Incontri di Geometria algebrica e aritmetica

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From Hilbert schemes of pts on a smooth surface to cohomological Hall algebras

Plan

1. Motivation: study of Hilbert schemes of pts via repr. theory

2. Hall algebras

3. Representations via torsion pairs and "doubling" Hall algebras

$\boxed{\text{Motivation: study of Hilbert schemes of pts via repr. theory}}$

S = smooth (quasi-) projective surface $/\mathbb{C}$
$n \in \mathbb{Z}, n \geq 0$

$\text{Hilb}^n(S)$ = Hilbert scheme of n-pts on S
= moduli space parametrizing zero-dim. subschemes
$Z \subset S$ such that $\dim_{\mathbb{C}} H^0(Z, \mathcal{O}_Z) = n$

Remarks

- $\text{Hilb}^n(S)$ is a smooth (quasi-) projective variety $/\mathbb{C}$ of dimension $2n$

- $$\pi : \text{Hilb}^n(S) \longrightarrow \text{Sym}^n(S) := \overbrace{S \times \cdots \times S}^{\text{n-th copies}} / \mathfrak{S}_n$$

  symmetric group of n letters

  is a resolution of singularities, i.e.,
  π is a proper morphism which is an iso over the smooth locus of $\text{Sym}^n(S)$.

Examples

- $n=1$: $\mathrm{Sym}^1(S) = S \simeq \mathrm{Hilb}^1(S)$

- $n=2$: $Z \in \mathrm{Hilb}^2(S) \rightsquigarrow \begin{cases} Z = \{x, y\},\ x \neq y & \Longrightarrow \pi(Z) = x+y \\ Z_{red} = \{x\} & \Longrightarrow \pi(Z) = x+x = 2x \end{cases}$

$$\Longrightarrow \mathrm{Hilb}^2(S) \simeq \underbrace{\mathrm{Blow}_{\Delta}}_{\Delta = \text{diagonal}}(S \times S)/\mathfrak{S}_2 \xrightarrow{\pi} \mathrm{Sym}^2(S)$$

"Bridge" between $\mathrm{Hilb}^n(S)$ and representation theory:

- First ingredient = Hecke correspondence

for $k>0$: $\mathrm{Hilb}^{n,n+k}(S) := \left\{ 0 \to J \to I \to Q_x \to 0 : \begin{array}{l} \bullet\ (I,J) \in \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^{n+k}(S) \\ \bullet\ \mathrm{supp}(Q_x) = \{x\} \end{array} \right\}$

$$\begin{array}{c} \mathrm{Hilb}^{n,n+k}(S) \\ \downarrow \\ \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^{n+k}(S) \times S \end{array}$$

reduced closed subscheme

for $k>0$: $\mathrm{Hilb}^{n+k,k}(S) \hookrightarrow \mathrm{Hilb}^{n+k}(S) \times \mathrm{Hilb}^{n}(S) \times S$

for $k=0$: $\mathrm{Hilb}^{n,n}(S)$ = diagonal $\hookrightarrow \mathrm{Hilb}^{n}(S) \times \mathrm{Hilb}^{n}(S)$

Attention ⚠ : $\mathrm{Hilb}^{n,n'}(S)$ is smooth $\Leftrightarrow |n-n'| \leq 1$.

- Second ingredient = Tautological bundles

$\mathcal{Z}_n \subset \mathrm{Hilb}^{n}(S) \times S$ universal family

$p: \mathrm{Hilb}^{n}(S) \times S \longrightarrow \mathrm{Hilb}^{n}(S)$ projection

Fact: $\tau_n := p_*(\mathcal{O}_{\mathcal{Z}_n})$ is a vector bundle of rank n.

Def. (Tautological bundles on Hecke correspondences)

$$\begin{cases} \tau_{n,n+1} := \mathrm{Ker}\left(pr_{n+1}^*(\tau_{n+1}) \twoheadrightarrow pr_n^*(\tau_n)\right) & \text{on } \mathrm{Hilb}^{n,n+1}(S) \\ \tau_{n+1,n} := \text{———}\,''\,\text{———} & \text{on } \mathrm{Hilb}^{n+1,n}(S) \\ \tau_{n,n} := pr_n^*(\tau_n) & \text{on } \mathrm{Hilb}^{n,n}(S) \end{cases}$$

▶ Third ingredient = G_0-theory = Grothendieck group of coherent sheaves

Set $\mathrm{Hilb}(S) := \bigsqcup_{n\geq 0} \mathrm{Hilb}^n(S)$.

From now on $S = \mathbb{C}^2 \circlearrowleft T = \mathbb{C}^* \times \mathbb{C}^*$. Define

$$f_{-1,\ell} = \prod_n \mathbb{R}(p_{n+1})_* \left([\tau_{n,n+1}]^{\otimes \ell} \overset{\mathbb{L}}{\otimes} pr_n^*(-) \right) \qquad \text{for } \ell \in \mathbb{Z}$$

$$f_{1,\ell} = \prod_n \mathbb{R}(p_n)_* \left([\tau_{n+1,n}]^{\otimes \ell} \overset{\mathbb{L}}{\otimes} pr_{n+1}^*(-) \right) \qquad \text{for } \ell \in \mathbb{Z}$$

$$e_{0,\ell} = \prod_n \mathbb{R}(p_n)_* \left([\Lambda^\ell \tau_{n,n}] \overset{\mathbb{L}}{\otimes} pr_n^*(-) \right) \qquad \text{for } \ell \in \mathbb{Z}, \ell > 0$$

$$e_{0,-\ell} = \prod_n \mathbb{R}(p_n)_* \left([\Lambda^\ell \tau_{n,n}^\vee] \overset{\mathbb{L}}{\otimes} pr_n^*(-) \right) \qquad \text{for } \ell \in \mathbb{Z}, \ell > 0$$

$$f_{\pm 1,\ell}, e_{0,\pm\ell} \in \mathrm{End}\left(G_0^T(\mathrm{Hilb}(\mathbb{C}^2))_{loc} \right)$$

localization

$$M_{loc} := M \otimes_{\mathbb{C}[q^{\pm1}, t^{\pm1}]} \mathbb{C}(q^{1/2}, t^{1/2})$$

Thm. (Schiffmann-Vasserot)

The algebra generated by $f_{\pm 1, \ell}, e_{0, \pm \ell}$ is isomorphic to the elliptic Hall algebra $\mathcal{E}$ $\left(= \text{quantum toroidal algebra } U(\hat{\hat{gl}}(1)) \text{ of } gl(1)\right)$:

As faithful reprs of $\mathcal{E} = U(\hat{\hat{gl}}(1))$,

$$G_0^T(\mathrm{Hilb}(\mathbb{C}^2))_{loc} \simeq \underbrace{\mathbb{C}(q^{1/2}, t^{1/2})[x_1, x_2, \dots]^{\mathfrak{S}_\infty}}_{\text{algebra of symmetric functions in infinitely many variables}}$$

Attention ⚠ : later on, I will give a "geometric" definition of $\mathcal{E}$.

Thm. (Neguț)

Let $S = K3$. The algebra generated by

$$f_{\pm 1, \ell}, e_{0, \pm \ell} \in \mathrm{End}(G_0(\mathrm{Hilb}(S)))$$

is isomorphic to the Ding-Iohara-Miki algebra modelled on $G_0(S)$.

Attention ⚠ : $\exists$ a surjective (but not injective) algebra map:

$$\text{Ding-Iohara-Miki algebra mod. on } G_0^T(\mathbb{C}^2) \twoheadrightarrow \mathcal{E}$$

Thm. (Neguț)

Let $S = K3$.

$\exists$ functors $\in \mathrm{End}(D^b(\mathrm{Coh}(\mathrm{Hilb}(S))))$ which categorify the previous result (i.e., after passing to $G_0(-)$, we recover the previous result).

Thm (Schiffmann-Vasserot)

The algebra generated by

$$\underbrace{f_{\pm 1, \ell},\ e_{0, \pm \ell}}_{\text{cohomological version: } \left\{ \begin{array}{l} [\tau_{n,n+1}] \longmapsto c_1(\tau_{n,n+1}) \\ [\Lambda^{\ell} \tau_{n,n}] \longmapsto c_\ell(\tau_{n,n}) \end{array} \right.} \in \mathrm{End}\left(H_T^*(\mathrm{Hilb}(\mathbb{C}^2))_{loc}\right)$$

is isomorphic to a "degenerate" version of $\mathcal{E}$ (= affine Yangian $Y(\widehat{gl}(1))$ of $gl(1)$)

This action induces Nakajima-Grojnowski's action of the Heisenberg algebra on $H_T^*(\mathrm{Hilb}(\mathbb{C}^2))_{loc}$.

"Bridge" between moduli spaces and repr. theory based on explicitly def. operators.

Advantages of this approach:

- it allows to compute relations between generators;
- it allows to characterize explicitly the "geometric" representation.
- it determines that $H_T^*(\mathrm{Hilb}(\mathbb{C}^2))_{loc}$ = irreducible h.w. repr. (of Heis)

Limit of this approach: it does NOT realize all possible geometric actions, e.g., the one with Hecke correspondence:

$$\left\{ 0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow i_* \mathcal{L}_C \longrightarrow 0 \right\}$$

where $C \overset{i}{\hookrightarrow} S$ is a smooth proj. curve inside a smooth surface, $\mathcal{L}_C$ line bundle on C.

In particular, it does NOT determine that

$$\bigoplus_{c_1, c_2} H^*\left(\mathcal{M}^{st}(K3; r, c_1, c_2)\right) \simeq \text{irreducible h.w. repr. (of some algebra)}$$

Hall algebras

S = smooth quasi-projective surface$/\mathbb{C}$.

$\underline{\mathrm{Coh}}_{ps}(S)$ = derived moduli stack of properly supported coherent sheaves on S

Consider the Hall convolution diagram

derived stack of short exact sequences

$$\begin{tikzcd} & \underline{\mathrm{Coh}}^{ext}_{ps}(S) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \underline{\mathrm{Coh}}_{ps}(S) \times \underline{\mathrm{Coh}}_{ps}(S) & & \underline{\mathrm{Coh}}_{ps}(S) \end{tikzcd}$$

$$p: 0 \to \mathcal{E}_2 \to \mathcal{E} \to \mathcal{E}_1 \to 0 \longmapsto (\mathcal{E}_1, \mathcal{E}_2)$$
$$q: \qquad\qquad \text{''} \qquad\qquad \longmapsto \mathcal{E}$$

Facts:

- p is derived l.c.i., i.e., $\mathbb{L}_p$ is perfect and in Tor-amplitude $[-1,1]$
- q is representable by proper schemes

Thm (Porta-S.)

1. $D^b_{coh}(\underline{Coh}_{ps}(S))$ has a monoidal structure induced by $q_* \circ p^*$.
(rather its dg enhanc.)
2. It descends to an associative algebra structure $KHA(S)$ on $G_0(\underline{Coh}(S))$

Remarks:

- By using the work of Adeel Khan on motivic Borel-Moore homology, (2) holds after replacing $G_0(-)$ with $H_*^{BM}(-) \rightsquigarrow COHA(S)$

- The above Theorem holds also for $\underline{Coh}_{\tau_\bullet}(S) \hookrightarrow \underline{Coh}_{ps}(S)$, where

$$\tau_{\leq 1} := \{ \mathcal{F} \in Coh_{ps}(S) : \dim(\mathrm{supp}(\mathcal{F})) \leq 1 \}$$

$$\tau_0 := \{ \mathcal{F} \in Coh_{ps}(S) : \dim(\mathrm{supp}(\mathcal{F})) = 0 \}$$

- (2) recovers the constructions by:
  - Kapranov-Vasserot via perfect obstruction theory
  - Schiffmann-Vasserot for $\mathbb{C}^2$ via the Lagrangian formalism
  - S.-Schiffmann for $T^*(\text{curve})$ —— // ——

- Diaconescu-Porta-S: the above Theorem holds for $(\mathscr{C}, \tau)$:
  - $\mathscr{C}$ is a "nice" triangulated category (e.g. Toën-Vaquié's moduli of objects is an Artin derived stack)
  - τ is a t-structure which satisfies openness of flatness
  - $\exists$ a Serre functor $S_{\mathscr{C}}$ such that $S_{\mathscr{C}}[-2]$ is t-exact
  - the Quot functor for $(\mathscr{C}, \tau)$ is represented by a proper algebraic space

Thm (Schiffmann-Vasserot)

$$\mathrm{KHA}^T(\mathbb{C}^2)_{loc} = \left(G_0^T(\underline{\mathrm{Coh}}_0(\mathbb{C}^2))_{loc}, \text{Hall product}\right) \simeq U^+(\hat{\mathfrak{gl}}(1))$$

$$\mathrm{COHA}^T(\mathbb{C}^2)_{loc} = \left(H_*^T(\underline{\mathrm{Coh}}_0(\mathbb{C}^2))_{loc}, \text{Hall product}\right) \simeq Y^+(\hat{\mathfrak{gl}}(1))$$

Attention: Hall algebras realize only halves of the full algebras we care.

Representations via torsion pairs and doubling Hall algebras

S = smooth projective surface$/\mathbb{C}$.

Consider

$$\mathcal{T}_{\leq 1} := \left\{ \mathcal{F} \in \mathrm{Coh}_{ps}(S) : \dim(\mathrm{supp}(\mathcal{F})) \leq 1 \right\}$$

$$\underline{\mathrm{Coh}}_{\leq 1}(S) := \underline{\mathrm{Coh}}_{\mathcal{T}_{\leq 1}}(S)$$

Introduce: $\mathcal{M}(S;1)$:= moduli space of rank-one torsion-free sheaves on S

Thm (Diaconescu-Porta-S.)
$D^b_{coh}(\mathcal{M}(S;1))$ is a left and right categorical module over $D^b_{coh}(\underline{\mathrm{Coh}}_{\leq 1}(S))$

In particular,
- $G_0(\mathcal{M}(S;1))$ is a left and right module over $G_0(\underline{\mathrm{Coh}}_{\leq 1}(S))$
- $H^{BM}_*(\mathcal{M}(S;1))$ is a left and right module over $H^{BM}_*(\underline{\mathrm{Coh}}_{\leq 1}(S))$

Def. (Algebras of "Hecke modifications along curves")
The Yangian of $\underline{\mathrm{Coh}}_{\leq 1}(S)$ is the subalgebra of $\mathrm{End}(H_*^{BM}(\mathcal{M}(S;1)))$ generated by the images of

$$\boxed{\begin{array}{l} \text{left action } a_\ell : \mathrm{COHA}_{\leq 1}(S) \longrightarrow \mathrm{End}(H_*^{BM}(\mathcal{M}(S;1))) \\[2ex] \text{right action } a_r : \mathrm{COHA}_{\leq 1}(S) \longrightarrow \mathrm{End}(H_*^{BM}(\mathcal{M}(S;1))) \end{array}}$$

Similarly, we define: quantum loop algebra of $\mathrm{Coh}_{\leq 1}(S)$ and its categorification

Remark

- The theorem above holds after replacing $\underline{\mathrm{Coh}}_{\leq 1}(S) \rightsquigarrow \underline{\mathrm{Coh}}_0(S)$ and

$$\mathcal{M}(S,1) \rightsquigarrow \text{moduli space of PT stable pairs on } S$$

- $S = K3$.
The theorem above holds after replacing $\underline{\mathrm{Coh}}_{\leq 1}(S) \rightsquigarrow \underline{\mathrm{Coh}}_0(S)$ and $\mathcal{M}(S,1) \rightsquigarrow \mathrm{Hilb}(S) \Longrightarrow$ we recover Neguț's construction

The previous theorem is a consequence of the following more general result.

$\mathcal{C}$ = "nice" triangulated category

τ = t-structure which satisfies openness of flatness

$v = (\mathcal{T}or, \mathcal{F})$ = torsion pair in $\mathrm{Coh}(S)$, i.e.,

- $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$
- $\forall E \in \mathrm{Coh}(S)\ \exists\ 0 \longrightarrow \underset{\mathcal{T}or}{\overset{}{T}} \longrightarrow E \longrightarrow F \longrightarrow 0$, with $T \in \mathcal{T}or$, $F \in \mathcal{F}$

$\Longrightarrow \tau_v$ = tilted t-structure on $\mathcal{C}$:

$$\mathcal{C}_v^{\heartsuit} = \left\{ E \in \mathcal{C} : \mathcal{H}_\tau^{-1}(E) \in \mathcal{F},\ \mathcal{H}_\tau^{0}(E) \in \mathcal{T}or,\ \mathcal{H}_\tau^{i}(E) = 0\ \forall i \neq 0, -1 \right\}$$

Thm (Diaconescu-Porta-S.)

Assume that

1. $\underline{\mathrm{Coh}}_{\mathcal{T}or}(\mathcal{C}, \tau)$, $\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C}, \tau)$ are open in $\underline{\mathrm{Coh}}(\mathcal{C}, \tau)$
2. p_τ is derived l.c.i., q_τ is proper
3. p_{τ_v} ——— " ———, q_{τ_v} ——— " ———
4. $\mathcal{T}or$ is a Serre subcategory and $\underline{\mathrm{Coh}}_{\mathcal{T}or}(S, \tau)$ is closed in both $\underline{\mathrm{Coh}}(\mathcal{C}, \tau)$ and $\underline{\mathrm{Coh}}(\mathcal{C}, \tau_v)$

Then

- $D^b_{coh}(\underline{Coh}_F(\mathcal{C},\tau))$ is a left categorical module of $D^b_{coh}(\underline{Coh}_{\tau_{or}}(\mathcal{C},\tau))$ induced by the monoidal structure of $D^b_{coh}(\underline{Coh}(\mathcal{C},\tau))$.
- $D^b_{coh}(\underline{Coh}_F(\mathcal{C},\tau))$ is a right categorical module of $D^b_{coh}(\underline{Coh}_{\tau_{or}}(\mathcal{C},\tau))$ induced by the monoidal structure of $D^b_{coh}(\underline{Coh}(\mathcal{C},\tau_v))$.

Similar statements hold for $G_0(-)$ and $H^{BM}_*(-)$.