Cohomological Hall algebras, their representations, and Nakajima operators

1. Cohomological Hall algebras of surfaces

S = smooth projective surface$/\mathbb{C}$.

$\underline{\mathrm{Coh}}(S)$ = moduli stack of coherent sheaves on S

$\underline{\mathbb{R}\mathrm{Coh}}(S)$ = derived enhancement of $\underline{\mathrm{Coh}}(S)$

Construction of COHA of S:

Consider the "convolution diagram":

$$\underline{\mathbb{R}\mathrm{Coh}}(S) \times \underline{\mathbb{R}\mathrm{Coh}}(S) \xleftarrow{ev_1 \times ev_3} \underline{\mathbb{R}\mathrm{Coh}}^{ext}(S) \xrightarrow{ev_2} \underline{\mathbb{R}\mathrm{Coh}}(S)$$

$\underline{\mathbb{R}\mathrm{Coh}}^{ext}(S)$ = stack of extensions

where:

$$ev_2 : \quad 0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0 \longmapsto E_2$$

$$ev_1 \times ev_3 : \quad \text{———— '' ————} \longmapsto (E_1, E_3)$$

- ev_2 is proper representable
- $ev_1 \times ev_3$ is derived lci (i.e., the cotangent complex $\mathbb{L}_{ev_1 \times ev_3}$ is perfect of amplitude $[-1,1]$)

Theorem (Kapranov-Vasserot, Yu Zhao for 0-dim. sheaves)

$H_*^{BM}(\underline{Coh}(S))$ = Borel-Moore homology of $\underline{Coh}(S)$

has the structure of an associative algebra, whose product is given by:

$$\boxed{\begin{aligned} m: H_*^{BM}(\underline{Coh}(S)) \times H_*^{BM}(\underline{Coh}(S)) &\xrightarrow{\boxtimes} \\ H_*^{BM}(\underline{Coh}(S)\times\underline{Coh}(S)) &\xrightarrow{(ev_2)_* \circ (ev_1 \times ev_3)^!} H_*^{BM}(\underline{Coh}(S)) \end{aligned}}$$

$\Longrightarrow$ Cohomological Hall algebra of S

Remark

In the above theorem, by replacing $H_*^{BM}(\underline{Coh}(S))$ with

$G_0(\underline{Coh}(S))$ = Grothendieck group of coh. sheaves on $\underline{Coh}(S)$

$\Longrightarrow$ K-theoretical Hall algebra of S

Theorem (Porta-S.)

$D^b_{coh}(\mathbb{R}\underline{Coh}(S))$ has the structure of a ($\mathbb{E}_1$-) monoidal dg-category, whose tensor product is given by:

$$\boxed{D^b_{coh}(\mathbb{R}\underline{Coh}(S)) \times D^b_{coh}(\mathbb{R}\underline{Coh}(S)) \xrightarrow{m} D^b_{coh}(\mathbb{R}\underline{Coh}(S))}$$

where $m = \left((ev_2)_* \circ (ev_1 \times ev_3)^*\right) \circ \boxtimes$

$\Longrightarrow$ Categorified Hall algebra of S

Remark

- The Thms hold also for
  - S only quasi-proj.
  - $\underline{Coh}_{ps}(S)$ = moduli stack of properly supported sheaves on S
- $\exists$ an equivariant version of the Thms w.r.t.

$$T = \text{torus} \curvearrowright S \rightsquigarrow T \curvearrowright \underline{Coh}_{ps}(S)$$

- Note that $D^b_{coh}(\mathbb{R}\underline{Coh}(S)) \not\simeq D^b_{coh}(\underline{Coh}(S))$

Moreover, $\nexists$ CatHA over $D^b_{coh}(\underline{Coh}(S))$

$\Longrightarrow$ "categorification" requires Derived Algebraic Geometry

Notation: in the following, some results hold for

$$H^{BM}_*(-), G_0(-), D^b_{coh}(-) \rightsquigarrow H(-)$$

Similarly, $HA(-)$ denotes $COHA(-), KHA(-), CatHA(-)$

2. Explicit description of HA of S

A central goal in the theory of COHAs is to achieve an explicit characterization of these algebras through their

Generators and Relations

This has been achieved for "smaller" COHAs, i.e., COHAs associated with subcategories of $Coh_{ps}(S)$.

More precisely, if

- $\mathcal{T}$ is a Serre subcategory of $\mathrm{Coh}_{ps}(S)$, and
- the moduli stack $\underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{T}}(S)$ of objects in $\mathcal{T}$ is open in $\underline{\mathbb{R}\mathrm{Coh}}_{ps}(S)$.

Then $\exists$ $\mathrm{COHA}_{\mathcal{T}}(S)$, $\mathrm{KHA}_{\mathcal{T}}(S)$, $\mathrm{catHA}_{\mathcal{T}}(S)$

For example,

- $\mathcal{T} = \mathrm{Coh}_0(S) = \{$0-dimensional sheaves on $S\} \subset \mathrm{Coh}_{ps}(S)$
- $\underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{T}}(S) = \mathbb{R}\underline{\mathrm{Coh}}_0(S) =$ derived moduli stack of 0-dimensional sheaves on S

$\Longrightarrow$ $\mathrm{COHA}_0(S)$, $\mathrm{KHA}_0(S)$, $\mathrm{CatHA}_0(S)$

In this case, we have a complete understanding of the algebraic structure of these algebras.

Let $\mathrm{HA}_0^{T,\mathrm{sph}}(S)$ be the subalgebra generated by $H^T(\mathbb{R}\underline{\mathrm{Coh}}_0(S;1))$

length 1

<u>Theorem</u> (Schiffmann-Vasserot)

Let $S = \mathbb{C}^2 \hookleftarrow T = \mathbb{C}^* \times \mathbb{C}^*$. Then

▶ $\begin{cases} \mathrm{COHA}_0^{\mathbb{C}^*\times\mathbb{C}^*, sph}(\mathbb{C}^2)_{loc} \simeq Y^+_{\varepsilon_1,\varepsilon_2}(\hat{\mathfrak{gl}}(1)) \\ \mathrm{COHA}_0^{\mathbb{C}^*\times\mathbb{C}^*}(\mathbb{C}^2)_{loc} \simeq \mathrm{COHA}_0^{\mathbb{C}^*\times\mathbb{C}^*, sph}(\mathbb{C}^2)_{loc} \end{cases}$ = (pos. part of affine Yangian of $\mathfrak{gl}(1)$)

given by generators and relations

▶ $\begin{cases} \mathrm{KHA}_0^{\mathbb{C}^*\times\mathbb{C}^*, sph}(\mathbb{C}^2)_{loc} \simeq U^+_{q,t}(\hat{\mathfrak{gl}}(1)) \simeq \mathcal{E}^+ \\ \mathrm{KHA}_0^{\mathbb{C}^*\times\mathbb{C}^*}(\mathbb{C}^2)_{loc} \simeq \mathrm{KHA}_0^{\mathbb{C}^*\times\mathbb{C}^*, sph}(\mathbb{C}^2)_{loc} \end{cases}$ = (pos. part of elliptic Hall algebra)

Thm (Mellit-Minets-Schiffmann-Vasserot)
Let S be a smooth surface (with pure cohomology). Then

▶ $\mathrm{COHA}_0(S) \simeq \mathrm{COHA}_0^{sph}(S)$

▶ $\mathrm{COHA}_0^{sph}(S) \simeq W^+(S)$ (algebra given by gens and rels that "resembles" $Y^+(\hat{\mathfrak{gl}}(1))$)

Remark

▶ $\exists$ a T-equivariant version of the above theorem w.r.t. T = torus acting on S.

- Neguț provided a description of $KHA_0^{sph}(S)$ by gens and rels (they resemble those of the elliptic Hall algebra).

Attention ⚠ :
The proof proceeds by studying representations of $HA_0(S)$ and using Nakajima type operators to determine the relations.

3. Representations

Let me explain the main ideas to construct representations of $HA_{\mathcal{T}}(S)$.

From now on, assume that S is projective.
Fix a torsion pair $v=(\mathcal{T},\mathcal{F})$ of $Coh(S)$, i.e.,

- $Hom(T,F)=0 \quad \forall\, T\in\mathcal{T},\ F\in\mathcal{F}$;
- $\forall\, E\ \exists\ 0\to \underset{\in \mathcal{T}}{T}\to E\to \underset{\in \mathcal{F}}{F}\to 0$

such that the

- $\mathcal{T}$ is a Serre subcategory, and
- the moduli stacks $\mathbb{R}\underline{Coh}_{\mathcal{T}}(S)$ and $\mathbb{R}\underline{Coh}_{\mathcal{F}}(S)$ are open in $\mathbb{R}\underline{Coh}(S)$.

Fix a stack $\mathfrak{M} \subseteq \mathbb{R}\underline{\mathrm{Coh}}_F(S)$.

Goal: define left or right action of $HA_\tau(S)$ on $H(\mathfrak{M})$

Set $\mathfrak{X} := \mathbb{R}\underline{\mathrm{Coh}}_\tau(S)$. Consider the induced diagram:

$$\begin{tikzcd}
\mathfrak{M} \times \mathfrak{X} \arrow[d] & \mathbb{R}\underline{\mathrm{Coh}}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S) \arrow[l, "ev_1^{\mathfrak{M}} \times ev_3^{\mathfrak{X}}"'] \arrow[r, "ev_2^{\mathfrak{M}}"] \arrow[d] & \mathfrak{M} \arrow[d] \\
\mathbb{R}\underline{\mathrm{Coh}}(S) \times \mathbb{R}\underline{\mathrm{Coh}}(S) & \mathbb{R}\underline{\mathrm{Coh}}^{ext}(S) \arrow[l, "ev_1 \times ev_3"] \arrow[r, "ev_2"'] & \mathbb{R}\underline{\mathrm{Coh}}(S)
\end{tikzcd}$$

If (RM): $\exists \, (ev_2^{\mathfrak{M}})_* \circ (ev_1^{\mathfrak{M}} \times ev_3^{\mathfrak{X}})^! \circ \boxtimes$

$\Longrightarrow$ $H(\mathfrak{M})$ is a right $HA_\tau(S)$-module

Similarly, consider

$$\begin{tikzcd}
\mathfrak{X} \times \mathfrak{M} \arrow[d] & \mathbb{R}\underline{\mathrm{Coh}}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S) \arrow[l, "ev_3^{\mathfrak{X}} \times ev_2^{\mathfrak{M}}"'] \arrow[r, "ev_1^{\mathfrak{M}}"] \arrow[d] & \mathfrak{M} \arrow[d] \\
\mathbb{R}\underline{\mathrm{Coh}}(S) \times \mathbb{R}\underline{\mathrm{Coh}}(S) & \mathbb{R}\underline{\mathrm{Coh}}^{ext}(S) \arrow[l, "ev_3 \times ev_2"] \arrow[r, "ev_1"'] & \mathbb{R}\underline{\mathrm{Coh}}(S)
\end{tikzcd}$$

If (LM): $\exists\, (ev_1^{\mathfrak{M}})_* \circ (ev_3^{\mathfrak{X}} \times ev_2^{\mathfrak{M}})^! \circ \boxtimes$

$\Longrightarrow$ $H(\mathfrak{M})$ is a left $HA_\tau(S)$-module

Attention: $H(\mathfrak{M})$ is NOT a bimodule of $HA_\tau(S)$.

Definition: We say that

- $\mathfrak{M}$ is a right Hecke pattern if it is open and

$$\underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S) \simeq \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\bullet,\mathfrak{M},\mathfrak{X}}(S)$$

- $\mathfrak{M}$ is a left Hecke pattern if it is open and

$$\underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S) \simeq \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\bullet,\mathfrak{X}}(S)$$

- $\mathfrak{M}$ is a 2-sided Hecke pattern if

$$\underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\bullet,\mathfrak{M},\mathfrak{X}}(S) \simeq \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S) \simeq \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\bullet,\mathfrak{X}}(S)$$

Proposition

If $\mathcal{M}$ is a left/right HP $\Longrightarrow$ $(L\mathcal{M})/(R\mathcal{M})$ holds.
If $\mathcal{M}$ is a 2-sided HP $\Longrightarrow$ $(L\mathcal{M})$ and $(R\mathcal{M})$ hold.

Examples of 2-sided HP for $\mathcal{X} = \mathbb{R}\underline{\mathrm{Coh}}_0(S)$

- $\mathcal{M} = \underline{\mathrm{Hilb}}(S) =$ Hilbert stack of pts of S

$$\simeq \mathrm{Hilb}(S) \times pt/\mathbb{C}^*$$

- Fix H ample divisor, $r \geq 1$, $c_1 \in NS(S)$ with $\gcd(r, c_1 \cdot H) = 1$.
  $\mathcal{M} = \mathbb{R}\underline{\mathrm{Coh}}^{H\text{-}s}(S; r, c_1)$.

To determine the algebraic structure of $HA_0(S)$:

- 1st step: work at the level of the left and right module $H(\underline{\mathrm{Hilb}}(S))$.
- 2nd step: consider Nakajima type operators.

4. Nakajima operators

- $\mathfrak{X} = S \times pt/\mathbb{C}^* \overset{cl}{=} \underline{\mathbb{R}\mathrm{Coh}}_0(S;1) \overset{i}{\hookrightarrow} \underline{\mathbb{R}\mathrm{Coh}}_0(S;1) \simeq \tilde{S} \times pt/\mathbb{C}^*$

  length 1

- $\mathfrak{M} = \mathrm{Hilb}(S)$, $\mathcal{U}$ = universal sheaf on $\mathfrak{M} \times S$

Consider

$$\begin{tikzcd}
\mathfrak{M} \times S & \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S) \arrow[l, "ev_1^{\mathfrak{M}} \times ev_3^{\mathfrak{X}}"'] \arrow[r, "ev_2^{\mathfrak{X}}"] & \mathfrak{M} \\
& \mathbb{P}(\mathcal{U}^\vee \otimes \omega_S[1]) \arrow[u, phantom, sloped, "\simeq"] \arrow[ul] &
\end{tikzcd}$$

$$\begin{tikzcd}
S \times \mathfrak{M} & \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S) \arrow[l, "ev_3^{\mathfrak{X}} \times ev_2^{\mathfrak{M}}"'] \arrow[r, "ev_1^{\mathfrak{M}}"] & \mathfrak{M} \\
& \mathbb{P}(\mathcal{U}) \arrow[u, phantom, sloped, "\simeq"] \arrow[ul] &
\end{tikzcd}$$

Def. (Schiffmann-Vasserot, Neguț — in BM homology and K-th.)
We define the Nakajima operators :

$$\boxed{\begin{aligned} e_d &:= (ev_2^{\mathcal{M}})_* \left((ev_1^{\mathcal{M}} \times ev_3^{\mathfrak{X}})^! \left((-) \boxtimes (-) \otimes \chi_d \right) \right) \\ f_d &:= (ev_1^{\mathcal{M}})_* \left((ev_3^{\mathfrak{X}} \times ev_1^{\mathcal{M}})^! \left((-) \boxtimes (-) \otimes \chi_d \right) \right) \end{aligned}}$$

$\Longrightarrow$ They can be seen as elements of $HA_0^{sph}(S)$.

Important: because of the appearance of $\mathbb{P}(-)$, one can explictly compute the relations.

5. What can we say beyond $HA_0(S)$?

Fix

$$\begin{cases} \mathcal{T} = Coh_{\leq 1}(S) := \left\{ \mathcal{F} \in Coh(S) : \dim(\operatorname{supp}(\mathcal{F})) \leq 1 \right\} \\ \mathcal{F} = Coh_{t.f.}(S) := \left\{ \text{torsion free sheaves on } S \right\} \end{cases}$$

Note that

- $\mathcal{T}$ is a Serre subcategory,
- $\mathbb{R}\underline{Coh}_{\mathcal{T}}(S) := \mathbb{R}\underline{Coh}_{\leq 1}(S)$ and $\mathbb{R}\underline{Coh}_{\mathcal{F}}(S) := \mathbb{R}\underline{Coh}_{t.f.}(S)$ are open in $\mathbb{R}\underline{Coh}(S)$,

▶ $\mathbb{R}\underline{\mathrm{Coh}}_{\leq 1}(S)$ is also closed in $\mathbb{R}\underline{\mathrm{Coh}}(S)$.

$\Longrightarrow \exists\, HA_{\leq 1}(S)$ = associative algebra structure on $H(\mathbb{R}\underline{\mathrm{Coh}}_{\leq 1}(S))$

<u>Goal</u>: construct representations of $HA_{\leq 1}(S)$.

<u>Attention</u>: 2-sided HPs for $\mathcal{X} := \mathbb{R}\underline{\mathrm{Coh}}_{\leq 1}(S)$ are rare

For example, set $\mathcal{M} := \mathbb{R}\underline{\mathrm{Coh}}_{t.f.}(S)$. Then

▶ a subsheaf of a torsion-free sheaf is torsion-free

$$\Longrightarrow \mathbb{R}\underline{\mathrm{Coh}}^{ext}_{\bullet,\mathcal{M},\mathcal{X}}(S) \simeq \mathbb{R}\underline{\mathrm{Coh}}^{ext}_{\mathcal{M},\mathcal{M},\mathcal{X}}(S)$$

$\Longrightarrow \mathcal{M}$ is a right Hecke pattern for $\mathcal{X}$

BUT

▶ an extension between a torsion and a torsion-free sheaves is **not** torsion-free:

$$0 \longrightarrow \underset{t.f.}{E_1} \longrightarrow G \longrightarrow \underset{\text{torsion}}{E_3} \longrightarrow 0 \quad \not\Longrightarrow \quad G = t.f.$$

i.e., $\underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\bullet,\mathfrak{X}}(S) \not\simeq \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S)$

i.e., $\mathfrak{M}$ is not a left Hecke pattern for $\mathfrak{X}$

On the other hand, we can "rotate" in $D^b(\mathrm{Coh}(S))$

$$E_3 \longrightarrow E_1[1] \longrightarrow G$$

This triangle is a short exact sequence in the tilted heart:

$$D^b(\mathrm{Coh}(S))^{\heartsuit_\tau} = \{E \in D^b(\mathrm{Coh}(S)) : \mathcal{H}^{-1}(E) = \text{t.f.},$$
$$\mathcal{H}^0(E) = \text{torsion}, \ \mathcal{H}^i(E) = 0 \ \forall i \neq -1, 0\}$$

Attention ⚠:

$(\mathrm{Coh}_{t.f.}(S)[1], \mathrm{Coh}_{\leq 1}(S))$ is a torsion pair of $D^b(\mathrm{Coh}(S))^{\heartsuit_\tau}$

In particular,

$$\underset{\text{torsion}}{E_3} \longrightarrow \underset{\text{t.f.}[1]}{E_1[1]} \longrightarrow G \implies G = E_2[1] \text{ with } E_2 = \text{t.f.}$$

i.e., $\underline{\mathbb{R}\mathrm{Coh}}^{\mathrm{ext}}_{\mathcal{M},\mathcal{M},\mathcal{X}}(S) \underset{\text{"rotation"}}{\simeq} \underline{\mathbb{R}\mathrm{Coh}}^{\mathrm{ext}}_{\mathcal{X},\mathcal{M}[1],\mathcal{M}[1]}(S;\heartsuit_\tau)$

$$\simeq \underline{\mathbb{R}\mathrm{Coh}}^{\mathrm{ext}}_{\mathcal{X},\mathcal{M}[1],\bullet}(S;\heartsuit_\tau)$$

i.e., $\mathcal{M}$ is a right Hecke pattern for $\mathcal{X}$ in the tilted heart

This observation is essential to prove:

Theorem (Diaconescu-Porta-S.)

$H(\underline{\mathbb{R}\mathrm{Coh}}_{\mathrm{t.f.}}(S))$ is a left and right module of $HA_{\leq 1}(S)$.

Question: What about Nakajima type operators in this case?

Fix an effective divisor D in S, an ample divisor H in S, and $\alpha \in \mathbb{Q}$.

Let

$$\mathrm{Coh}^{(s)s}_{\alpha}(S) = \{E \in \mathrm{Coh}_{\leq 1}(S) : E \text{ is } H\text{-(semi)semistable of fixed slope } \alpha\}$$

$$\mu(-) = \frac{\chi(-)}{H \cdot ch_1(-)}$$

Definition

A subcategory $\mathcal{X} \subset Coh^{s}_{\alpha}(S)$ is admissible if

- any sheaf $\mathcal{E} \in \mathcal{X}$ is scheme-theoretically supported on D
- $\mu_{H\text{-max}}(\mathcal{E} \otimes \mathcal{O}_S(D)) \leq \alpha \leq \mu_{H\text{-min}}(\mathcal{E} \otimes \mathcal{O}_S(-D))$

We assume that the corresponding moduli stack $\mathfrak{X}$ is open and closed in $\mathbb{R}\underline{Coh}^{ss}_{\alpha}(S)$.

Let $i : D \hookrightarrow S$ be the inclusion.
Let $\mathcal{M} \subset Coh_{t.f.}(S)$ be the subcategory of locally free sheaves $\mathcal{F}$ on S s.t.

$$\boxed{\mu_{H\text{-max}}(i_* i^* \mathcal{F} \otimes \mathcal{O}_S(D)) \leq \alpha \leq \mu_{H\text{-min}}(i_* i^* \mathcal{F})}$$

$\Longrightarrow \mathfrak{M} \subseteq \mathbb{R}\underline{Coh}_{t.f.}(S)$ is open.

Lemma

$\mathfrak{M}$ is a 2-sided HP for $\mathfrak{X}$.

Consider:

$$\boxed{\begin{array}{c} \mathfrak{M}\times\mathfrak{X} \xleftarrow{ev_1^{\mathfrak{M}}\times ev_3^{\mathfrak{X}}} \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S) \xrightarrow{ev_2^{\mathfrak{X}}} \mathfrak{M} \\ \mathfrak{X}\times\mathfrak{M} \xleftarrow{ev_3^{\mathfrak{X}}\times ev_2^{\mathfrak{M}}} \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S) \xrightarrow{ev_1^{\mathfrak{M}}} \mathfrak{M} \end{array}}$$

Definition

We define the Nakajima operators :

$$\boxed{e := (ev_2^{\mathfrak{M}})_* \left((ev_1^{\mathfrak{M}} \times ev_3^{\mathfrak{X}})^! \circ \boxtimes \right) \qquad f := (ev_1^{\mathfrak{M}})_* \left((ev_3^{\mathfrak{X}} \times ev_1^{\mathfrak{M}})^! \circ \boxtimes \right)}$$

Furthermore, $\exists$ $B\mathbb{G}_m$-action on $\mathfrak{X} \Longrightarrow$

$$\boxed{e = \bigoplus_{d\in\mathbb{Z}} e_d \quad \text{and} \quad f = \bigoplus_{d\in\mathbb{Z}} f_d}$$

By combining techniques from Hall algebras with the approach of Negut and Y. Zhao, we effectively computed the relations.

For example, we obtain:

Theorem (Diaconescu-Porta-S.-Y. Zhao)
Let $\pi: S \longrightarrow B$ be a relatively minimal smooth projective elliptic surface, which admits a section.
Assume that π admits a unique singular fiber D such that

D_{red} = affine ADE configuration of (-2)-rational curves E_i

with at least 3 irreducible components E_i.

Assume that $\exists$ a (possibly trivial) torus T acting on S.
Let

$$\mathcal{X} = \{ \mathcal{O}_{E_i}(-1) : E_i = \text{irreducible comp. of } D_{red} \} \subset \mathrm{Coh}^s_0(S)$$

Then the action of the Nakajima type operators associated with $\mathcal{X}$ on $H^T(\mathfrak{M})$ gives rise to a representation of

- the affine Yangian of type ADE in BM homology
- the quantum toroidal algebra of type ADE in K-theory

Attention:

This provides a cohomological and K-theoretical version of an old construction of Ginzburg-Kapranov-Vasserot concerning operators arising from

$$\boxed{\text{Hecke modifications associated to } \mathcal{O}_{E_i}(-1)}$$

Remark

Our framework works also when
D = smooth projective curve of genus ≥ 1 with $D^2 < 0$

Current Goal: compute the relations in this case.