Cohomological Hall algebras of 1-dimensional sheaves and Yangians

DPSSV-1, arXiv:2502.19445
Diaconescu-Porta-S.-Schiffmann-Vasserot,
Nilpotent cohomological Hall algebras of surfaces

DPSSV-2, arXiv:2603.03386
Diaconescu-Porta-S.-Schiffmann-Vasserot,
Cohomological Hall algebras of one-dimensional sheaves on surfaces and Yangians

What topics will I cover?

2-dimensional Cohomological Hall algebras (COHAs)

$\Downarrow$

Construction of COHAs

$\Downarrow$

Description of the "quantum nature" of a COHA (i.e., its relation to Yangians)

$\Downarrow$

Quivers $\longleftarrow$ Examples $\longrightarrow$ Surfaces

$\Downarrow$

McKay correspondence

$\downarrow$

affine ADE quivers

$\downarrow$

minimal resolutions $X \longrightarrow \mathbb{C}^2/G$

In particular, the following structures will be introduced:

- braid group actions on Yangians and COHAs,
- Bridgeland's stability conditions
- limiting COHAs

Lecture 1

1. 2d COHAs of quivers

1.1 Construction

$$\boxed{\begin{array}{c}\text{Geometry}\\ \text{Moduli stack}\end{array}} \longrightarrow \text{COHA} \longrightarrow \boxed{\begin{array}{c}\text{Algebra}\\ \text{Associative algebra}\end{array}}$$

Fix a quiver $Q = (I = \{\text{vertices}\}, \Omega = \{\text{edges}\})$, i.e., an oriented graph.

Notation: source of $e = s(e) \bullet \xrightarrow{\; e \in \Omega \;} \bullet\, t(e) =$ target of e

Examples

- $Q =$ one-loop quiver : 

- $Q = A_N^{(1)}$-quiver :

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r] & \overset{2}{\bullet} \arrow[r] & \cdots \arrow[r] & \overset{N-1}{\bullet} \arrow[r] & \overset{N}{\bullet} \\ & & \overset{0}{\bullet} \arrow[ull] \arrow[urr] & & \end{tikzcd}$$

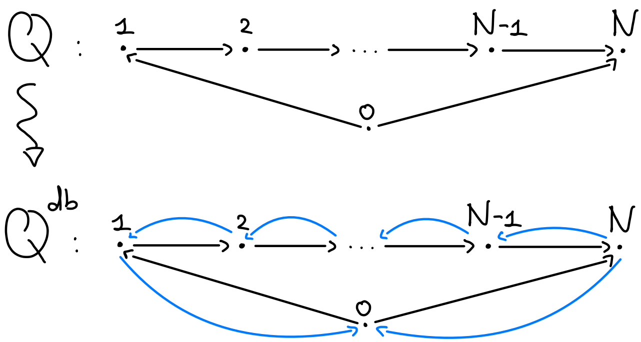
Define

- the double quiver $Q^{db} = (I, \Omega^{db} := \Omega \sqcup \Omega^{op})$ where

$$\Omega^{op} := \{ e^* : j \longrightarrow i \mid e : i \longrightarrow j \in \Omega \}$$

Examples

- Q : 

- 

- the path algebra $\mathbb{C}Q^{db}$ of Q^{db}

  = the $\mathbb{C}$-vector space spanned by all the paths in Q^{db} (including those of length zero) with multiplication given by concatenation of paths, whenever possible.

- a representation of Q^{db} is a right $\mathbb{C}Q^{db}$-module, i.e., the datum $\left(V = \bigoplus_{i \in I} V_i , \left(x_e : V_{s(e)} \longrightarrow V_{t(e)}\right)_{e \in \Omega^{db}}\right)$

If $\dim V_i < +\infty$, the repr. of Q^{db} is finite-dimensional

Notation:

$\underline{d} = (\dim V_i)_{i \in I} \in \mathbb{N} I$ is called a dimension vector.

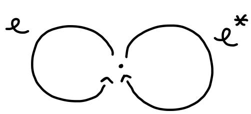
- the preprojective algebra Π_Q of Q = the quotient

$$\Pi_Q := \mathbb{C}Q^{db} \Big/ \sum_{e \in \Omega} (ee^* - e^*e)$$

(preprojective relations)

Examples

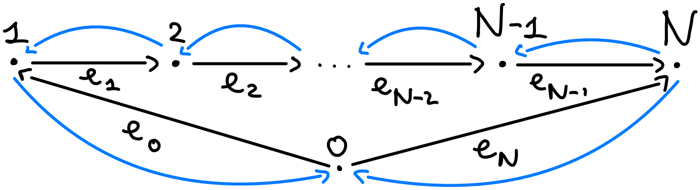
- Q = one-loop quiver : 

$\Longrightarrow Q^{db} =$ 

$\Longrightarrow \mathbb{C}Q^{db} = \mathbb{C}\langle e, e^* \rangle$

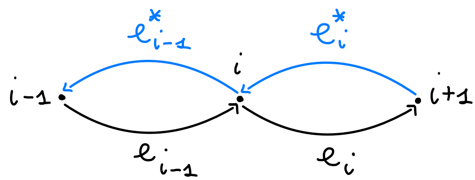
$\Longrightarrow \Pi_Q = \mathbb{C}\langle e, e^* \rangle \Big/ \underbrace{[e, e^*]}_{\text{preprojective rel.}} = \mathbb{C}[e, e^*]$

- Q^{db} :



$\Longrightarrow \sum_{e\in\Omega}(ee^* - e^*e)$ is equivalent to the family of rels

$$\{e_{i-1}e_{i-1}^* - e_i^*e_i \;:\; i=0,\dots,N\}$$



- a (finite-dimensional) representation of Π_Q is a (finite-dimensional) right Π_Q-module, i.e., the datum

$$\left(V=\bigoplus_{i\in I}V_i\,,\,\left(x_e : V_{s(e)}\longrightarrow V_{t(e)}\right)_{e\in\Omega^{db}}\right)$$

such that $\sum_{e\in\Omega}[x_e, x_{e^*}]=0$ (with each V_i finite-dim.).

Set

$$\boxed{\underline{\mathrm{Rep}}(\Pi_Q) := \text{moduli stack of finite-dimensional representations of } \Pi_Q}$$

Explicitly,
$$\underline{\mathrm{Rep}}(\Pi_Q) = \bigsqcup_{\underline{d}\in\mathbb{N}I} \underline{\mathrm{Rep}}(\Pi_Q;\underline{d})$$
(dimension vector)

for any $\underline{d}\in\mathbb{N}I$:
$$\underline{\mathrm{Rep}}(\Pi_Q;\underline{d}) = \text{quotient stack } C_{\underline{d}}/GL_{\underline{d}}$$
where

- $GL_{\underline{d}} := \prod_{i\in I} GL_{d_i}(\mathbb{C})$ acts by conjugation on:
- $C_{\underline{d}}$ = commuting variety :=

$$\left\{ (A_e)_{e\in\Omega^{db}} \in \bigoplus_{e\in\Omega^{db}} \mathrm{Hom}\left(\mathbb{C}^{d_{s(e)}}, \mathbb{C}^{d_{t(e)}}\right) : \sum_{e\in\Omega} [A_e, A_{e^*}] = 0 \right\}$$

Example: Q = one-loop quiver

For $d\in\mathbb{N}$, we get
$$\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}};d) = C_d/GL_d(\mathbb{C})$$

where
$$C_d = \{(A_1, A_2) \in \mathrm{Mat}_d(\mathbb{C})^{\times 2} : [A_1, A_2] = 0\}$$
is the "usual" commuting variety.

The moduli stack $\underline{\mathrm{Rep}}(\Pi_Q)$ admits an action of the torus $T_{max} := (\mathbb{C}^*)^{\Omega} \times \mathbb{C}^*$ induced by
$$((t_e)_{e\in\Omega}, t) \cdot (A_e, A_{e^*})_{e\in\Omega} := (t_e A_e, t_e^{-1} t A_{e^*})_{e\in\Omega}$$

Schiffmann-Vasserot, Yang-Zhao:

Fix a subtorus $T \subseteq T_{max} = (\mathbb{C}^*)^{\Omega} \times \mathbb{C}^*$. Then

$\exists$ COHA_Q^T = T-equivariant cohomological Hall algebra associated to finite-dimensional rep.s of Π_Q

= associative $\mathbb{Z} \times \mathbb{N}I$-graded $H_T^*(pt)$-algebra structure on

$$\underline{H^{T}_{*}}\left(\underline{\mathrm{Rep}}(\Pi_Q)\right) = \bigoplus_{\underline{d}\in \mathbb{N}I} \underline{H^{GL_{\underline{d}}\times T}_{*}}\left(C_{\underline{d}}\right)$$

(equivariant Borel-Moore homology)

where the multiplication $p_* \circ q^!$ is induced by

$$\underline{\mathrm{Rep}}(\Pi_Q)\times\underline{\mathrm{Rep}}(\Pi_Q) \xleftarrow{\;q\;} \underline{\mathrm{Rep}}^{ext}(\Pi_Q) \xrightarrow{\;p\;} \underline{\mathrm{Rep}}(\Pi_Q)$$

(stack of extensions)

Here,

$$p:\; 0 \longrightarrow E_2 \longrightarrow E \longrightarrow E_1 \longrightarrow 0 \longmapsto E$$

$$q:\; \text{———————— // ————————} \longmapsto (E_2, E_1)$$

Remarks

1. p is a proper representable map $\Longrightarrow \exists\; p_*$

2. Since $\underline{\mathrm{Rep}}(\Pi_Q)$ is singular, constructing

$$q^! = \text{Gysin pullback}$$

is technically nontrivial

It can be defined either:

- by using $\underline{\mathrm{Rep}}(\Pi_Q;\underline{d}) = C_{\underline{d}}/GL_{\underline{d}} \hookrightarrow T^*\mathbb{A}_{\underline{d}}/GL_{\underline{d}}$ (affine space: $\mathbb{A}_{\underline{d}}$)

and Fulton's construction in equivariant BM homology
(viewpoint of Schiffmann-Vasserot, Yang-Zhao)

or

- by observing that $\exists$ derived enhancements

$$\underline{\mathbb{R}\mathrm{Rep}}(\Pi_Q) \quad \text{and} \quad \underline{\mathbb{R}\mathrm{Rep}}^{ext}(\Pi_Q)$$

$\Longrightarrow \mathbb{R}q$ is quasi-smooth $\Longrightarrow \exists\, \mathbb{R}q^!$
(viewpoint of Varagnolo-Vasserot, Diaconescu-Porta-S.)

3. $H_*^T \rightsquigarrow G_0^T$ = T-equivariant G_0-theory ;
—————//————— Chow groups ; ...

4. $H_*^T \rightsquigarrow D^b\mathrm{Coh}(-)$
(following Pădurariu, Varagnolo-Vasserot, Diaconescu-Porta-S.)

We may also define a "nilpotent variant" of 2d COHAs.
Assume that Q has no edge-loops.

Let I be the two-sided ideal of Π_Q which is generated by all edges of Q^{db}.

Definition

A repr. M of Π_Q is nilpotent if $\exists\, m \in \mathbb{N}$ such that $M I^m = 0$.

Set

$$\boxed{\Lambda_Q := \text{moduli stack of finite-dimensional nilpotent representations of } \Pi_Q}$$

Remarks

- Λ_Q is a reduced closed substack of $\underline{\mathrm{Rep}}(\Pi_Q)$
- $\Lambda_Q = \bigsqcup_{\underline{d} \in \mathbb{N}I} \Lambda_{\underline{d}}$

Schiffmann-Vasserot:

$$\boxed{\exists\ \mathrm{COHA}_Q^{T,nil} = T\text{-equivariant COHA associated to } \Lambda_Q}$$