Cohomological Hall algebras of 1-dimensional sheaves and Yangians

DPSSV-1, arXiv: 2502.19445
Diaconescu-Porta-S.-Schiffmann-Vasserot,
Nilpotent cohomological Hall algebras of surfaces

DPSSV-2, arXiv: 2603.03386
Diaconescu-Porta-S.-Schiffmann-Vasserot,
Cohomological Hall algebras of one-dimensional sheaves on surfaces and Yangians

Lecture 2

A difference between $COHA^T_Q$ and $COHA^{T,nil}_Q$

- Schiffmann-Vasserot: $COHA^{T,nil}_Q$ is generated by the subspaces

$$COHA^{T,nil}_{Q,\alpha_i} := \alpha_i\text{-graded piece of } COHA^{T,nil}_Q$$

as i varies over I. Here, α_i = i-th coordinate vector of $\mathbb{N}I$

- Davison: $COHA^T_Q$ is not generated by $COHA^T_{Q,\alpha_i}$ unless Q = finite ADE quiver

Remark: If Q has edge-loops, we need to replace

"nilpotent" $\rightsquigarrow$ "(strongly) semi-nilpotent"

(following Bozec-Schiffmann-Vasserot)

1. What is the "quantum" nature of COHA_Q^T and $\mathrm{COHA}_Q^{T,nil}$?

Fix a subgroup $T \subseteq T_{max} = (\mathbb{C}^*)^{\Omega} \times \mathbb{C}^*$.

Maulik-Okounkov defined a Hopf algebra $/ H_T^*(pt)$

$$\boxed{\mathbb{Y}_{Q,T}^{MO} = \text{Yangian of } Q}$$

for which $\exists$ an increasing $\mathbb{N}$-filtration $F_\bullet$ of $\mathbb{Y}_{Q,T}^{MO}$ such that

$$\mathrm{gr}_{F_\bullet} \mathbb{Y}_{Q,T}^{MO} \simeq U(\mathfrak{g}_{Q,T}^{MO}[u])$$

Here, $\mathfrak{g}_{Q,T}^{MO}$ = Maulik-Okounkov Lie algebra of Q

Facts:

- Botta-Davison: $\mathfrak{g}_{Q,T}^{MO} \simeq \mathfrak{g}_Q^{MO} \otimes H_T^*(pt)$
- Schiffmann-Vasserot: $\exists$ a triangular decomposition

$$\mathbb{Y}_{Q,T}^{MO,+} \otimes_{H_T^*(pt)} \mathbb{Y}_{Q,T}^{MO,0} \otimes_{H_T^*(pt)} \mathbb{Y}_{Q,T}^{MO,-} \xrightarrow{\sim} \mathbb{Y}_{Q,T}^{MO}$$

The following has been proved by Schiffmann-Vasserot, Neguț for $Q = Q_{\cdot}$, and by Botta-Davison, Schiffmann-Vasserot for arbitrary quivers:

Theorem

Let Q be a quiver and let $T \subseteq T_{max} = (\mathbb{C}^*)^{\Omega} \times \mathbb{C}^*$ be a torus.
$\exists$ an isomorphism of $H_T^*(pt)$-algebras

$$COHA_Q^T \xrightarrow{\sim} \mathbb{Y}_{Q,T}^{MO,+} \quad \text{and} \quad COHA_Q^{T,nil} \xrightarrow{\sim} \mathbb{Y}_{Q,T}^{MO,-}$$

What's next?

We give a different presentation of $COHA_Q^{T,nil}$ when Q has no edge-loops

From now on, we fix a quiver $Q = (I, \Omega)$, which has no edge-loops.

2. Multi-parameter Yangian

Recall that

$$T_{max} := \left\{ (\gamma_e) \in (\mathbb{C}^*)^{\Omega^{db}} : \gamma_e \gamma_{e^*} = \gamma_f \gamma_{f^*} \quad \forall e, f \in \Omega \right\}$$

$$\Longrightarrow H^*_{T_{max}}(pt) \simeq \mathbb{Q}[\varepsilon_e : e \in \Omega^{db}] / \{ \varepsilon_e + \varepsilon_{e^*} = \hbar \quad \forall e \in \Omega \}.$$

We introduce

$$\boxed{\mathfrak{Z}_{i,j}(t) := \prod_{\substack{e \in \Omega^{db} \\ i \xrightarrow{e} j}} (t + \varepsilon_e) \qquad \forall i, j \in I, i \neq j}$$

<u>Definition</u> (DPSSV-2)

The multi-parameter Yangian $\mathbb{Y}_{Q,T_{max}}$ of Q is the unital associative $H^*_{T_{max}}(pt)$-algebra generated by

$$\boxed{x^{\pm}_{i,\ell} \text{ and } h_{i,\ell} \quad \text{with } i \in I \text{ and } \ell \in \mathbb{N}}$$

subject to the following rels:

- $\forall i,j \in I, \forall r,s \in \mathbb{N}$:

$$[h_{i,r}, h_{j,s}] = 0 \quad , \quad [x^+_{i,r}, x^-_{j,s}] = \delta_{i,j} h_{i,r+s}$$

- $\forall i \in I$:

$(*)$ $$(u-w\mp\hbar)h_i(u)x_i^\pm(w) \doteq (u-w\pm\hbar)x_i^\pm(w)h_i(u)$$

$(*)$ with $x_i^\pm(u)$ instead of $h_i(u)$

- $\forall i,j \in I, i\neq j$:

$$(**)\begin{cases} \zeta_{i,j}(u-w)\,h_i(u)x_j^+(w) \doteq \zeta_{i,j}(u-w-\hbar)x_j^+(w)h_i(u) \\ \zeta_{i,j}(u-w-\hbar)h_i(u)x_j^-(w) \doteq \zeta_{i,j}(u-w)\,x_j^-(w)h_i(u) \end{cases}$$

$(**)$ with $x_i^\pm(u)$ instead of $h_i(u)$

- Serre rels and cubic rels

Here,

$$h_i(u) := 1 + \hbar\sum_{\ell\in\mathbb{N}} h_{i,\ell}\,u^{-\ell-1} \quad \text{and} \quad x_i^\pm(u) := \sum_{\ell\in\mathbb{N}} x^\pm_{i,\ell}\,u^{-\ell-1}$$

and

$$\boxed{P_1(u_1,\dots,u_s) \doteq P_2(u_1,\dots,u_s)}$$

means an equality of the coefficients of each monomial involving strictly negative powers of the variables associated to the $x^{\pm}_{i,\ell}$'s.

Examples

- Fix $r,s\in\mathbb{N}$. Taking the $u^{-r-1}w^{-s-1}$ coefficient in $(*)$, we get

$$[h_{i,r+1}, x^{\pm}_{i,s}] - [h_{i,r}, x^{\pm}_{i,s+1}] = \pm\hbar\{h_{i,r}, x^{\pm}_{i,s}\}$$

(Here, $\{a,b\} := ab+ba$)

- Let $i,j\in I$, $i\neq j$. The $u^{-r-1}w^{-s-1}$ coefficient in $(**)$ is

  - if $\nexists\, i \xrightarrow{e} j$: $[h_{i,r+1}, x^{\pm}_{j,s}] - [h_{i,r}, x^{\pm}_{j,s+1}] = 0$

  - if $\exists!\, e\in\Omega$ joining i and j :

$$[h_{i,r+1}, x^{\pm}_{j,s}] - [h_{i,r}, x^{\pm}_{j,s+1}] = \mp\frac{\hbar}{2}\{h_{i,r}, x^{\pm}_{j,s}\} - m_{i,j}\frac{\varepsilon_e - \varepsilon_{e^*}}{2}[h_{i,r}, x^{\pm}_{j,s}]$$

where $m_{i,j} := \begin{cases} 1 & \text{if } i \longrightarrow j \in \Omega \\ -1 & \text{if } j \longrightarrow i \in \Omega \end{cases}$.

Notation

$\mathbb{Y}^{\pm}_{Q,T_{max}}$:= the subalgebra of $\mathbb{Y}_{Q,T_{max}}$ generated by $\{x^{\pm}_{i,\ell}\}$

$\mathbb{Y}^{0}_{Q,T_{max}}$:= ——— // ——— $\{h_{i,\ell}\}$

Lemma

$\exists$ a triangular decomposition

$$\mathbb{Y}^{+}_{Q,T_{max}} \otimes_{H^*_{T_{max}}(pt)} \mathbb{Y}^{0}_{Q,T_{max}} \otimes_{H^*_{T_{max}}(pt)} \mathbb{Y}^{-}_{Q,T_{max}} \xrightarrow{\sim} \mathbb{Y}_{Q,T_{max}}$$

Remark

Thanks to the Lemma, $\exists$ canonical projection

$$\boxed{pr : \mathbb{Y}_{Q,T_{max}} \longrightarrow \mathbb{Y}^{-}_{Q,T_{max}}}$$

3. Relation between $\mathbb{Y}^{-}_{Q,T_{max}}$ and $\mathbb{HA}^{T}_{Q}$

Define

$$\mathbb{S} := H^*_{T_{max}}(pt)\left[P_\ell(z_i) : i \in I,\ \ell \in \mathbb{N}\right]$$

Set

$$ch(z_i) := P_0(z_i) + \sum_{\ell \geq 1} \frac{P_\ell(z_i)}{\ell!}$$

Attention ⚠:
For $\ell \geq 1$, we shall think of $P_\ell(z_i)$ as the ℓ-th power sum in an infinite set of variables $z_{i,1}, z_{i,2}, \ldots$

$\forall \underline{d} \in \mathbb{N}I$ $\exists$ canonical algebra homomorphism

$$\boxed{\begin{aligned} i : \mathbb{S} &\longrightarrow H^*_{T_{max} \times GL(\underline{d})}(pt) \\ ch(z_i) &\longmapsto ch(\mathcal{V}_i) \end{aligned}}$$

Tautological v.b.

Moreover, $\exists$ $\cap : \bigoplus_{\underline{d} \in \mathbb{N}I} H^*_{T_{max} \times GL(\underline{d})}(pt) \curvearrowright COHA_Q^{T,nil}$

$\Longrightarrow$ action $\cap : \mathbb{S} \curvearrowright COHA_Q^{T,nil}$

Furthermore, $\exists$ action $\cap : \mathbb{S} \curvearrowright \mathbb{Y}_{Q,T_{max}}$ by

$$p_\ell(z_i) \cap x^\pm_{j,k} = \pm \delta_{i,j} x^\pm_{i,k+\ell} \text{ and } p_\ell(z_i) \cap h_{j,k} = 0$$

In order to state the relation between the COHA and the Yangian, I need to

- introduce a refined version of the generation result,
- twist the Hall product.

For any $i \in I$,

- $\alpha_i \in \mathbb{N}I$ is the i-th coordinate vector,
- Λ_{α_i} is the moduli stack of nilpotent reprs of Π_Q of dimension α_i,
- $[\Lambda_{\alpha_i}]$ denotes the fundamental class of Λ_{α_i}.

Theorem (Schiffmann-Vasserot)

The algebra $COHA_Q^{T,nil}$ is generated by $(z_{i,1})^{\ell} \cap [\Lambda_{\alpha_i}]$ for any $i \in I$ and $\ell \in \mathbb{N}$.

Now, we "twist" the Hall product. Denote by

$$m_{\underline{d}_1,\underline{d}_2} : COHA_{Q,\underline{d}_1}^{T,nil} \otimes COHA_{Q,\underline{d}_2}^{T,nil} \longrightarrow COHA_{Q,\underline{d}_1+\underline{d}_2}^{T,nil}$$

the $(\underline{d}_1,\underline{d}_2)$-th graded part of the Hall product.

Let $\langle -,- \rangle$ be the Euler form $= \sum_{k=0}^{2} (-1)^k \dim \mathrm{Ext}^k_{\mathbb{C}Q}(-,-)$

$\Longrightarrow$ it descends to a bilinear form on the Grothendieck group of nilpotent repr.s of Π_Q $(\simeq \mathbb{Z}I)$

Definition

The twisted COHA HA_Q^T is the algebra $COHA_Q^{T,nil}$ with the twisted product:

$$\boxed{m^{tw} := \bigoplus_{\underline{d}_1,\underline{d}_2} m^{tw}_{\underline{d}_1,\underline{d}_2} \quad \text{and} \quad m^{tw}_{\underline{d}_1,\underline{d}_2} := (-1)^{\langle \underline{d}_1,\underline{d}_2 \rangle} m_{\underline{d}_1,\underline{d}_2}}$$

Theorem (DPSSV-2)

$\exists$ a surjective $H^*_{T_{max}}(pt)$-algebra homomorphism

$$\Phi : \mathbb{Y}^-_{Q,T_{max}} \twoheadrightarrow \mathbb{HA}^{T_{max}}_Q$$

$$x^-_{i,\ell} \longmapsto (z_{i,1})^\ell \cap [\Lambda_{\alpha_i}]$$

intertwining the $\mathfrak{S}$-actions on $\mathbb{Y}^-_{Q,T_{max}}$ and $\mathbb{HA}^{T_{max}}_Q$

Attention ⚠ :

The proof uses the theory of shuffle algebras.