Cohomological Hall algebras and affine Yangians

Categorified Enumerative Geometry and Representation Theory

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Plan:

1. Overview of 2d Cohomological Hall algebras

2. COHA of a surface and affine Yangians

1. Overview of 2d Cohomological Hall algebras

▶ Quivers

Q = quiver $= (I = \{\text{vertices}\}, \Omega = \{\text{edges } e\})$

$\rightsquigarrow Q^{db}$ = double quiver $= (I, \underbrace{\Omega \sqcup \Omega^{op}}_{})$

$\Omega^{op} = \{e^* : j \longrightarrow i \mid e : i \longrightarrow j \in \Omega\}$

$\rightsquigarrow \mathbb{C}Q^{db}$ = path algebra of Q^{db}

$\rightsquigarrow \Pi_Q$ = preprojective algebra of $Q = \mathbb{C}Q^{db} / \sum_{e \in \Omega} [e, e^*]$

preprojective rels

Denote:

$$\boxed{\underline{\mathrm{Rep}}(\Pi_Q) = \text{moduli stack of finite-dimensional representations of } \Pi_Q}$$

Schiffmann-Vasserot, Yang-Zhao:

$$\boxed{\exists\ \mathrm{COHA}_Q^{(T)} = (T\text{-equivariant})\ \mathrm{COHA}\ \text{associated to finite-dim. reprs of } \Pi_Q}$$

= unital associative algebra structure on

$$H_*^{(T)}(\underline{\mathrm{Rep}}(\Pi_Q))$$

with multiplication $p_* \circ q^*$ induced by:

$$\underline{\mathrm{Rep}}(\Pi_Q) \times \underline{\mathrm{Rep}}(\Pi_Q) \xleftarrow{q} \underline{\mathrm{Rep}}^{ext}(\Pi_Q) \xrightarrow{p} \underline{\mathrm{Rep}}(\Pi_Q)$$

$\underline{\mathrm{Rep}}^{ext}(\Pi_Q)$ = stack of extensions

where:

$$p: \ 0 \to E_2 \to E \to E_1 \to 0 \ \longmapsto E$$

$$q: \ \qquad \qquad \text{''} \qquad \qquad \longmapsto (E_2, E_1)$$

Finally, the torus action is given as:

- $(\mathbb{C}^*)^\Omega \times \mathbb{C}^* \curvearrowright \underline{\mathrm{Rep}}(\Pi_Q ; \underline{d})$ — $\in$ dimension vector $\in \mathbb{N}I$

$$(t_e, t) \cdot (x_e, x_e^* := x_{e^*})_{e\in\Omega} = (t_e x_e, t_e^{-1} t x_{e^*})_{e\in\Omega}$$

- $T \subseteq (\mathbb{C}^*)^\Omega \times \mathbb{C}^*$ subtorus

Also, Schiffmann-Vasserot:

$\exists$ $\mathrm{COHA}_Q^{(T),nil}$ = (T-equivariant) COHA associated to strongly semi-nilpotent reprs of Π_Q

= nilpotent (i.e., x_e and x_{e^*} both nil.) if Q is without ede-loops

Set

$$\boxed{A_Q^T := \mathrm{COHA}_Q^{T,nil}}$$

Now, let's recall two results relating COHAs of quivers with Yangians:

Theorem (Schiffmann-Vasserot)

Let Q be a quiver.

$\exists$ an injective morphism of $H_T^*(pt)$-algebras:

$$\boxed{\Psi : \mathbb{A}_Q^T \hookrightarrow (\mathbb{Y}_Q^{MO})^+}$$

= pos. part of Maulik-Okounkov Yangian of Q (given via R-matrix)

Ψ is iso for Q = 1-loop quiver or finite ADE quivers.

$\exists$ also "another" Yangian, originally due to Drinfeld.

Given

- Q = either the 1-loop quiver or a quiver without edge-loops
- $T_{max} = (\mathbb{C}^*)^{\Omega} \times \mathbb{C}^*$

Schiffmann-Vasserot: $\exists$ Yangian $\mathbb{Y}_{1\text{-loop}}$ = unital associative algebra over $H_{T_{max}}^*(pt)$

DPSSV: Q without edge-loops $\exists$ Yangian $\mathbb{Y}_Q$ = ——//——

Attention: $\mathbb{Y}_{1\text{-loop}}$ and $\mathbb{Y}_Q$ given by gen.s and rels

Remark

- when restricted to $\mathbb{C}^* \subset T_{max}$, and
- Q = finite or affine ADE quiver

$\Big\} \Longrightarrow$ $\mathbb{Y}_Q$ = Drinfeld's Yangian

Theorem (DPSSV)

Let Q be either the 1-loop quiver or a quiver without edge-loops.
$\exists$ a surjective morphism of $H^*_{T_{max}}(pt)$-algebras

$$\boxed{\Phi : \mathbb{Y}_Q^+ \twoheadrightarrow \mathbb{A}_Q^{T_{max}}}$$

Φ is an iso for

- Q = 1-loop quiver (proved previously by Schiffmann-Vasserot)
- Q = finite ADE quiver (proved previously by Yang-Zhao)
- Q = affine ADE quiver

Summarizing: $\mathbb{Y}_{1\text{-loop}} \simeq \mathbb{Y}^{MO}_{1\text{-loop}}$; $\mathbb{Y}_{ADE} \simeq \mathbb{Y}^{MO}_{ADE}$; $\mathbb{Y}_{\substack{\text{affine}\\ \text{ADE}}} \subseteq \mathbb{Y}^{MO}_{\substack{\text{affine}\\ \text{ADE}}}$

▶ Curves

X = smooth projective curve$/\mathbb{C}$

S.-Schiffmann, Minets (for $rk=0$):

$\exists\ \mathrm{COHA}_X^{\mathrm{Dol}}$ = COHA associated to Higgs sheaves $(\mathcal{E}, \mathcal{E} \xrightarrow{\phi} \mathcal{E} \otimes \Omega_X^1)$ on X

$\exists\ \mathrm{COHA}_X^{\mathrm{Dol,nil}}$ = COHA associated to nilpotent Higgs sheaves $(\mathcal{E}, \mathcal{E} \xrightarrow{\phi} \mathcal{E} \otimes \Omega_X^1)$ on X

Remark

- Porta-S.: $\exists$ COHA associated to flat bundles on X
- Porta-S., Davison: $\exists$ COHA associated to finite-dim. repr.s of $\pi_1(X)$
- Porta-S.: $\exists$ COHA versions of Riemann-Hilbert and non-abelian Hodge correspondences

▸ Surfaces

S = smooth quasi-projective surface $/\mathbb{C}$.

Kapranov-Vasserot, Yu Zhao (in dim=0):

$\exists$ COHA_S = COHA associated to properly supported sheaves on S

In dim = 0, we have a complete characterization:

Theorem (Mellit-Minets-Schiffmann-Vasserot)

$\mathrm{COHA}_{S,0\text{-dim}} \simeq$ pos. part of $W_{1+\infty}$-algebra modelled on $H_*^{BM}(S)$

(i.e., explicit description by gen.s and rel.s)

Remark

$\exists$ K-theoretical Hall algebras of quivers, curves, surfaces

$\exists$ categorified ________ // ________

2. COHA of a surface and affine Yangians

We saw that Yangians are related to COHAs of nilpotent representations.
First, we introduce a "nilpotent" version of COHA_S.

Nilpotent COHA_S

- S = smooth quasi-projective surface$/\mathbb{C}$
- $C \subset S$ reduced closed subscheme

Consider

$\mathrm{Coh}(S, C)$ = moduli stack of coherent sheaves on S set-theoretically supported on C

sheaf analog of nilpotency

Example: X = smooth projective curve$/\mathbb{C}$

$\mathrm{Coh}(T^*X, X) \simeq$ moduli stack of nilpotent Higgs sheaves on X

seen as zero section

Theorem 1 (DPSSV)

1. $\exists$ an associative algebra structure on $H_*^{BM}(\underline{\mathrm{Coh}}(S,C))$
$\Longrightarrow \mathrm{COHA}_{S,C}$

If T = torus $\curvearrowright S$, C T-invariant $\Longrightarrow \exists\ \mathrm{COHA}_{S,C}^T$

2. Given (S_1, C_1) and (S_2, C_2) s.t. $\widehat{(S_1)}_{C_1} \simeq \widehat{(S_2)}_{C_2}$, we have

$$\mathrm{COHA}_{S_1, C_1} \simeq \mathrm{COHA}_{S_2, C_2}$$

formal completions

i.e., $\mathrm{COHA}_{S,C}$ depends only on $\widehat{(S)}_C$

Moreover, the same holds equivariantly.

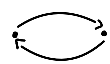
Relations to affine Yangians

- $G \subset SL(2,\mathbb{C})$ finite group
$\updownarrow$
ADE quiver $Q_{fin} \subset$ affine ADE quiver Q

- $\pi : Y \longrightarrow X := \mathbb{C}^2/G$ Kleinian resolution of singularities

$C := \pi^{-1}(0) = C_1 \cup \dots \cup C_e$; $C_i \simeq \mathbb{P}^1$; $(C_i \cdot C_j) = -$ Cartan matrix of Q_{fin}

- torus $T \subset GL(2,\mathbb{C})$ centralizing G (T = trivial or $\mathbb{C}^*$ or $\mathbb{C}^* \times \mathbb{C}^*$)

Example: $G = \mathbb{Z}_2 \Longrightarrow Q_{fin} = \bullet = A_1$, $Q =$  $= A_1^{(1)}$

$\Longrightarrow \mathbb{C}^* \times \mathbb{C}^* \curvearrowright Y = T^*\mathbb{P}^1 \supset C = \mathbb{P}^1$ = zero section

Recall the (derived) McKay correspondence:

$$\tau : D^b(\mathrm{Coh}(Y)) \xrightarrow{\sim} D^b(\mathrm{Mod}(\Pi_Q))$$

Question:

Is there a relation between

$$\boxed{COHA^T_{Y,C} \quad \text{and} \quad A^T_Q \simeq \mathbb{Y}^+_Q}$$

induced by τ ?

Equivalently, can we describe $COHA^T_{Y,C}$ via gen.s and rel.s "arising" from $\mathbb{Y}^+_Q$?

Attention ⚠ :

1. τ is not t-exact w.r.t. the standard t-structures

$$\Longrightarrow \underline{\mathrm{Coh}}(Y, C) \cancel{\simeq} \Lambda_Q = \text{stack of nilpotent repr.s of } \Pi_Q$$

$$\Longrightarrow \mathrm{COHA}^T_{Y,C} \cancel{\simeq} A^T_Q \simeq Y^+_Q \quad !$$

2. if we were dealing with classical Hall algebras, then:

<u>Cramer's theorem:</u> $\mathcal{A}, \mathcal{B}$ abelian categories:

$$D^b(\mathcal{A}) \simeq D^b(\mathcal{B}) \Longrightarrow D(\mathcal{H}_{\mathcal{A}}) \simeq D(\mathcal{H}_{\mathcal{B}})$$

Drinfeld double

<u>Example:</u> $D^b(\mathrm{Coh}(\mathbb{P}^1)) \simeq D^b(\mathrm{Mod}(\cdot \rightrightarrows \cdot))$

- $\mathcal{H}_{\mathbb{P}^1} \simeq U_q(\widehat{\mathfrak{sl}(2)})^{D+}$ in the new Drinfeld's presentation
- $\mathcal{H}_{\cdot \rightrightarrows \cdot} \simeq U_q(\widehat{\mathfrak{sl}(2)})^{DJ+}$ in the Drinfeld-Jimbo's presentation

$$\Longrightarrow D(\mathcal{H}_{\mathbb{P}^1}) \simeq U_q(\widehat{\mathfrak{sl}(2)}) \simeq D(\mathcal{H}_{\cdot \rightrightarrows \cdot})$$

but $\nexists$ Cramer theorem for COHAs!

Expectation $\mathrm{COHA}^T_{Y,C}$ realizes a new positive half of a completion of $\mathbb{Y}_Q$!

Here, the completion arises from the fact that the stack $\underline{\mathrm{Coh}}(Y,C)$ is NOT quasi-compact

Consider

- $\check{\theta}$ = dominant coweight of Q_{fin}
  (e.g. $\check{\theta} = \check{\rho}_0 = \sum$ fundamental coweights of Q_{fin})
- $\mu_{\check{\theta}}(\underline{d}) = \dfrac{(\check{\theta}, \underline{d})}{(\check{\rho}, \underline{d})}$ (Here, $\check{\rho} = \sum$ fundamental coweights of Q)
- $\begin{cases} t_{-2\check{\theta}} \in W^{ex}_Q = \text{extended affine Weyl group} \\ T_{-2\check{\theta}} \in B^{ex}_Q = \text{———}\,\text{''}\,\text{——— braid group} \end{cases}$

Theorem 2 (DPSSV)

$$\boxed{\mathrm{COHA}^T_{Y,C} \simeq \widehat{\mathbb{Y}_Q} := \bigoplus_{\underline{d}} \widehat{\mathbb{Y}_{Q,\underline{d}}}}$$

where

$$\widehat{\mathcal{Y}}_{Q,\underline{d}} := \lim_{\ell \leq 0} \left(T^{-\ell}_{-2\check{\theta}} \left(\mathcal{Y}^{+}_{t^{\ell}_{-2\check{\theta}}(\underline{d})} \right) \Big/ T^{-\ell}_{-2\check{\theta}} \left(J_{\check{\theta}, t^{\ell}_{-2\check{\theta}}(\underline{d})} \right) \right)$$

$$J_{\check{\theta},\underline{d}} := \sum_{\mu_{\check{\theta}}(\underline{d})>0} \mathcal{Y}^{+}_{Q}\, \mathcal{Y}^{+}_{\underline{d}} \subset \mathcal{Y}^{+}_{\underline{d}}$$

Theorem 3 (DPSSV) - PBW type Theorem

Let S be a smooth quasi-projective surface$/\mathbb{C}$.
Consider a type A_2 chain $C = C_1 \cup C_2 \subset S$ with $C_i \simeq \mathbb{P}^1$ for $i=1,2$.
Then the Hall multiplication induces the map

$$\operatorname*{colim}_{a_1} \bigoplus_{n_1 \geq a_1} \left(\mathrm{COHA}^{(T)}_{S,C_1}(n_1) \otimes \mathrm{COHA}^{(T)}_{S,C_2}(n_2) \right) \xrightarrow{\varphi} \mathrm{COHA}^{(T)}_{S,C}(n_1+n_2)$$

grading by the Euler characteristic

which is weakly surjective, i.e., $\mathrm{res} \circ \varphi$ is surjective, where

$$\mathrm{res}: H_*^{(T)}(\underline{\mathrm{Coh}}(S,C)) \longrightarrow H_*^{(T)}(U)$$

and $U \subset \underline{\mathrm{Coh}}(S,C)$ is any quasi-compact open substack.

Moreover, a similar statement holds for

- $C =$ type ADE chain of $\mathbb{P}^1$'s
- $C =$ affine type DE chain of $\mathbb{P}^1$'s

Remark: $\exists$ also a description of $\mathrm{Ker}(\varphi)$.

Long-term goals

We aim at proving:

- Thanks to Thm 2,

$$\mathrm{COHA}^T_{Y,C} \simeq \langle \mathrm{gen.s} \rangle / \mathrm{rel.s}$$

▸ Let $S \longrightarrow \mathbb{P}^1$ be an elliptic fibration.
Then the natural algebra map

$$\mathrm{COHA}_{S,\text{smooth fiber}} \longrightarrow \mathrm{COHA}_{S,\text{sing. fiber}}$$

is injective.

affine type chain of $\mathbb{P}^1$'s

▸ Thanks to Thm 3,

$$\boxed{\begin{aligned} \mathrm{COHA}_{S,\text{affine type chain of } \mathbb{P}^1\text{'s}} &\simeq \langle \text{gen.s} \rangle / \text{rel.s} \\ &\supset \mathrm{COHA}_{S,\text{elliptic curve}} \supseteq U(L\hat{\mathfrak{g}}) \end{aligned}}$$

Idea of the proof of Thm 2

Step 1: $\mathrm{COHA}^T_{Y,C}$ via Harder-Narasimhan strata

The dominant coweight $\check{\theta}$ corresponds to an ample line bundle $\omega_{\check{\theta}} \in \mathrm{Pic}(Y)$:

$$\boxed{\underline{\mathrm{Coh}}(Y,C) \simeq \operatorname{colim}_{\ell \leq 0} \underline{\mathrm{Coh}}^{>\ell}_{\omega_{\check{\theta}}}(Y,C) \text{ via open embeddings}}$$

where

$\underline{\mathrm{Coh}}^{>\ell}_{\omega_{\check{\theta}}}(Y,C) \subset \underline{\mathrm{Coh}}(Y,C)$ open substack consisting of those sheaves $\mathcal{F}$ on Y s.t. $\mu_{\omega\text{-min}}(\mathcal{F}) > \ell$

$$\Longrightarrow \boxed{\mathrm{COHA}^T_{Y,C} \simeq \lim_{\ell \leq 0} H^T_*\left(\underline{\mathrm{Coh}}^{>\ell}_{\omega_{\check{\theta}}}(Y,C)\right)}$$

Step 2: "interpolation" sheaves vs. repr.s

Consider the central charge

$$Z_{\check{\theta}} : K_0(\mathrm{mod}(\Pi_Q)) \simeq \mathbb{Z}I \longrightarrow \mathbb{C}$$
$$\underline{d} \longmapsto -(\check{\theta}, \underline{d}) + i(\check{\rho}, \underline{d})$$

$\Longrightarrow$ it defines a Bridgeland's stability condition $(Z_{\check{\theta}}, \mathcal{P}_{\check{\theta}})$ on $D^b(\mathrm{mod}(\Pi_Q))$ with

slicing

$$\mathcal{P}_{\check{\theta}}((0,1]) \simeq \mathrm{mod}(\Pi_Q) \quad \text{and} \quad \mathcal{P}_{\check{\theta}}((-\tfrac{1}{2}, \tfrac{1}{2})) \simeq \mathrm{Coh}_{ps}(Y)$$

induced by τ

The McKay correspondence τ yields equivalences:

$$\begin{tikzcd}
\underline{\mathrm{Coh}}^{>0}_{\omega_{\check{\theta}}}(Y,C) \arrow[r, hook] \arrow[d, "\simeq"] & \underline{\mathrm{Coh}}^{>-2}_{\omega_{\check{\theta}}}(Y,C) \arrow[r, hook] \arrow[d, "\simeq"] & \underline{\mathrm{Coh}}^{>-4}_{\omega_{\check{\theta}}}(Y,C) \arrow[r, hook] \arrow[d, "\simeq"] & {} \\
\mathcal{M}^{nil}_{\mathcal{P}_{\check{\theta}}((0,\frac{1}{2}))} \arrow[r, hook] & \mathcal{M}^{nil}_{\mathcal{P}_{\check{\theta}}((\nu_1,\frac{1}{2}))} \arrow[r, hook] & \mathcal{M}^{nil}_{\mathcal{P}_{\check{\theta}}((\nu_2,\frac{1}{2}))} \arrow[r, hook] & {}
\end{tikzcd}$$

moduli stack of nilpotent objects $\in \mathcal{P}_{\check{\theta}}((-,\tfrac{1}{2}])$

Here, $\nu_\ell := \frac{1}{\pi}\arctan(2h\ell)$ for $\ell \le 0$

$(\check{\rho},\delta)=h$, δ = minimal imaginary root

$$\Longrightarrow \boxed{\mathrm{COHA}^{T}_{Y,C} \simeq \lim_{\ell\le 0} H^{T}_{*}\left(\mathcal{M}^{nil}_{\mathcal{P}_{\check{\theta}}((\nu_\ell,\frac{1}{2}))}\right)}$$

Step 2: Action of the extended affine braid group

Iyama-Reiten: $\exists$ group homomorphism

$$\boxed{\rho : B^{ex}_{Q} \longrightarrow \mathrm{Aut}(D^b(\mathrm{Mod}(\Pi_Q)))}$$

Attention

Under the McKay corr. $\tau : D^b(\mathrm{Coh}(Y)) \simeq D^b(\mathrm{Mod}(\Pi_Q))$

$$\text{action of } T_{-2\check{\theta}} \longleftrightarrow -\otimes \mathcal{O}_Y(-2\omega_{\check{\theta}})$$

$\mathrm{mod}(\Pi_Q)^{\leq 0} \subset \mathrm{mod}(\Pi_Q)$ consisting of those finite-dim. reprs M s.t. $\mu_{\check{\theta}\text{-max}}(M) \leq 0$

Lemma A: $T_{-2\check{\theta}}^{-\ell} : \mathrm{mod}(\Pi_Q)^{\leq 0} \xrightarrow{\sim} \mathcal{P}_{\check{\theta}}((\nu_\ell, \tfrac{1}{2}]))$ for $\ell \leq 0$

Lemma B: $H_*^T(\Lambda_{\underline{d}}^{\leq 0}) \simeq H_*^T(\Lambda_{\underline{d}})/J_{\check{\theta},\underline{d}} \simeq \mathbb{Y}_{\underline{d}}^+/J_{\check{\theta},\underline{d}}$, where

$$J_{\check{\theta},\underline{d}} := \sum_{\mu_{\check{\theta}}(\underline{d})>0} \mathbb{Y}_Q^+ \mathbb{Y}_{\underline{d}}^+$$

Attention: $\exists$ also an action of B_Q^{ex} on $\mathbb{Y}_Q$

Lemma C: $\exists$ a "compatibility" between the braid group action on Yangians and the braid group action on COHAs induced by IR

$$\Longrightarrow \boxed{\begin{aligned} \mathrm{COHA}_{Y,C}^T &\simeq \lim_{\ell \leq 0} H_*^T\left(\mathcal{M}^{nil}_{\mathcal{P}_{\check{\theta}}((\nu_\ell, \frac{1}{2}))}\right) \\ &\simeq \bigoplus_{\underline{d}} \lim_{\ell \leq 0} \left(T_{-2\check{\theta}}^{-\ell}\left(\mathbb{Y}^+_{t^\ell_{-2\check{\theta}}(\underline{d})}\right) \Big/ T_{-2\check{\theta}}^{-\ell}\left(J_{\check{\theta}, t^\ell_{-2\check{\theta}}(\underline{d})}\right)\right) \end{aligned}}$$

$\square$