightarrow{\sim} \Lambda^{\ell-k,0}_{Q} \simeq \Lambda^{>\nu_{-\ell+k}}_{Q} = $ HN stratum

Lemma

1. The vector space

$$\boxed{HA^{(T)}_{\check{\theta}} := \lim_{\ell} \operatorname{colim}_{k \geq \ell} H^{(T)}_{*}\left(\Lambda^{\ell,k}_{Q}\right)}$$

has the structure of an unital associative algebra with multiplication induced from that of $HA^{(T)}_{Q}$.

2. $\exists$ an algebra isomorphism $HA^{(T)}_{S,C} \simeq HA^{(T)}_{\check{\theta}}$

Corollary

$HA^{(T)}_{\check{\theta}}$ does not depend on the specific choice of strictly dominant finite coweight.

Finally, a careful analysis of the compatibility between the action of $B_{ex}$ on Yangians and on $HA^{T}_{Q}$ yields:

Lemma

$\exists$ an algebra isomorphism $HA^{T}_{\check{\theta}} \xrightarrow{\sim} \mathbb{Y}^{+}_{S,C}$, where

$$\boxed{\mathbb{Y}^{+}_{S,C} := \lim_{\ell} T^{2\ell}_{\check{\theta}}\left(\mathbb{Y}^{-}_{Q}\right) \Big/ T^{2\ell}_{\check{\theta}}(J)}$$

where $J := \sum_{\mu_{\check{\theta}}(\underline{d})>0} \mathbb{Y}^{-}_{Q} \cdot \mathbb{Y}^{-}_{Q,-\underline{d}}$.

The proof of Thm 2 follows from the above lemmas. $\square$

<u>Conjecture</u> $\mathbb{Y}^{+}_{S,C}$ is a new half of $\widehat{\mathbb{Y}}^{MO}_{Q}$.

<u>Question:</u> what does it happen if we drop "strictly" from $\check{\theta}$?

Fix $J \subseteq I_f$ and let $\check{\theta} \in \check{X}_f$ be any dominant coweight s.t.

$$\check{\theta}(\alpha_i) = 0 \quad \forall\, i \in I_f \setminus J$$

simple root

<u>Rmk</u>: Shimpi: $\inf_{n \geq 0} L^{n}_{\check{\theta}}(\mathrm{nilp}(\Pi_Q)) = P_C(S/S_J)$

where

- contraction of $C_i,\ i \in I_f \setminus J$ :

$$\begin{tikzcd} S \arrow[rr, "\pi_J"] \arrow[dr, "\pi"'] & & S_J \arrow[dl] \\ & \mathbb{C}^2/G & \end{tikzcd}$$

- $\mathcal{P}_C(S/S_J) \subset D^b(\mathrm{Coh}_C(S))$ = Van der Bergh's category of perverse coherent sheaves set-th. supp. on $C$

$\Longrightarrow$ COHA$^{(T)}$ of $\mathcal{P}_C(S/S_J) := \mathbb{HA}^{(T)}_{\check{\Theta}}$

Theorem (S-Schiffmann-Shimpi)

$\mathbb{HA}^{T}_{\check{\Theta}} \simeq$ filtered deformation of $\widehat{U}(\mathfrak{g}^J_{ell})$