Cohomological Hall algebras and affine Yangians

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Plan:

1. Overview of 2d Cohomological Hall algebras

2. Nilpotent COHA of a surface and affine Yangians

1. Overview of 2d Cohomological Hall algebras

▸ Quivers

Q = quiver $= (I = \{\text{vertices}\}, \Omega = \{\text{edges}\})$

$\rightsquigarrow Q^{db}$ = double quiver $= (I, \Omega \sqcup \Omega^{op} =: \Omega^{db})$

$\Omega^{op} = \{e^* : j \to i \mid e : i \to j \in \Omega\}$

$\rightsquigarrow \mathbb{C}Q^{db}$ = path algebra of Q^{db}

$\rightsquigarrow \Pi_Q$ = preprojective algebra of $Q = \mathbb{C}Q^{db} / \underbrace{\sum_{e \in \Omega} [e, e^*]}$

(preprojective rels)

Example

Q : $\overset{e}{\circlearrowleft}$ = one-loop quiver $\Longrightarrow \Pi_{1\text{-loop}} \simeq \mathbb{C}\langle e, e^* \rangle / [e, e^*] \simeq \mathbb{C}[e, e^*]$

Denote:

$$\boxed{\underline{\text{Rep}}(\Pi_Q) = \text{moduli stack of finite-dimensional representations of } \Pi_Q}$$

Remark

- $$\underline{\mathrm{Rep}}(\Pi_Q) = \coprod_{\underline{d} \in \mathbb{Z}I} \underline{\mathrm{Rep}}(\Pi_Q ; \underline{d})$$

dimension vector

- $\underline{\mathrm{Rep}}(\Pi_Q ; \underline{d})$ = quotient stack $C_{\underline{d}} / GL(\mathbb{C} ; \underline{d})$

where

- $$GL(\mathbb{C} ; \underline{d}) := \prod_{i \in I} GL(\mathbb{C} ; d_i)$$
- $$C_{\underline{d}} = \left\{ (A_e)_{e \in \Omega^{db}} \in \bigoplus_{e \in \Omega^{db}} \mathrm{Hom}\left(\mathbb{C}^{d_{s(e)}}, \mathbb{C}^{d_{t(e)}}\right) : \sum_{e \in \Omega} [A_e, A_{e^*}] = 0 \right\}$$

Example

Q = 1-loop quiver $\Longrightarrow \underline{\mathrm{Rep}}(\Pi_{1\text{-loop}} ; d) = C_d / GL(\mathbb{C} ; d)$, $d \in \mathbb{Z}$

$$\boxed{C_d = \left\{ (A_1, A_2) \in \mathrm{Mat}(\mathbb{C}, d)^{\times 2} : [A_1, A_2] = 0 \right\} = \text{commuting variety}}$$

Schiffmann-Vasserot, Yang-Zhao:

$\exists\ \mathrm{COHA}_Q^T$ = T-equivariant COHA associated to finite-dim. reprs of Π_Q

= unital associative algebra structure on

$$H_*^T(\underline{\mathrm{Rep}}(\Pi_Q)) \simeq \bigoplus_{\underline{d}\in\mathbb{Z}I} H_*^{GL(\mathbb{C};\underline{d})\times T}(C_{\underline{d}})$$

with multiplication $p_*\circ q^!$ induced by:

$$\underline{\mathrm{Rep}}(\Pi_Q)\times\underline{\mathrm{Rep}}(\Pi_Q) \xleftarrow{\ q\ } \underline{\mathrm{Rep}}^{ext}(\Pi_Q) \xrightarrow{\ p\ } \underline{\mathrm{Rep}}(\Pi_Q)$$

$\underline{\mathrm{Rep}}^{ext}$ = stack of extensions

where:

$$p:\ 0\longrightarrow E_2\longrightarrow E\longrightarrow E_1\longrightarrow 0\ \longmapsto E$$

$$q:\ \qquad = \qquad \longmapsto (E_2, E_1)$$

and the torus action is given as:

- $(\mathbb{C}^*)^\Omega\times\mathbb{C}^* \curvearrowright \underline{\mathrm{Rep}}(\Pi_Q;\underline{d})$

$$\underset{(t_e,t)}{\cup\!\!\!\!\!\!\quad} \cdot \left(A_e, A_e^* := A_{e^*}\right)_{e\in\Omega} = \left(t_e A_e,\ t_e^{-1} t A_{e^*}\right)_{e\in\Omega}$$

- $T\subseteq(\mathbb{C}^*)^\Omega\times\mathbb{C}^*$ subtorus

Also, <u>Schiffmann-Vasserot</u>:

$\exists$ $\mathrm{COHA}_Q^{(T),nil}$ = (T-equivariant) COHA associated to the moduli stack Λ_Q of strongly semi-nilpotent reprs of Π_Q

<u>Attention</u> ⚠:
strongly semi-nilpotent = nilpotent (i.e., both A_e and A_{e^*} nil.) if Q is without edge-loops

Set

$$\boxed{\mathrm{HA}_Q^T := \mathrm{COHA}_Q^{T,nil}}$$

Now, let's recall the main result relating COHAs of quivers and Yangians:

<u>Theorem</u> (Schiffmann-Vasserot, Botta-Davison)
Let Q be an arbitrary quiver and $T = T_{max} = (\mathbb{C}^*)^{\Omega} \times \mathbb{C}^*$.
$\exists$ an isomorphism of $H_T^*(pt)$-algebras:

$$\Psi : \mathrm{HA}_Q^T \xrightarrow{\sim} \mathbb{Y}_Q^{MO,-}$$

Here, $\mathbb{Y}_Q^{MO,-}$ = negative part of Maulik-Okounkov Yangian $\mathbb{Y}_Q^{MO}$ of Q

w.r.t. triangular dec.

Remark

Maulik-Okounkov: definition of $\mathbb{Y}_Q^{MO}$ via R-matrix

= filtered deformation of $U(\mathfrak{g}_Q^{MO}[t])$

Here

- $\mathfrak{g}_Q^{MO}$ = $\mathbb{Z}$-graded Lie algebra
- when Q is without edge-loops : $\mathfrak{g}_Q^{MO}[0] = \mathfrak{g}_Q^{KM}$

McBreen: for $\mathbb{C}^* \subset T_{max}$ and Q = finite ADE

$$\Longrightarrow \begin{cases} \mathfrak{g}_Q^{MO} = \mathfrak{g}_{ADE} \\ \mathbb{Y}_Q^{MO} = \text{Drinfeld's Yangian} = \text{ass. algebra given by generators and relations} \end{cases}$$

Schiffmann-Vasserot: $\mathbb{Y}_{1\text{-loop}}^{MO}$ = ass. algebra given by gen.s and rel.s

DPSSV: for Q = affine ADE, Q_{fin} = finite ADE, $(\mathbb{C}^*)^{\times 2} \subset T_{max}$

$$\Longrightarrow \begin{cases} \mathfrak{g}_Q^{MO}[t] = \text{Universal central extension of } \mathfrak{g}_{Q_{fin}}[s^{\pm 1}, t] =: \mathfrak{g}_{ell} \\ \mathbb{Y}_Q^{MO} = \text{ass. algebra given by gen.s and rel.s} \end{cases}$$

▸ Surfaces

S = smooth quasi-projective surface $/\mathbb{C}$
T = (possibly trivial) torus $\curvearrowright S$

$\boxed{\underline{\mathrm{Coh}}_{ps}(S) = \text{moduli stack of properly supported coherent sheaves on } S}$

Remark

We can also define:

- $\underline{\mathrm{Coh}}_0(S) \subset \underline{\mathrm{Coh}}_{ps}(S)$ corresponding to 0-dim. sheaves
- $\underline{\mathrm{Coh}}_{\leq 1}(S) \subset \underline{\mathrm{Coh}}_{ps}(S)$ corresponding to sheaves of dim ≤ 1

Kapranov-Vasserot, Yu Zhao (in dim=0):

$\exists\ \mathrm{COHA}^{(T)}_S$ = (T-equivariant) COHA associated to properly supported sheaves on S

= unital associative algebra structure on

$$H^{(T)}_*\left(\underline{\mathrm{Coh}}_{ps}(S)\right)$$

with multiplication $p_* \circ q^!$ induced by:

$$\mathbb{R}\underline{\mathrm{Coh}}_{ps}(S) \times \mathbb{R}\underline{\mathrm{Coh}}_{ps}(S) \xleftarrow{q} \mathbb{R}\underline{\mathrm{Coh}}^{ext}_{ps}(S) \xrightarrow{p} \mathbb{R}\underline{\mathrm{Coh}}_{ps}(S)$$

Attention ⚠: Derived Algebraic Geometry is needed to define $p^!$

$\Longrightarrow$ $\mathbb{R}\underline{\mathrm{Coh}}_{ps}(S)$ = derived moduli stack

Remark

- $\exists\ \mathrm{COHA}^{(T)}_{S,0\text{-dim}}$ associated to $\underline{\mathrm{Coh}}_0(S)$
- $\exists\ \mathrm{COHA}^{(T)}_{S,\leq 1}$ associated to $\underline{\mathrm{Coh}}_{\leq 1}(S)$

In dim = 0, we have a complete characterization:

Theorem (Mellit-Minets-Schiffmann-Vasserot)
$COHA^{(T)}_{S,0\text{-dim}}$ can be described explicitly by generators and relations.
In particular, if $\omega_S \simeq \mathcal{O}_S$: $COHA_{S,0\text{-dim}} \simeq U(\mathfrak{g}_S^{BPS}[t])$

The questions I would like to address today are:

Question 1: can we describe $COHA^T_{S,\leq 1}$ by generators and relations?

Question 2: is $COHA^T_{S,\leq 1}$ related to Yangians?

2. COHA of a surface and affine Yangians

We saw that Yangians are related to COHAs of nilpotent representations.
First, we introduce a "nilpotent" version of $COHA_S$.

- S = smooth quasi-projective surface$/\mathbb{C}$
- $C \subset S$ reduced closed subscheme

Consider

$$\boxed{\begin{array}{l}\underline{\mathrm{Coh}}(S,C) = \text{moduli stack of coherent sheaves on } S \\ \qquad\qquad \text{set-theoretically supported on } C\end{array}}$$

sheaf analog of nilpotency

Example: X = smooth projective curve$/\mathbb{C}$

$\underline{\mathrm{Coh}}(T^*X, X) \simeq$ moduli stack of nilpotent Higgs sheaves on X

seen as zero section

Theorem 1 (DPSSV)

1. $\exists$ an associative algebra structure on $H_*^{BM}(\underline{\mathrm{Coh}}(S,C))$

$$\Longrightarrow \mathrm{COHA}_{S,C} =: \mathrm{HA}_{S,C}$$

If T = torus $\curvearrowright S$, C T-invariant $\Longrightarrow \exists\ \mathrm{COHA}^T_{S,C} =: \mathrm{HA}^T_{S,C}$

2. Given (S_1, C_1) and (S_2, C_2) s.t. $\widehat{(S_1)}_{C_1} \simeq \widehat{(S_2)}_{C_2}$, we have

$$\mathrm{HA}_{S_1,C_1} \simeq \mathrm{HA}_{S_2,C_2}$$

formal completions

i.e., $HA_{S,C}$ depends only on $\widehat{(S)}_C$

Moreover, the same holds equivariantly.

The first relation between $HA^T_{S,C}$ and Yangians is when

$$\boxed{S = \text{minimal resolution of ADE singularity}}$$

- $G \subset SL(2,\mathbb{C})$ finite group
  $\updownarrow$
  ADE quiver Q_{fin} $\subset$ affine ADE quiver Q

- $\pi : S \longrightarrow \mathbb{C}^2/G$ Kleinian resolution of singularities

$C := \pi^{-1}(0) = C_1 \cup \dots \cup C_e$; $C_i \simeq \mathbb{P}^1$; $(C_i \cdot C_j) = -$ Cartan matrix of Q_{fin}

- torus $T \subset GL(2,\mathbb{C})$ centralizing G (T = trivial or $\mathbb{C}^*$ or $\mathbb{C}^* \times \mathbb{C}^*$)

Example: $G = \mathbb{Z}_2 \implies Q_{fin} = \bullet = A_1$, $Q = \begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd} = A_1^{(1)}$

$\implies \mathbb{C}^* \times \mathbb{C}^* \curvearrowright S = T^*\mathbb{P}^1 \supset C = \mathbb{P}^1 = $ zero section

central elements

Recall

$$\mathfrak{g}_{ell} = \mathfrak{g}_{Q_{fin}}[s^{\pm 1}, t] \oplus K \ , \quad K := \bigoplus_{\ell \in \mathbb{N}} \mathbb{Q}\, c_\ell \oplus \bigoplus_{\substack{k \in \mathbb{Z}_{\neq 0} \\ \ell \in \mathbb{N}_{\geq 1}}} \mathbb{Q}\, c_{k,\ell}$$

Define

$$\mathfrak{g}_{ell}^{+} := \mathfrak{n}^{-}_{Q_{fin}}[s^{\pm 1}, t] \oplus s^{-1}\mathfrak{h}_{Q_{fin}}[s^{-1}, t] \oplus \bigoplus_{k<0} \mathbb{Q}\, c_{k,\ell}$$

completion

Theorem 2 (DPSSV)

- $\exists$ a canonical algebra isomorphism $\mathbb{HA}_{S,C} \simeq \hat{U}(\mathfrak{g}_{ell}^{+})$
- $\exists$ a canonical algebra isomorphism $\mathbb{HA}_{S,C}^{T} \simeq \mathbb{Y}_{\infty}^{+}$

where $\mathbb{Y}_{\infty}^{+}$ is a filtered deformation of $\hat{U}(\mathfrak{g}_{ell}^{+})$

Question: how do we prove this theorem?

Recall the derived McKay correspondence:

$$\tau : D^b(\mathrm{Coh}(S)) \xrightarrow{\sim} D^b(\mathrm{Mod}(\Pi_Q))$$

τ is not t-exact w.r.t. the standard t-structures

$$\Longrightarrow \underline{\mathrm{Coh}}(Y, C) \not\simeq \Lambda_Q = \text{stack of nilpotent repr.s of } \Pi_Q$$

$$\Longrightarrow HA^T_{Y,C} \not\simeq HA^T_Q$$

Attention ⚠ : we will "interpolate" between the 2 hearts by using:

- braid group actions on bounded derived cat.s
- Bridgeland stability conditions

Recall that

finite coweight lattice

- extended affine braid group $B_{ex} \simeq B_{fin} \cup \{L_{\check{\lambda}} : \check{\lambda} \in \check{X}_{fin}\} / \text{rels}$
- $\check{X}_{fin} \xrightarrow{\sim} \mathrm{Pic}(S)$, $\check{\lambda} \longmapsto \mathcal{L}_{\check{\lambda}}$

Lemma

$\exists$ a group homorphism

$$\rho : B_{ex} \longrightarrow \mathrm{Aut}(D^b(\mathrm{Coh}(S,C)))$$

such that

- $\rho(T_i) =$ dual twist functor associated to $\mathcal{O}_{C_i}(-1)$
- $\rho(L_{\check{\lambda}}) = (\mathcal{L}_{\check{\lambda}} \otimes -) =: L_{\check{\lambda}} \quad \forall\, \check{\lambda} \in \check{X}_{fin}$

$\Longrightarrow$ by the McKay equivalence, $L_{\check{\lambda}} \in \mathrm{Aut}(D^b(\mathrm{nilp}(\Pi_Q)))$

Now fix a coweight $\check{\theta} = \sum_{i \in I} \check{\theta}_i \check{\omega}_i \in \check{X}_{aff}$ s.t.

$$\boxed{\theta_i > 0 \;\; \forall\, i \neq 0 \text{ and } \check{\theta}_0 = -\sum_{i=1}^{e} \check{\theta}_i}$$

$\Longrightarrow \theta_{fin} := \sum_{i \neq 0} \check{\theta}_i \check{\omega}_i \in \check{X}_{fin}$

Consider the stability function on $\mathrm{nilp}(\Pi_Q)$:

$$Z_{\check{\theta}} : K_0(\mathrm{nilp}(\Pi_Q)) \simeq \mathbb{Z}I \longrightarrow \mathbb{C}$$

$$\underline{d} \longmapsto -(\check{\theta}, \underline{d}) + (\underbrace{\check{\rho}}_{=\sum_i \check{\omega}_i}, \underline{d})$$

$\Longrightarrow \exists$ associated $(Z_{\check{\Theta}}, \mathcal{P}_{\check{\Theta}})$ = Bridgeland's stability condition on $D^b(\mathrm{nilp}(\Pi_Q))$

Lemma

1. $\exists\, f_{\check{\Theta}_{fin}} : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\forall\, I \subset \mathbb{R}$ interval :

$$L_{\check{\Theta}_{fin}} : \mathcal{P}_{\check{\Theta}}(I) \xrightarrow{\sim} \mathcal{P}(f_{\check{\Theta}_{fin}}(I))$$

In particular $\forall\, k \in \mathbb{Z}$:

$$L_{-2k\check{\Theta}_{fin}} : \mathrm{nilp}(\Pi_Q) \xrightarrow{\sim} \mathcal{P}_{\check{\Theta}}((\nu_{-k}, \nu_{-k}+1))$$

with $\nu_\ell := \frac{1}{\pi}\arctan(2h\ell)$

Coxeter number

2. $\mathcal{P}_{\check{\Theta}}((-\frac{1}{2}, \frac{1}{2}]) \simeq \mathrm{Coh}_C(S)$

Remark

- $\nu_k \xrightarrow{k \to +\infty} \frac{1}{2}$
- we have a "sequence" of t-structures $\{\tau_k\}_{k \in \mathbb{N}}$ all equivalent to $\tau_0 :=$ standard t-structure s.t.

$$\boxed{\tau_k \xrightarrow{k \longrightarrow +\infty} \tau_\infty = \text{t-structure with heart } \mathrm{Coh}_C(S)}$$

For $\ell, k \in \mathbb{N}$, $k \geq \ell$, set

$$\Lambda_Q^{\ell,k} := \text{(derived) moduli stack of objects in } \mathcal{P}_{\check{\Theta}}((\nu_{-\ell}, \nu_{-k}+1])$$

Lemma

1. The vector space

$$\boxed{HA_\infty^T := \lim_{\ell} \operatorname{colim}_{k \geq \ell} H_*^T(\Lambda_Q^{\ell,k})}$$

has the structure of an unital associative algebra with multiplication induced from that of A_Q^T.

2. $\exists$ an algebra isomorphism $HA_{S,C}^T \simeq HA_{\check{\Theta},\infty}^T$

Finally, a careful analysis of the compatibility between the action of B_{ex} on Yangians and on HA_Q^T yields:

Lemma

$\exists$ an algebra isomorphism $HA_\infty^T \xrightarrow{\sim} \mathbb{Y}_\infty^+$, where

$$\mathbb{Y}_\infty^+ := \lim_{\ell} T_{2\ell\check{\Theta}_{fin}}(\mathbb{Y}_Q^-) / T_{2\ell\check{\Theta}_{fin}}(J)$$

The proof of Thm 2 follows from the above lemmas. $\square$

<u>Conjecture</u> $\mathbb{Y}^{+}_{\infty}$ is a new half of $\mathbb{Y}^{MO}_{Q}$.