Cohomological Hall algebras and their representations

versus

Nakajima operators

1. Cohomological Hall algebras of surfaces

S = smooth projective surface$/\mathbb{C}$.

$$\boxed{\mathrm{Coh}(S) = \text{moduli stack of coherent sheaves on } S}$$

Attention ⚠ : we work within the framework of Derived Algebraic Geometry

$\Longrightarrow$ $\mathbb{R}\mathrm{Coh}(S)$ = derived enhancement of $\mathrm{Coh}(S)$

Construction of COHA of S:

Consider the "convolution diagram":

$$\mathbb{R}\mathrm{Coh}(S) \times \mathbb{R}\mathrm{Coh}(S) \xleftarrow{\;q\;} \mathbb{R}\mathrm{Coh}^{ext}(S) \xrightarrow{\;p\;} \mathbb{R}\mathrm{Coh}(S)$$

= stack of extensions

where:

$$p = ev_2 : \quad 0 \to E_1 \to E_2 \to E_3 \to 0 \longmapsto E_2$$

$$q = ev_3 \times ev_1 : \quad \text{———— '' ————} \longmapsto (E_3, E_1)$$

- p is proper representable
- q is derived lci

Theorem

- Kapranov-Vasserot (Yu Zhao for 0-dim. sheaves):

$H_*^{BM}(\underline{Coh}(S))$ = Borel-Moore homology of $\underline{Coh}(S)$

(resp. $G_0(\underline{Coh}(S))$ = Grothendieck group of coh. sheaves on $\underline{Coh}(S)$)

has the structure of an associative algebra, whose product is given by:

$$\boxed{\begin{aligned} m : H_*^{BM}(\underline{Coh}(S)) \times H_*^{BM}(\underline{Coh}(S)) &\xrightarrow{\boxtimes} \\ H_*^{BM}(\underline{Coh}(S) \times \underline{Coh}(S)) &\xrightarrow{p_* \circ q^*} H_*^{BM}(\underline{Coh}(S)) \end{aligned}}$$

(and similarly for $G_0(\underline{Coh}(S))$).

$\Longrightarrow$ COHA of S (resp. K-theoretical HA of S)

▶ Porta-S.: $D^b_{coh}(\underline{\mathbb{R}Coh}(S))$ has the structure of a $(\mathbb{E}_1\text{-})$ monoidal dg-category, whose tensor product is given by:

$$\boxed{D^b_{coh}(\underline{\mathbb{R}Coh}(S)) \times D^b_{coh}(\underline{\mathbb{R}Coh}(S)) \xrightarrow{m} D^b_{coh}(\underline{\mathbb{R}Coh}(S))}$$

where $m = (p_* \circ q^*) \circ \boxtimes$

$\Longrightarrow$ Categorified HA of S

Remark

▶ the Thm holds also for
- S only quasi-proj.
- $\underline{Coh}_{ps}(S)$ = moduli stack of properly supported sheaves on S

▶ $\exists$ an equivariant version of the Thm w.r.t.

$$T = \text{torus} \curvearrowright S \rightsquigarrow T \curvearrowright \underline{Coh}_{ps}(S)$$

- Note that $D^b_{coh}(\underline{\mathbb{R}Coh}(S)) \not\simeq D^b_{coh}(\underline{Coh}(S))$

Moreover, $\nexists$ CatHA over $D^b_{coh}(\underline{Coh}(S))$

$\Longrightarrow$ "categorification" requires DAG

- $\underline{\mathbb{R}Coh}_0(S)$ = derived moduli stack of 0-dimensional sheaves on S

$\Longrightarrow$ $COHA_0(S)$, $KHA_0(S)$, $CatHA_0(S)$

it is a Serre subcategory

Theorem (Schiffmann-Vasserot)

$\mathbb{C}^* \times \mathbb{C}^* \curvearrowright \mathbb{C}^2$:

- $KHA_0^{\mathbb{C}^*\times\mathbb{C}^*}(\mathbb{C}^2) \simeq U^+_{q,t}(\hat{gl}(1)) \simeq \mathcal{E}^+$ = (pos. part of elliptic Hall algebra)

- $COHA_0^{\mathbb{C}^*\times\mathbb{C}^*}(\mathbb{C}^2) \simeq Y^+(\hat{gl}(1))$ = (pos. part of affine Yangian of $gl(1)$)

Notation: In the following, some results hold for

$$H_*^{BM}(-),\ G_0(-),\ D^b_{coh}(-) \rightsquigarrow H(-)$$

Similarly, $HA(-)$ denotes $COHA(-)$, $KHA(-)$, $CatHA(-)$

2. Representations

They are essential to obtain a description of HAs by generators and relations

Let us consider again $\mathfrak{X} := \mathbb{R}\underline{Coh}_0(S)$ and $HA_0(S)$.

Construction of a geometric representation of $HA_0(S)$:

Fix $\mathfrak{M} \subset \mathbb{R}\underline{Coh}(S)$.

$\mathbb{R}\underline{Coh}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S)$ = substack of $\mathbb{R}\underline{Coh}^{ext}(S)$ consisting of

$$\boxed{\begin{array}{c} 0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0 \\ E_1 \in \mathfrak{M} \ni E_2, \quad E_3 \in \mathfrak{X} \end{array}}$$

Consider the induced convolution diagram:

$$\begin{tikzcd}
\mathfrak{X} \times \mathfrak{M} \arrow[d] & \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M},\mathfrak{M},\mathfrak{X}}(S) \arrow[l, "q_L"'] \arrow[r, "p_L"] \arrow[d] & \mathfrak{M} \arrow[d] \\
\underline{\mathbb{R}\mathrm{Coh}}(S) \times \underline{\mathbb{R}\mathrm{Coh}}(S) & \underline{\mathbb{R}\mathrm{Coh}}^{ext}(S) \arrow[l, "q"'] \arrow[r, "p"] & \underline{\mathbb{R}\mathrm{Coh}}(S)
\end{tikzcd}$$

(L) If $\begin{cases} q_L \text{ is derived lci} \\ p_L \text{ is proper} \end{cases}$ $\Longrightarrow$ $H(\mathfrak{M})$ is a left module of $HA_0(S)$ via

$$\boxed{HA_0(S) \times H(\mathfrak{M}) \xrightarrow{(p_{L*} \circ q_L^*) \circ \boxtimes} H(\mathfrak{M})}$$

Consider the induced convolution diagram:

$$\mathfrak{M} \times \mathfrak{X} \xleftarrow{q_R := ev_2 \times ev_3} \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathfrak{M}\,\mathfrak{M},\mathfrak{X}}(S) \xrightarrow{p_R := ev_1} \mathfrak{M}$$

(R) If $\begin{cases} q_R \text{ is derived lci} \\ p_R \text{ is proper} \end{cases}$ $\Longrightarrow$ $H(\mathfrak{M})$ is a right module of $HA_0(S)$ via

$$\boxed{H(\mathfrak{M}) \times HA_0(S) \xrightarrow{(p_{R*} \circ q_R^*) \circ \boxtimes} H(\mathfrak{M})}$$

Important fact:

If $\mathfrak{M}$ is a left/right Hecke pattern for $\mathfrak{X}$

$\Longrightarrow$ Condition (L)/(R) holds.

Consider a s.e.s.

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0 \qquad (*)$$

with 0-dimensional sheaf E_3, and

$E_1, E_2 \in Coh_{\geq 1}(S) = \{\mathcal{E} \in Coh(S) : \mathcal{E}$ does not contain any 0-dimensional subsheaf$\}$

Def.

Let $\mathfrak{M}$ be a substack of $\underline{\mathbb{R}Coh_{\geq 1}}(S)$.

- $\mathfrak{M}$ is a left HP for $\mathfrak{X}$ if for any s.e.s. $(*)$

$$\boxed{E_3 \in \mathfrak{X},\ E_2 \in \mathfrak{M} \Longrightarrow E_1 \in \mathfrak{M}}$$

- $\mathcal{M}$ is a right ——————//—————— $\circledast$

$$\boxed{E_3 \in \mathcal{H},\ E_1 \in \mathcal{M} \Longrightarrow E_2 \in \mathcal{M}}$$

- $\mathcal{M}$ is a 2-sided ——————//—————— if it is both a left and right Hecke pattern.

Remark

If $\mathcal{M}$ is a 2-sided HP, $H(\mathcal{M})$ is a left and right module of $HA_0(S)$.

Examples of 2-sided HP

- $\mathcal{M} = \underline{\mathrm{Hilb}}(S) =$ Hilbert stack of pts of S

$$\simeq \mathrm{Hilb}(S) \times pt/\mathbb{C}^*$$

- Fix H ample divisor, $r \geq 1$, $c_1 \in NS(S)$ with $\gcd(r, c_1 \cdot H) = 1$.

$\mathcal{M} = \underline{\mathbb{R}\mathrm{Coh}}^{H\text{-}s}(S; r, c_1)$.

3. Nakajima operators

- $\mathfrak{X} = S = {}^{cl}\tilde{S} \xhookrightarrow{i} \tilde{S} \xleftarrow{\vartheta} \tilde{S} \times pt/\mathbb{C}^* \simeq \underline{\mathbb{R}\mathrm{Coh}}_0(S; 1)$

  length 1

- $\mathcal{M} = \mathrm{Hilb}(S)$, $\mathcal{U}$ = universal sheaf on $\mathcal{M} \times S$

We have:

$$\begin{tikzcd}
\mathfrak{X} \times \mathcal{M} & & \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{M},\mathcal{M},\mathfrak{X}}(S) \arrow[ll, "q_L"'] \arrow[r, "p_L"] & \mathcal{M} \\
& & \mathbb{P}(\mathcal{U}^\vee \otimes \omega_S[1]) \arrow[u, phantom, "\wr"] \arrow[ull, "\tilde{q}_L"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{M} \times \mathfrak{X} & & \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{M},\mathcal{M},\mathfrak{X}}(S) \arrow[ll, "q_R"'] \arrow[r, "p_R"] & \mathcal{M} \\
& & \mathbb{P}(\mathcal{U}) \arrow[u, phantom, "\wr"] \arrow[ull, "\tilde{q}_R"] &
\end{tikzcd}$$

Def. (Schiffmann-Vasserot, Neguț)

We define the Nakajima operators:

$$\begin{cases} \mu_d^+ := \tilde{q}_{R!}\left(p_R^*(-) \otimes \mathcal{O}(d)\right) \\ \mu_k^- := \tilde{q}_{L*}\left(p_L^*(-) \otimes \mathcal{O}(-k)\right) \end{cases} : H(\mathcal{M}) \longrightarrow H(\mathcal{M} \times S)$$

Remark

- Relation to Hall product: $\forall\, \mathcal{E} \in H(S)$:

$$\boxed{pr_{m,*}\left(\mu_k^-(-) \otimes pr_S^* \mathcal{E}\right) = m\left(-, \vartheta_{i_*}^*(\mathcal{E}) \otimes \mathcal{O}(k)\right)}$$

$\Longrightarrow$ negative Nakajima operators are elements in $HA_0(S)$

- $\exists\ \mu^+_{d_1,\dots,d_n}$ and $\mu^-_{k_1,\dots,k_m}$ defined by Neguț$_S$ ("iterated" Nakajima operators)

Thm (Neguț$_S$ for G_0, Mellit-Minets-Schiffmann-Vasserot for H_*^{BM})
Nakajima operators induce an action of

$$\boxed{\begin{array}{ll} \text{the elliptic Hall algebra of } S & \text{on } G_0(Hilb(S)) \\ \text{the affine Yangian of } S & \text{on } H_*^{BM}(Hilb(S)) \end{array}}$$

Moreover, $COHA_0(S)$ is generated by $H_*^{BM}(\underline{\mathbb{R}Coh}_0(S;1))$
i.e., $COHA_0(S)$ is spherically generated.

Remark

- Neguț has computed rels between categorified $\mu_\bullet^+$'s
- Yu Zhao ——— " ——— $\mu_\bullet^+$'s and $\mu_\bullet^-$'s

4. Cohomological Hall algebras (revised)

Fix a t-structure $\tau = (\mathcal{C}^{\le 0}, \mathcal{C}^{\ge 0})$ on

$$\mathcal{C} = D(S)$$ = unbounded derived category of quasi-coh. sheaves on S

Fact: For any $A \in \mathrm{Aff}_{\mathbb{C}}$, $\exists$ a t-structure on $\tau_A = (\mathcal{C}_A^{\le 0}, \mathcal{C}_A^{\ge 0})$ on

$$\mathcal{C}_A := D(S \times \operatorname{Spec} A)$$

given by

$$\mathcal{C}_A^{\le 0} = \{ E \in \mathcal{C}_A : \mathbb{R}p_* E \in \mathcal{C}^{\le 0} \}$$

$$\mathcal{C}_A^{\ge 0} = \{ E \in \mathcal{C}_A : \mathbb{R}p_* E \in \mathcal{C}^{\ge 0} \}$$

where $p : S \times \operatorname{Spec} A \longrightarrow S$

Def.
$\underline{\mathrm{Coh}}(S,\tau)$ = moduli stack of families of perfect complexes on S which are flat w.r.t. τ

i.e., $\forall A \in \mathrm{Aff}_{\mathbb{C}}$:

$$\underline{\mathrm{Coh}}(S,\tau)(A) = \{ E \in \mathcal{C}_A : E \in \mathrm{Perf}(S \times \mathrm{Spec}\, A),$$
$$\forall M \text{ A-module}, \; E \otimes p_A^* M \in \mathcal{C}_A^{\heartsuit_\tau} \}$$

$p_A : S \times \mathrm{Spec}\, A \longrightarrow \mathrm{Spec}\, A$

heart of the t-structure

Remark
- $\exists$ a derived enhancement $\underline{\mathbb{R}\mathrm{Coh}}(S,\tau)$ of $\underline{\mathrm{Coh}}(S,\tau)$
- $\underline{\mathbb{R}\mathrm{Coh}}(S,\tau) \subset \underline{\mathbb{R}\mathrm{Perf}}(S)$ = derived moduli stack of perfect complexes on S

Def.
We say that τ satisfies openness of flatness if $\underline{\mathbb{R}\mathrm{Coh}}(S,\tau)$ is an open substack of $\underline{\mathbb{R}\mathrm{Perf}}(S)$

Prop:
$\underline{\mathbb{R}\mathrm{Coh}}(S,\tau)$ is Artin $\Leftrightarrow$ τ satisfies openness of flatness

Fix a t-structure τ.
Consider the "convolution diagram" as before:

$$\boxed{\underline{\mathbb{R}\mathrm{Coh}}(S,\tau)\times\underline{\mathbb{R}\mathrm{Coh}}(S,\tau) \xleftarrow{q_\tau} \underline{\mathbb{R}\mathrm{Coh}}^{ext}(S,\tau) \xrightarrow{p_\tau} \underline{\mathbb{R}\mathrm{Coh}}(S,\tau)}$$

Thm (Porta-S., Diaconescu-Porta-S.)
Assume that
1. τ satisfies openness of flatness,
2. p_τ proper,
3. Serre functor $S_{\mathscr{C}}$ s.t. $S_{\mathscr{C}}[-2]$ is t-exact w.r.t. τ.

Then
- $\exists$ a ($\mathbb{E}_1$-)monoidal structure on $D^b_{coh}(\mathbb{R}\underline{\mathrm{Coh}}(S,\tau))$ induced by $((p_\tau)_* \circ q_\tau^*)\circ \boxtimes$.
- $G_0(\underline{\mathrm{Coh}}(S,\tau))$ and $H_*^{BM}(\underline{\mathrm{Coh}}(S,\tau))$ have the structures of unital associative algebras.

Furthermore, for any open substack $\mathfrak{X}$ of $\underline{\mathbb{R}\mathrm{Coh}}(S,\tau)$ s.t. $\mathfrak{X}(\mathrm{Spec} K)$ is a Serre subcategory $\forall$ field K, then $\exists$ induced $HA_{\mathfrak{X}}(S,\tau)$.

Remark

We proved a more general version of the theorem. For example, if

$$S \rightsquigarrow \text{Kuznetsov component } Ku$$

(of a smooth cubic 4fold $\subset \mathbb{P}^5$, Fano 3fold of $rk(Pic)=1$, GM variety, ...)

$$\tau = t\text{-structure associated to a stability condition } \sigma$$

Then, Assumptions 1 and 2 hold $\Longrightarrow \exists\ HA(Ku, \tau)$

Examples of $HA(S,\tau)$

Fix a torsion pair $v = (\mathcal{T}or, \mathcal{F})$ of $Coh(S)$, i.e.,

- $Hom(T,F) = 0 \quad \forall\, T \in \mathcal{T}or,\ F \in \mathcal{F}$;
- $\forall\, E \ \exists\ 0 \to \underset{\mathcal{T}or}{\underset{\cap}{T}} \to E \to \underset{\mathcal{F}}{\underset{\cap}{F}} \to 0$

$\Longrightarrow \exists$ new t-structure τ_v such that its heart is:

$$\boxed{\begin{array}{l}\mathcal{C}_v^{\heartsuit} = \{ E \in \mathcal{C} : \mathcal{H}^{-1}(E) \in \mathcal{F}, \mathcal{H}^0(E) \in \mathcal{T}or, \mathcal{H}^i(E) = 0 \ \forall i \neq -1, 0 \} \\ \text{and } (\mathcal{F}[1], \mathcal{T}or) \text{ torsion pair of } \mathcal{C}_v^{\heartsuit}\end{array}}$$

$\exists\ \underline{\mathbb{R}Coh}_{\mathcal{T}or}(S), \underline{\mathbb{R}Coh}_{\mathcal{F}}(S) \subseteq \underline{\mathbb{R}Coh}(S), \subseteq \underline{\mathbb{R}Coh}(S, \tau_v)$

Def.

We say that the torsion pair $v = (\mathcal{T}or, \mathcal{F})$ is open if $\underline{\mathbb{R}Coh}_{\mathcal{T}or}(S)$ and $\underline{\mathbb{R}Coh}_{\mathcal{F}}(S)$ are open substacks of $\underline{\mathbb{R}Coh}(S)$

Facts:

- Lieblich: τ_v satisfies openness of flatness
- DPS: $P\tau_v$ is proper.

$$\boxed{\Longrightarrow \exists\ HA(S, \tau_v) \text{ associated to the tilted t-structure } \tau_v}$$

Moreover,
if $\mathcal{T}or$ is a Serre subcategory $\Longrightarrow \exists\ HA_{\mathcal{T}or}$ associated to $\mathcal{T}or$

5. Representations (revisited)

Fix a t-structure τ on $\mathcal{C} = D(S)$.

Fix $v = (\mathcal{T}or, \mathcal{F})$ torsion pair of $\mathcal{C}^\heartsuit$ = heart of τ

Note that

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0 \quad \text{in } \mathcal{C}^\heartsuit$$

if $E_3 \in \mathcal{T}or, E_2 \in \mathcal{F} \Longrightarrow E_1 \in \mathcal{F}$

$$\boxed{\Longrightarrow \mathcal{F} \text{ left HP for } \mathcal{T}or \text{ w.r.t. } \tau}$$

but

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0 \quad \text{in } \mathcal{C}^\heartsuit$$

if $E_3 \in \mathcal{T}or, E_1 \in \mathcal{F} \not\Longrightarrow E_2 \in \mathcal{F}$

$\Longrightarrow \mathcal{F}$ not right HP for $\mathcal{T}or$ w.r.t. τ

On the other hand, if we "rotate":

$$0 \longrightarrow E_3 \longrightarrow E_1[1] \longrightarrow G \longrightarrow 0 \quad \text{in } \mathcal{C}_v^\heartsuit = \text{tilted heart}$$

because $\mathcal{F}[1]$ is torsion in $\mathcal{C}_v^\heartsuit$

if $E_3 \in \mathcal{T}or,\ E_1 \in \mathcal{F} \Longrightarrow G = E_2[1] \in \mathcal{F}[1]$

This observation is essential to prove:

Thm (Diaconescu-Porta-S.)
Assume that:

1. τ and τ_v satisfy openness of flatness

2. p_τ and p_{τ_v} are proper

3. Serre functor S_φ s.t. $S_\varphi[-2]$ is t-exact w.r.t. τ and τ_v,

4. $\mathbb{R}\underline{Coh}_{\mathcal{T}or}(S,\tau)$ and $\mathbb{R}\underline{Coh}_{\mathcal{F}}(S,\tau)$ are open in both $\mathbb{R}\underline{Coh}(S,\tau)$ and $\mathbb{R}\underline{Coh}(S,\tau_v)$

5. $\mathcal{T}or$ is a Serre subcategory

6. $\mathbb{R}\underline{Coh}_{\mathcal{T}or}(S,\tau)$ is closed in both $\mathbb{R}\underline{Coh}(S,\tau)$ and $\mathbb{R}\underline{Coh}(S,\tau_v)$

Then

- $H(\underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau))$ is a left module of $HA_{\mathcal{T}or}$ induced by:

$$\underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{T}or}(S,\tau)\times\underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{F}}(S,\tau)\xleftarrow{q_\tau}\underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{F},\mathcal{F},\mathcal{T}or}(S,\tau)\xrightarrow{p_\tau}\underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{F}}(S,\tau)$$

- $H(\underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau))$ is a right module of $HA_{\mathcal{T}or}$ induced by

$$\underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{F}}(S,\tau)\times\underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{T}or}(S,\tau)\xleftarrow{q_{\tau_r}}\underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{T}or,\mathcal{F}[1],\mathcal{F}[1]}(S,\tau)\xrightarrow{p_{\tau_r}}\underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{F}}(S,\tau)$$

Attention ⚠ :

the use of the tilted heart "compensate" the lack of 2-sided Hecke patterns.

Remark

The Thm holds if we replace $\underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{F}}(S,\tau)$ by a substack

$$\mathfrak{M}\subset\underline{\mathbb{R}\mathrm{Coh}}_{\mathcal{F}}(S,\tau)$$

Moreover, we proved a more general version of the Thm for $(\mathcal{C}, \tau, v = (\mathcal{T}or, \mathcal{F}))$ satisfying some natural conditions.

Examples

$$\begin{cases} \mathcal{T}or = Coh_{\leq 1}(S) := \{ \mathcal{F} \in Coh(S) : \dim(\mathrm{supp}(\mathcal{F})) \leq 1 \} \\ \mathcal{F} = Coh_{t.f.}(S) := \{ \text{torsion free sheaves on } S \} \end{cases}$$

or $\mathcal{T}or = Coh_0(S)$, $\mathcal{F} = Coh_{\geq 1}(S)$

$\Longrightarrow$ we recover the examples discussed before.

Moreover, we get:

$P(S) :=$ moduli space of Pandharipande-Thomas stable pairs on S

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \parallel$

$(\mathcal{F}, s : \mathcal{O}_S \longrightarrow \mathcal{F})$ with $\mathcal{F}$ pure 1-dimen.
$\mathrm{Coker}(s)$ 0-dimen.

Thm (DPS)

- $H(P(S))$ is a right and left module of $HA_0(S)$
- ——— " ——— a left module of $HA_{\leq 1}(S)$.

6. Nakajima operators (revisited)

$\mathcal{M} \subset \underline{\mathbb{R}\mathrm{Coh}}_{t.f.}(S)$ a 2-sided HP for $\mathcal{X} \subseteq \underline{\mathbb{R}\mathrm{Coh}}_{tor}(S)$ open

Then

$$\boxed{\begin{array}{c}
\begin{tikzcd}
\mathcal{X} \times \mathcal{M} & \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{M},\mathcal{M},\mathcal{X}}(S) \arrow[l, "q_L"'] \arrow[r, "p_L"] \arrow[d] & \mathcal{M} \\
& \mathbb{P}(\text{complex in tor-ampl. } [0,1]) \arrow[ul, "\tilde{q}_L"] &
\end{tikzcd}
\\
\begin{tikzcd}
\mathcal{M} \times \mathcal{X} & \underline{\mathbb{R}\mathrm{Coh}}^{ext}_{\mathcal{M},\mathcal{M},\mathcal{X}}(S) \arrow[l, "q_R"'] \arrow[r, "p_R"] \arrow[d] & \mathcal{M} \\
& \mathbb{P}(\text{complex in tor-ampl. } [0,1]) \arrow[ul, "\tilde{q}_R"] &
\end{tikzcd}
\end{array}}$$

$\Longrightarrow$ p_L, p_R are derived lci and $\mathbb{C}^*$-equiv., $\tilde{q}_L, \tilde{q}_R$ are proper

Def.

We define Nakajima operators as:

$$\boxed{\begin{cases} \mu^+_d := \tilde{q}_{R!}\left(p_R^*(-) \otimes \mathcal{O}(d)\right) : \\ \mu^-_k := \tilde{q}_{L*}\left(p_L^*(-) \otimes \mathcal{O}(-k)\right) : \end{cases} H(\mathcal{M}) \longrightarrow H(\mathcal{M} \times \mathcal{X})}$$

Main example (work in progress - Diaconescu-Porta-S-Zhao)

Let $D \xhookrightarrow{i} S$ be an effective divisor.

- $\mathcal{X} \subset \underline{\mathbb{R}\mathrm{Coh}}_{\alpha}^{H\text{-}s}(S)$ parametrizing sheaves G scheme-theoretically supported on D such that

$$\boxed{\mu_{H\text{-}max}(G(D)) \leq \alpha \quad \left(\text{resp. } \mu_{H\text{-}min}(G(-D)) \geq \alpha\right)}$$

- $\mathcal{M} \subseteq \underline{\mathbb{R}\mathrm{Coh}}_{l.f.}(S)$ parametrizing locally free sheaves F on S with

$$\boxed{\mu_{H\text{-}max}\left(i_* i^* F(D)\right) \leq \alpha \quad \left(\text{resp. } \mu_{H\text{-}min}\left(i_* i^* F\right) \geq \alpha\right)}$$

Examples of D

- D = a configuration of (-2)-curves of genus zero.
- D = smooth proj. curve of genus ≥ 1 and $D^2 < 0$.