Cohomological Hall algebras of 1-dimensional sheaves and Yangians

DPSSV-1, arXiv: 2502.19445
Diaconescu-Porta-S.-Schiffmann-Vasserot,
Nilpotent cohomological Hall algebras of surfaces

DPSSV-2, arXiv: 2603.03386
Diaconescu-Porta-S.-Schiffmann-Vasserot,
Cohomological Hall algebras of one-dimensional sheaves on surfaces and Yangians

Lecture 4

1. Lie theory associated to affine quivers

Fix

- $Q_{fin} = (I_{fin} = \{1, 2, \dots, e\}, \Omega_{fin}) =$ finite ADE quiver
- $Q = (I := \{0\} \cup I_{fin}, \Omega) =$ affine ADE quiver

Example (type A)

$Q_{fin} = A_N$-quiver :

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r] & \overset{2}{\bullet} \arrow[r] & \cdots \arrow[r] & \overset{N-1}{\bullet} \arrow[r] & \overset{N}{\bullet} \end{tikzcd}$$

$Q = A_N^{(1)}$-quiver :

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r] & \overset{2}{\bullet} \arrow[r] & \cdots \arrow[r] & \overset{N-1}{\bullet} \arrow[r] & \overset{N}{\bullet} \\ & & \overset{0}{\bullet} \arrow[ull] \arrow[urr] & & \end{tikzcd}$$

We have:

$R_{fin} := R_{Q_{fin}}$

$\Delta_{fin} := \Delta_{Q_{fin}} = \{\alpha_1, \dots, \alpha_e\}$

$R := R_Q$

$\Delta := \Delta_Q = \{\alpha_0, \alpha_1, \dots, \alpha_e\}$

$$\check{X}_{fin} := \check{X}_{Q_{fin}}$$

$$\check{R}_{fin} := \bigoplus_{i=1}^{\ell} \mathbb{Z}\check{\alpha}_i$$

$$W_{fin} := W_{Q_{fin}}$$

$$B_{fin} := B_{Q_{fin}}$$

δ = indivisible positive imaginary root

$$= \sum_{i=0}^{\ell} r_i \alpha_i \quad (r_0 = 1)$$

$$\check{X} = \check{X}_Q$$

$$\check{R}_{fin} \hookrightarrow \check{X}_{fin} \hookrightarrow \check{X}$$

via the Cartan pairing

since $R \simeq \mathbb{Z}\delta \oplus R_{fin}$

$$W := W_Q$$

$$B := B_Q$$

Facts

- $W_{fin} \subset W$, $B_{fin} \subset B$
- $W = W_{fin} \ltimes \check{R}_{fin}$

Definition

The extended affine Weyl group is

$$W_{ex} := W_{fin} \ltimes \check{X}_{fin}$$

Fact

$W_{ex} = \Gamma \ltimes W$, where Γ is the group of automorphisms of the underlying diagram of Q

$\Longrightarrow$ length $\ell(\gamma \ltimes w) := \ell(w)$

Definition

The extended affine braid group is

$$B_{ex} := \langle \{ T_w : w \in W_{ex} \} \mid T_v T_w = T_{vw} \text{ whenever } \ell(vw) = \ell(v) + \ell(w) \rangle$$

Facts

- $B_{ex} = \Gamma \ltimes B$

- $\mathbb{B}_{ex}$ is generated by $\mathbb{B}_{fin}$ and $\{L_{\check{\lambda}} : \check{\lambda} \in \check{X}_{fin}\}$ subject to the rels:

$$\boxed{\begin{aligned} & L_{\check{\lambda}_1} L_{\check{\lambda}_2} = L_{\check{\lambda}_2} L_{\check{\lambda}_1} \\ & T_i L_{\check{\lambda}} = L_{\check{\lambda}} T_i \qquad \text{if } s_i(\check{\lambda}) = \check{\lambda} \\ & L_{\check{\lambda}} = T_i L_{\check{\lambda}-\check{\alpha}_i} T_i \quad \text{if } s_i(\check{\lambda}) = \check{\lambda} - \check{\alpha}_i \end{aligned}}$$

- The actions of $\mathbb{B}^+$ on Yangians and COHAs extend to $\mathbb{B}^+_{ex}$-actions.

2. Affine Yangians

Definition

The affine Yangian associated to Q is

$$\mathbb{Y} = \mathbb{Y}_{Q,\varepsilon_1,\varepsilon_2} := \mathbb{Y}_{Q,T_{max}} \otimes_{H^*_{T_{max}}(pt)} H^*_T(pt)$$

where

- $T := (\mathbb{C}^*)^{\times 2} \hookrightarrow T_{max}$
  $(\gamma_1, \gamma_2) \longmapsto (\gamma_e = \gamma_1, \gamma_{e^*} = \gamma_e)_{e \in \Omega}$
  $\Longrightarrow H^*_{T_{max}}(pt) \longrightarrow H^*_T(pt)$
- $\varepsilon_1 = \varepsilon_e$ and $\varepsilon_2 = \varepsilon_{e^*}$ $\forall e \in \Omega$.

Proposition (DPSSV-2)
$\mathbb{Y}$ coincides with the affine Yangian of Kodera and Bershtein-Tsymbaliuk.

Now, we describe the classical limit of $\mathbb{Y}$.

Consider

- $\mathfrak{g}_{fin} := \mathfrak{g}_{Q_{fin}}$
- $\mathfrak{g} := \mathfrak{g}_Q$ = the universal central extension of the Lie algebra $\mathfrak{g}_{fin} \otimes \mathbb{Q}[s^{\pm 1}]$

Define the elliptic Lie algebra $\mathfrak{g}_{ell}$ as follows:

$$\boxed{\mathfrak{g}_{ell} := \text{the universal central extension of the Lie algebra } \mathfrak{g}_{fin} \otimes \mathbb{Q}[s^{\pm 1}, t]}$$

As v.s.:

$$\mathfrak{g}_{ell} = \left(\mathfrak{g}_{fin} \otimes \mathbb{Q}[s^{\pm 1}, t]\right) \oplus \left(\bigoplus_{\ell \in \mathbb{N}} \mathbb{Q} c_\ell \oplus \bigoplus_{\substack{\ell \in \mathbb{N}, \ell \geq 1 \\ k \in \mathbb{Z}, k \neq 0}} \mathbb{Q} c_{k,\ell} \right)$$

central

Proposition (Guay-Regelskis-Wendlandt, DPSSV-2)
$\exists$ an increasing $\mathbb{N}$-filtration $F^\bullet$ of $\mathbb{Y}$ such that

$$\mathrm{gr}_{F^\bullet} \mathbb{Y} \simeq U(\mathfrak{g}_{ell}) \otimes_{\mathbb{Q}} H_T^*(pt) \quad \text{as algebras}$$

A similar result also holds for $\mathbb{Y}^-$.

Let

- $\mathfrak{n}_{fin} :=$ the 'standard' negative half of $\mathfrak{g}_{fin}$

- $\mathfrak{n} :=$ the 'standard' negative half of $\mathfrak{g}_{aff}$
$= s^{-1}\mathfrak{g}_{fin}[s^{-1}] \oplus \mathfrak{n}_{fin}$

- $\mathfrak{n}_{ell} := \mathfrak{n}[t] \oplus \bigoplus_{k<0, \ell \geq 1} c_{k,\ell}$

Proposition (DPSSV-2)

$\exists$ an increasing $\mathbb{N}$-filtration $F^\bullet$ of $\mathbb{Y}^-$ such that

$$\mathrm{gr}_{F^\bullet} \mathbb{Y}^- \simeq U(\mathfrak{n}_{ell}) \otimes_{\mathbb{Q}} H^*_T(pt) \quad \text{as algebras}$$

3. Relation between $\mathbf{HA}^T_Q$ and $\mathbb{Y}^-_{Q,T}$ when Q is affine ADE

Let us consider the 2d COHA $\mathbf{HA}^T := \mathbf{HA}^T_Q$ of Q with respect to the torus $T := (\mathbb{C}^*)^{\times 2}$

Theorem (DPSSV-2)

The canonical homomorphism $\Phi : \mathbb{Y}^- \longrightarrow \mathbf{HA}^T$ is an isomorphism of unital associative algebras.

Remarks

- DPSSV-2: $\mathbb{Y}^- \simeq \mathbb{Y}^{MO,-}_{Q,T}$ as algebras
- Jindal: ${}^J\widetilde{\mathbb{Y}}^+ \simeq \mathbb{Y}^{MO,+}_{Q,T}$ ——— // ——— (only in type A)

Here ${}^J\widetilde{\mathbb{Y}}$ is obtained from $\mathbb{Y}$ by adding an additional set of gen.s and rel.s

(due to the fact that $COHA^T_Q$ is not generated by the $COHA^T_{Q,\alpha_i}$ $\forall i \in I$)

- Jindal: ${}^J\widetilde{\mathbb{Y}}^e \simeq \mathbb{Y}^{MO}_{Q,T}$ as algebras (only in type A)

(adding additional central elements)

- Nakajima: $\mathbb{Y}^{MO}_{Q,T}$ is isomorphic, as a Hopf algebra, to the "extended affine Yangian" ${}^N\mathbb{Y}^e$, in terms of R-matrices, for $Q \neq A^{(1)}_1$.