From 4d Gauge Theories to Cohomological Hall Algebras, and back (again)

Plan

- Alday-Gaiotto-Tachikawa conjectures and Cohomological Hall algebras
- Cohomological Hall algebras of ADE singularities

Gauge theories on $\mathbb{R}^4$ and Alday-Gaiotto-Tachikawa conjecture

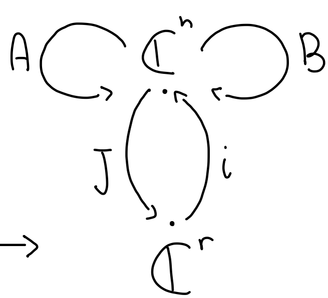
Fix $r, n \in \mathbb{N}$, $r \geq 1$.

We introduce the instanton moduli space:

$\mathcal{M}(r,n) :=$ moduli space of torsion-free sheaves $\mathcal{E}$ on $\mathbb{P}^2_{\mathbb{C}} = \mathbb{C}^2 \cup \ell_\infty$ with $\mathrm{rk}(\mathcal{E}) = r$, $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = n$, framed at ℓ_∞, i.e., $\mathcal{E}|_{\ell_\infty} \simeq \mathcal{O}_{\ell_\infty}^{\oplus r}$

$\simeq$ Nakajima quiver variety associated to the 1-loop quiver G and dimensions r, n

$$:= \left\{ (A,B,i,j) : \begin{array}{l} 1)\ [A,B] + ij = 0 \\ 2)\ \nexists\ 0 \subsetneq S \subsetneq \mathbb{C}^n \text{ s.t.} \\ \quad A(S) \subseteq S,\ B(S) \subseteq S,\ \mathrm{Im}(i) \subseteq S \end{array} \right\} \Big/ GL_n(\mathbb{C})$$



via conjugation:
$(gAg^{-1}, gBg^{-1}, gi, jg^{-1})$

Facts:

- $\mathcal{M}(r,n)$ is a smooth (quasi-proj.) variety$/\mathbb{C}$ of dim. $2rn$
- Donaldson: $\mathcal{M}(r,n) \underset{\text{open}}{\supset} \mathcal{M}^{ASD}(r,n)$,

$\mathcal{M}^{ASD}(r,n) :=$ moduli space of $SU(r)$-instantons P on $S^4 = \mathbb{R}^4 \cup \{\infty\}$ with instanton charge n, framed at ∞, i.e., together with fixed $\phi \in P_\infty$.

To introduce Nekrasov's partition functions, one needs to "turn on" the Ω-deformation.

Mathematically, we consider the action of the torus

$$T = \underbrace{(\mathbb{C}^*)^2}_{\text{torus of } \mathbb{P}^2} \times \underbrace{(\mathbb{C}^*)^r}_{\hookrightarrow GL_r(\mathbb{C})} \curvearrowright \mathcal{M}(r,n):$$

$(\mathbb{C}^*)^2 \ni (t_1,t_2)$, $(\mathbb{C}^*)^r \ni D$ = diagonal matrix

$$\boxed{(t_1,t_2,D)\cdot(A,B,i,j) = (t_1 A, t_2 B, iD^{-1}, t_1 t_2 D j)}$$

Definition

(Instanton part of) Nekrasov's partition function of the pure $\mathcal{N}=2$ SUSY $SU(r)$-gauge theory on $\mathbb{C}^2$ is

$$\boxed{Z^{\text{inst,pure}}_{\mathbb{C}^2} := \sum_{n\geq 0} q^n \left([M(r,n)]_T , [M(r,n)]_T \right)}$$

where

- $[M(r,n)]_T$:= fundamental class of $M(r,n)$
- $(-,-)$:= intersection pairing on

$$\mathbb{L}^{(r)}_{loc} := H^*_T(M(r,n)) \otimes_{H^*_T(pt)} \mathrm{Frac}(H^*_T(pt))$$

Remark: $Z^{\text{inst,pure}}_{\mathbb{C}^2}$ could be also informally interpreted as

$$Z^{\text{inst,pure}}_{\mathbb{C}^2} = \sum_{n\geq 0} q^n \int^{\text{equiv}}_{M(r,n)} 1$$

For an arbitrary gauge theory, the Nekrasov's partition function is

$$Z^{\text{inst,quiver}}_{\mathbb{C}^2} = \sum_{n\geq 0} q^n \int^{\text{equiv}}_{M(r,n)} \text{euler class of "bi-fundamental bundle"}$$

depending on the gauge theory

The Alday-Gaiotto-Tachikawa conjecture states an identification

Instanton part of Nekrasov's partition function of a quiver (e.g., $A_N, A_N^{(1)}$) gauge theory on $\mathbb{R}^4$

$$\|$$

conformal block of conformal Toda field theory on a Riemann surface (e.g., sphere, torus) with punctures

Now, we state a mathematical reformulation of the AGT conjecture.
First, note that the "chiral algebra" of the Toda CFT is:

$$\boxed{\mathcal{W}(\mathfrak{gl}(r)) := \mathcal{W}(\mathfrak{sl}(r)) \otimes \mathcal{H}eis}$$

- $\mathcal{H}eis$ = infinite-dimensional Heisenberg algebra

▸ $\mathcal{W}(\mathfrak{sl}(r))$ = $\mathbb{Z}$-graded vertex algebra generated by

$$\widetilde{W}_i(z) = \sum_{\ell\in\mathbb{Z}} \widetilde{W}_{i,\ell}\, z^{-\ell-i} \quad \text{for } i=2,\dots,r$$

Example: For $r=2$, $\mathcal{W}(\mathfrak{gl}(2)) = \text{Virasoro} \otimes \text{Heis}$

A mathematical formulation of the AGT conjecture for pure gauge theories (stated by Gaiotto) is:

Theorem (Schiffmann-Vasserot '12; Maulik-Okounkov; '12)

1. $\exists$ a representation of $\mathcal{W}(\mathfrak{gl}(r))$ of level $-\frac{\varepsilon_2}{\varepsilon_1}-r$ on

$$\mathbb{L}^{(r)}_{loc} := \bigoplus_{n\geq 0} H^*_T(\mathcal{M}(r,n))_{loc}$$

identifying it with the Verma module of highest weight $\frac{\varepsilon_1}{\varepsilon_2}\left((a_1,\dots,a_r)+\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)(0,-1,\dots,1-r)\right)$. (Here, $H^*_T(pt)=\mathbb{C}[\varepsilon_1,\varepsilon_2,a_1,\dots,a_r]$)

2. The Gaiotto state $G=\sum_{n\geq 0}[\mathcal{M}(r,n)]_T \in \prod_{n\geq 0} H^*_T(\mathcal{M}(r,n))_{loc}$ is a Whittaker vector, i.e.,

$$\widetilde{W}_{i,\ell}(G) = \begin{cases} \varepsilon_2^{1-r}\varepsilon_1^{-1}\, G & \text{if } i=r,\ \ell=1 \\ 0 & \text{otherwise} \end{cases}$$

Remark (Proofs of AGT conjectures for quiver gauge theories)

- Neguţ, '15: $\mathcal{N}=2^*$ $SU(2)$ gauge theory
- Mironov-Morozov-Shakirov, '11: A_1 $SU(2)$ gauge theory with 4 masses and $\varepsilon_1+\varepsilon_2=0$
- Ghosal-Remy-Sun-Sun, '20: A_1 $SU(2)$ gauge theory with 4 masses
- Yuan-Hu-Huang-Zhang, '24: A_n $SU(r)$ gauge theory with $2n+2$ masses and $\varepsilon_1+\varepsilon_2=0$

Let us return to SV and MO:

Attention ⚠: They did not construct directly the representation of $\mathcal{W}(\mathfrak{gl}(r))$, rather...

Theorem (Schiffmann-Vasserot '12; Maulik-Okounkov, '12)

1. $\exists$ a faithful representation of $\left(\mathbb{Y}^{MO}_{1\text{-loop}}\right)_{loc(r)}$ on $\mathbb{L}^{(r)}_{loc}$ such that it is generated by $[M(r,0)]_T$.

a certain localization

2. $\exists$ an embedding $\left(Y^{MO}_{1\text{-loop}}\right)_{loc(r)} \hookrightarrow U(\mathcal{W}(gl(r)))$

such that $\exists$ an equivalence of categories:

$$\{\text{admissible } U(\mathcal{W}(gl(r)))\text{-modules}\}$$
$$\cong$$
$$\{\text{admissible } \left(Y^{MO}_{1\text{-loop}}\right)_{loc(r)}\text{-modules}\}$$

(2) can be derived from the following more recent result:

<u>Theorem</u> (Gaberdiel-Gopakumar, G-G-Li-Peng, Linshaw)

$$Y^{MO}_{1\text{-loop}} \simeq U(\text{universal 2-parameter } \mathcal{W}_\infty\text{-algebra})$$

<u>Remark</u>: all $\mathcal{W}(gl(r))$ can be obtained as quotients of

Remark: equivalent geometric realizations of $\mathbb{Y}^{MO}_{1\text{-loop}}$

- Maulik-Okounkov: stable envelopes + R-matrix realization

already addressed in Ben Davison's talk

- Schiffmann-Vasserot: double of the (nilpotent) COHA of the preprojective algebra of the 1-loop quiver

- Schiffmann-Vasserot: as the subalgebra of $\mathrm{End}(\mathbb{L}^{(r)}_{loc})$ generated by Nakajima type operators associated to

$$\begin{tikzcd} & \mathrm{Hecke} = \{0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{O}_x \to 0\} \arrow[dl] \arrow[dr] & \\ M(r,n) \ni \mathcal{E} & & \mathcal{E} \in M(r,n+1) \end{tikzcd}$$

Gauge theories on ALE spaces and COHAs

Fix an affine ADE quiver $Q=(I=\{\text{vertices}\}, \Omega=\{\text{edges}\})$

- Q_{fin} = corresponding finite ADE quiver
- $\Gamma \subset SL(2,\mathbb{C})$ = the finite group associated to Q_{fin}
- $X \xrightarrow{\pi} \mathbb{C}^2/\Gamma$ is the minimal resolution of the ADE singularity
- $C=\pi^{-1}(0)=C_1\cup\cdots\cup C_e$, $C_i\simeq\mathbb{P}^1$

Note that:

Instanton moduli space of Gauge theories on $\mathbb{R}^4$	Instanton moduli space of Gauge theories on ALE
$\mathcal{M}(r,n)$ = Nakajima quiver variety of the 1-loop quiver	$\mathcal{M}_Q^\theta(\vec{v},\vec{w})$ = Nakajima quiver variety of Q

Attention ⚠ : $\mathcal{M}_Q^{\theta}(\vec{v},\vec{w})$ depends on a stability parameter θ

- $\exists\, \theta^{res}$ such that

$$\mathcal{M}_Q^{\theta^{res}}(\vec{v},\vec{w}) = \text{instanton moduli space of } U(r) \text{ gauge theories on the minimal resolution } X$$

$\sum w_i$

- $\exists\, \theta^{orb}$ such that

$$\mathcal{M}_Q^{\theta^{orb}}(\vec{v},\vec{w}) = \text{instanton moduli space of } U(r) \text{ gauge theories on the orbifold } [\mathbb{C}^2/\Gamma]$$

- $\mathcal{M}_Q^{\theta^{res}}(\vec{v},\vec{w}) \not\simeq \mathcal{M}_Q^{\theta^{orb}}(\vec{v},\vec{w})$

Note that:

Theorem (Maulik-Okounkov, Schiffmann-Vasserot)
$\exists$ a faithful representation of $\mathbb{Y}_Q := \mathbb{Y}(\mathfrak{g}_Q^{KM})$ on

$$\mathbb{L}^{(\vec{w})}_{\theta^{orb}} := \bigoplus_{\vec{v}} H_T^*\left(\mathcal{M}_Q^{\theta^{orb}}(\vec{v},\vec{w})\right)$$

Attention ⚠ : an equivalent result for $\mathcal{M}_Q^{\theta res}(\vec{v},\vec{w})$ is missing.

More pressing questions are:

- what is the correct COHA that should act on

$$\bigoplus_{\vec{v}} H_T^*\left(\mathcal{M}_Q^{\theta res}(\vec{v},\vec{w})\right) \; ?$$

- Does this action induces an action of $\mathbb{Y}_Q$?

Consider $T \subset GL(2,\mathbb{C})$ centralizing Γ (e.g., T = trivial, $\mathbb{C}^*$, or $\mathbb{C}^* \times \mathbb{C}^*$)

$\boxed{\text{COHA}^T_{X,C} = T\text{-equivariant COHA of sheaves on } X \text{ set-theoretically supported on } C}$

Remark: $\text{COHA}^T_{X,C}$ is the "sheaf analog" of the nilpotent COHA of a quiver

Attention ⚠ : $\text{COHA}^T_{X,C}$ depends only on the formal completion of X along C

Theorem 1 (Diaconescu–Porta–S.–Schiffmann–Vasserot)

$$\boxed{\mathrm{COHA}^{T}_{X,C} \simeq \widehat{\mathbb{Y}}^{-}_{Q}}$$

where

$$\boxed{\begin{aligned} \widehat{\mathbb{Y}}^{-}_{Q} &:= \lim_{\ell} \left(T_{2\ell\check{\theta}}\left(\mathbb{Y}^{-}_{Q}\right) \Big/ T_{2\ell\check{\theta}}\left(J_{\check{\theta}}\right)\right) \\ J_{\check{\theta}} &:= \sum_{\mu_{\check{\theta}}(\underline{d})>0} \mathbb{Y}^{-}_{Q}\, \mathbb{Y}^{-}_{-\underline{d}} \end{aligned}}$$

where

- $\check{\theta}$ = dominant coweight of Q_{fin}
  (e.g. $\check{\theta} = \check{\rho}_0 = \sum$ fundamental coweights of Q_{fin})
- $\mu_{\check{\theta}}(\underline{d}) = \dfrac{(\check{\theta}, \underline{d})}{(\check{\rho}, \underline{d})}$ (Here, $\check{\rho} = \sum$ fundamental coweights of Q)
- $T_{2\ell\check{\theta}} \in B^{ex}_{Q}$ = extended affine braid group

In particular,

a certain completion

$$\boxed{\mathrm{COHA}_{X,C} \simeq \widehat{U}(\mathfrak{n}_Q)}$$

where

$$\mathfrak{n}_Q := \mathfrak{n}^+_{fin}[s^{\pm 1}, t] \oplus s^{-1}\mathfrak{h}_{fin}[s^{-1}, t] \oplus \bigoplus_{\substack{k<0\\ \ell>0}} \mathbb{C}c_{k,\ell}$$

central

with $\mathfrak{g}_{Q_{fin}} = \mathfrak{n}^-_{fin} \oplus \mathfrak{h}_{fin} \oplus \mathfrak{n}^+_{fin}$.

Attention ⚠: $\mathbb{Y}^-_Q\big|_0 \simeq U\Big(s^{-1}\mathfrak{g}_{Q_{fin}}[s^{-1}, t] \oplus \mathfrak{n}^-_{fin}[t] \oplus \bigoplus_{\substack{k<0\\ \ell>0}} \mathbb{C}c_{k,\ell}\Big)$

Expectations:

- The Drinfeld double of $\mathrm{COHA}^T_{X,C}$ is a completion $\widehat{\mathbb{Y}}_Q$ of $\mathbb{Y}_Q$ with a new "Drinfeld type" coproduct

This is also motivated by Beck's result for $U_q(\mathfrak{g}^{KM}_Q)$:

$$\text{Drinfeld-Jimbo presentation} \xleftrightarrow{\quad} \text{new Drinfeld's presentation}$$

action of B^{ex}_Q

- As formulated by Belavin-Feigin, B-F-Bershtein-Litvinov-Tarnopolsky AGT type conjectures for ALE spaces (for $\Gamma = \mathbb{Z}_p$) predict that

$$\underbrace{\mathrm{Heis} \times \widehat{\mathfrak{sl}}(p)_r \times \left(\left(\widehat{\mathfrak{sl}}(r)_p \times \widehat{\mathfrak{sl}}(r)_{n-p}\right) \big/ \widehat{\mathfrak{sl}}(r)_n \right)}_{\parallel}$$

$$\bigoplus_{\vec v} H_T^*\left(\mathcal{M}_Q^{\theta^{orb}}(\vec v, \vec w)\right) \curvearrowleft V \curvearrowright \bigoplus_{\vec v} H_T^*\left(\mathcal{M}_Q^{\theta^{res}}(\vec v, \vec w)\right)$$

$\Longrightarrow$ completions of Y_Q endowed with different coproducts should yield different realizations of the vertex algebra V.

- $\widehat{Y}_Q$ should be realized as an algebra of Nakajima type operators associated to

line bundle on C_i

$$\mathrm{Hecke} = \left\{ 0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow j_{i*}\mathcal{L} \longrightarrow 0 \right\}$$

$j_i : C_i \subset X$

At the moment, Diaconescu-Porta-S-Yu Zhao are studying these Nakajima type operators with $\mathcal{E}', \mathcal{E} \in \mathfrak{M} \neq \mathcal{M}_Q^{\theta}(\vec v, \vec w)$

- $COHA^T_{\text{surface, curve}}$ is a "building block" for COHAs of surfaces, more precisely, one expects the following:
  - S smooth qproj. surface, endowed with a T-action, (possibly trivial)
  - D smooth curve $\subset S$ such that $\exists\, \phi \subset D^1 \subset \cdots \subset D^s = D$

The Hall multiplication induces a map

$$COHA^T_{S,D_s} \otimes COHA^T_{S,D_s^{s-1}} \otimes \cdots \otimes COHA^T_{S,D_2^1} \otimes COHA^T_{S,D_1} \xrightarrow{\Phi} COHA^T_{S,D}$$

where $D_i := \overline{D^i \setminus D^{i-1}}$ and $D_i^{i-1} := D^{i-1} \cap D_i$.

Conjecture

Φ is injective and (topologically) surjective.

Theorem 2 (Diaconescu-Porta-S.-Schiffmann-Vasserot)

Φ is topologically surjective for $S =$ elliptic surface and $D =$ special fiber of type DE.

Attention ⚠: In this case, the building blocks are

$$\boxed{COHA_{\text{elliptic s.}, D_i} \simeq COHA_{T^*\mathbb{P}^1, \mathbb{P}^1}}$$

Idea of the proof of Thm 1

Step 1: $COHA^T_{X,C}$ via Harder-Narasimhan strata

The dominant coweight $\check{\theta}$ of Q_{fin} corresponds to a line bundle $\omega_{\check{\theta}} \in \mathrm{Pic}(X)$:

$$\boxed{\underline{\mathrm{Coh}}(X,C) \simeq \operatorname*{colim}_{\ell} \underline{\mathrm{Coh}}^{>\ell}_{\omega_{\check{\theta}}}(X,C) \text{ via open embeddings}}$$

where

$$\underline{\mathrm{Coh}}^{>\ell}_{\omega_{\check{\theta}}}(X,C) \subset \underline{\mathrm{Coh}}(X,C) \text{ open substack consisting of}$$

those sheaves $\mathcal{F}$ on X s.t. $\mu_{\omega_{\check{\theta}}\text{-min}}(\mathcal{F}) > \ell$

$$\Longrightarrow \boxed{COHA^T_{X,C} \simeq \lim_{\ell} H^T_*\left(\underline{\mathrm{Coh}}^{>\ell}_{\omega_{\check{\theta}}}(X,C)\right)}$$

Step 2: "interpolation": sheaves vs. repr.s

Consider the central charge

$$Z_{\check{\theta}} : K_0(\mathrm{mod}(\Pi_Q)) \simeq \mathbb{Z}I \longrightarrow \mathbb{C}$$
$$\underline{d} \longmapsto -(\check{\theta}, \underline{d}) + i(\check{\rho}, \underline{d})$$

$\Longrightarrow$ it defines a Bridgeland's stability condition $(Z_{\check{\theta}}, \mathcal{P}_{\check{\theta}})$ on $D^b(\mathrm{mod}(\Pi_Q))$ with

slicing

$$\mathcal{P}_{\check{\theta}}((0,1]) \simeq \mathrm{mod}(\Pi_Q) \quad \text{and} \quad \mathcal{P}_{\check{\theta}}\left(\left(-\tfrac{1}{2}, \tfrac{1}{2}\right]\right) \simeq \mathrm{Coh}_{ps}(X)$$

induced by τ

The McKay correspondence τ yields equivalences:

$$\begin{tikzcd}
\underline{\mathrm{Coh}}^{>0}_{\omega_{\check{\theta}}}(X,C) \arrow[r, hook] \arrow[d, "\simeq"] & \underline{\mathrm{Coh}}^{>-2}_{\omega_{\check{\theta}}}(X,C) \arrow[r, hook] \arrow[d, "\simeq"] & \underline{\mathrm{Coh}}^{>-4}_{\omega_{\check{\theta}}}(X,C) \arrow[r, hook] \arrow[d, "\simeq"] & {} \\
\mathcal{M}^{nil}_{\mathcal{P}_{\check{\theta}}((0,\frac{1}{2}])} \arrow[r, hook] & \mathcal{M}^{nil}_{\mathcal{P}_{\check{\theta}}((\nu_1,\frac{1}{2}])} \arrow[r, hook] & \mathcal{M}^{nil}_{\mathcal{P}_{\check{\theta}}((\nu_2,\frac{1}{2}])} \arrow[r, hook] & {}
\end{tikzcd}$$

moduli stack of nilpotent objects $\in \mathcal{P}_{\check{\theta}}((-,\tfrac{1}{2}])$

Here, $\nu_\ell := \frac{1}{\pi}\arctan(-2h\ell)$ for $\ell \in \mathbb{Z}$

$(\check{\rho},\delta) = h$, δ = minimal imaginary root

$$\Longrightarrow \boxed{\mathrm{COHA}^T_{X,C} \simeq \lim_{\ell} H^T_*\left(\mathcal{M}^{nil}_{\mathcal{P}_{\check{\theta}}((\nu_\ell,\frac{1}{2}))}\right)}$$

Step 2 : Action of the extended affine braid group

Iyama-Reiten : $\exists$ group homomorphism

$$\boxed{\rho : B^{ex}_Q \longrightarrow \mathrm{Aut}(D^b(\mathrm{Mod}(\Pi_Q)))}$$

Attention

Under the McKay corr. $\tau : D^b(\mathrm{Coh}(Y)) \simeq D^b(\mathrm{Mod}(\Pi_Q))$

$$\text{action of } T_{2\ell\check{\theta}} \longleftrightarrow - \otimes \mathcal{O}_Y(2\ell\omega_{\check{\theta}})$$

$\mathrm{mod}(\Pi_Q)^{\leq 0} \subset \mathrm{mod}(\Pi_Q)$ consisting of those finite-dim. reprs M s.t. $\mu_{\check{\theta}\text{-max}}(M) \leq 0$

Lemma A: $T_{2\ell\check{\theta}} : \mathrm{mod}(\Pi_Q)^{\leq 0} \xrightarrow{\sim} \mathcal{P}_{\check{\theta}}((\nu_\ell, \tfrac{1}{2}]))$ for $\ell \in \mathbb{Z}$

Lemma B: $H_*^T(\Lambda^{\leq 0}) \simeq H_*^T(\Lambda)/J_{\check{\theta}} \simeq \mathbb{Y}_Q^- / J_{\check{\theta}}$

Attention: $\exists$ also an action of B_Q^{ex} on $\mathbb{Y}_Q$

Lemma C: $\exists$ a "compatibility" between the braid group action on Yangians and the braid group action on COHAs induced by IR

$$\Longrightarrow \boxed{\begin{aligned} \mathrm{COHA}^T_{Y,C} &\simeq \lim_{\ell} H_*^T\Big(\mathcal{M}^{nil}_{\mathcal{P}_{\check{\theta}}((\nu_\ell, \frac{1}{2}))}\Big) \\ &\simeq \lim_{\ell} \Big(T_{2\ell\check{\theta}}(\mathbb{Y}_Q^-) / T_{2\ell\check{\theta}}(J_{\check{\theta}})\Big) \end{aligned}}$$

$\square$