Cohomological Hall algebras of 1-dimensional sheaves and Yangians

DPSSV-1, arXiv: 2502.19445
Diaconescu-Porta-S.-Schiffmann-Vasserot,
Nilpotent cohomological Hall algebras of surfaces

DPSSV-2, arXiv: 2603.03386
Diaconescu-Porta-S.-Schiffmann-Vasserot,
Cohomological Hall algebras of one-dimensional sheaves on surfaces and Yangians

Lecture 3

1. Braid group actions on $\mathcal{Y}^-_{Q,T_{max}}$ and HA^T_Q

1.1 Weyl and braid groups of Q

Let us fix some notation.

- $R_Q := \mathbb{Z}I$ = root lattice
- $\Delta_Q := \{\alpha_i\ \text{i-th coordinate vector of } \mathbb{Z}I\ \forall i \in I\}$
  = set of simple roots
- $\check{X}_Q := (\mathbb{Z}I)^\vee$ = coweight lattice
- $\{\check{\omega}_i\}_{i\in I}$:= dual basis of Δ_Q
  = set of fundamental coweights
- canonical pairing $(-,-) : \check{X}_Q \times R_Q \longrightarrow \mathbb{Z}$
- $\check{\alpha}_i \in \check{X}_Q$ = coroot defined by

$$\boxed{(\check{\alpha}_i,\alpha_i)=2 \text{ and } (\check{\alpha}_i,\alpha_j):=-\#\{e:i\longrightarrow j \text{ or } e:j\longrightarrow i\}}$$

- $A=(a_{ij})_{ij}$ = Cartan matrix, with $a_{ij}=(\check{\alpha}_i,\alpha_j)$

$\Longrightarrow \mathfrak{g}_Q^{KM}$ = Kac-Moody Lie algebra of Q

Definition

The Weyl group of Q is the subgroup W_Q of $GL(R_Q)$ generated by $s_i:\alpha_j\longmapsto\alpha_j-(\check{\alpha}_j,\alpha_i)\alpha_i$

Remark

$W_Q\simeq\langle s_i : i\in I\rangle \Big/ {\scriptstyle s_i^2=1 \atop \text{braid rels}}$, where the braid rels are:

$$\boxed{\begin{array}{ll} s_is_j=s_js_i & \text{if } a_{ij}=0 \\ s_is_js_i=s_js_is_j & \text{if } a_{ij}=-1 \end{array}}$$

Notation: $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced expression if ℓ is minimal among all possible expressions of $w \in W_Q$. We call ℓ the length $\ell(w)$ of w.

Definition

The braid group of Q is $B_Q := \langle T_i : i \in I \rangle / \text{braid rels for } T_i$

Notation: If $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced expression, set

$$\boxed{T_w := T_{i_1} \cdots T_{i_\ell}}$$

1.2 $B_Q \curvearrowright Y_Q$

Since $\mathrm{ad}(x_{i,0}^{\pm})$ is a locally nilpotent operator on Y_Q, $\exists$ algebra homomorphism

$$\boxed{\begin{aligned} B_Q &\longrightarrow \mathrm{Aut}(Y_Q) \\ T_i &\longmapsto \exp(\mathrm{ad}(x_{i,0}^{+}))\exp(\mathrm{ad}(x_{i,0}^{-}))\exp(\mathrm{ad}(x_{i,0}^{+})) \end{aligned}}$$

Now, $\forall w \in W_Q$ define

$$\boxed{\overline{T}_w : Y_Q^- \hookrightarrow Y_Q \xrightarrow{T_w} Y_Q \xrightarrow{pr} Y_Q^-}$$

We call $\overline{T}_w$ the truncated braid operator ass. to w

Let $B_Q^+ \subset B_Q$ be the monoid generated by the elements T_w $\forall w \in W_Q$.

Lemma (DPSSV-2)

$\exists\, B_Q^+ \curvearrowright Y_Q^-$ given by the assignment $T_w \longmapsto \overline{T}_w$.

1.3 $B_Q \curvearrowright HA_Q^T$

Consider the bounded derived category $D^b(\mathrm{mod}\,\Pi_Q)$ of finite-dimensional reprs of Π_Q.

Let S_i be the repr. of Π_Q given by:

$$V_i = \mathbb{C},\ V_j = 0\ \ \forall j \neq i, \text{ and } x_e = 0\ \ \forall e \in \Omega^{db}$$

Facts:

- S_i is simple
- $\mathbb{R}\mathrm{Hom}(S_i, S_i) \simeq \mathbb{C} \oplus \mathbb{C}[-2]$ i.e., S_i is 2-spherical

Definition

The twist functor T_{S_i} is the endofunctor defined as

$$\mathbb{R}\mathrm{Hom}(S_i, M) \overset{\mathbb{L}}{\underset{\mathbb{C}}{\otimes}} S_i \xrightarrow{ev} M \longrightarrow T_{S_i}(M)$$

while the dual twist functor T'_{S_i} is the endofunctor defined as

$$T'_{S_i}(M) \longrightarrow M \longrightarrow \mathbb{R}\mathrm{Hom}(M, S_i)^\vee \overset{\mathbb{L}}{\underset{\mathbb{C}}{\otimes}} S_i$$

Facts

- Seidel-Thomas: T_{S_i} is an equivalence with inverse given by T'_{S_i}
- Buan-Iyama-Reiten-Scott: the assignments

$$T_i \longmapsto T'_{S_i} \quad \text{and} \quad T_i^{-1} \longmapsto T_{S_i}$$

define a group homomorphism

$$B_Q \longrightarrow \mathrm{Aut}(D^b(\mathrm{mod}\,\Pi_Q))$$

$$\Longrightarrow B_Q^+ \curvearrowright \Lambda_Q \Longrightarrow B_Q^+ \curvearrowright HA_Q^T$$

<u>Attention ⚠</u>: In the final lecture, we will see the relation between these actions via the map Φ.