Cohomological Hall algebras of 1-dimensional sheaves and Yangians

DPSSV-1, arXiv:2502.19445
Diaconescu-Porta-S.-Schiffmann-Vasserot,
Nilpotent cohomological Hall algebras of surfaces

DPSSV-2, arXiv:2603.03386
Diaconescu-Porta-S.-Schiffmann-Vasserot,
Cohomological Hall algebras of one-dimensional sheaves on surfaces and Yangians

Lecture 5

1. COHAs of surfaces

Geometry		Algebra
Moduli stack	$\longrightarrow$ COHA $\longrightarrow$	Associative algebra

S = smooth quasi-projective surface $/\mathbb{C}$

T = (possibly trivial) torus $\curvearrowright S$

$\boxed{\underline{\mathrm{Coh}}_{ps}(S) = \text{moduli stack of properly supported coherent sheaves on } S}$

Remark

We can also define

- $\underline{\mathrm{Coh}}_0(S) \subset \underline{\mathrm{Coh}}_{ps}(S) \longleftrightarrow$ 0-dim. sheaves
- $\underline{\mathrm{Coh}}_{\leq 1}(S) \subset \underline{\mathrm{Coh}}_{ps}(S) \longleftrightarrow$ sheaves of dim ≤ 1

Attention ⚠

$\exists$ a derived enhancement

$$\underline{\mathbb{R}\mathrm{Coh}}_{ps}(S)\times\underline{\mathbb{R}\mathrm{Coh}}_{ps}(S)\xleftarrow{\mathbb{R}q}\underline{\mathbb{R}\mathrm{Coh}}^{ext}_{ps}(S)\xrightarrow{\mathbb{R}p}\underline{\mathbb{R}\mathrm{Coh}}_{ps}(S)$$

such that $\mathbb{R}q$ is quasi-smooth $\Longrightarrow \exists\, (\mathbb{R}q)^!$

Kapranov-Vasserot, Yu Zhao in dim $=0$, Porta-S. :

$$\boxed{\exists\ \mathrm{COHA}^{(T)}_S = \text{(T-equivariant) COHA associated to properly supported sheaves on } S}$$

$=$ associative $H^*_T(pt)$-algebra structure on

$$H^{(T)}_*\left(\underline{\mathbb{R}\mathrm{Coh}}_{ps}(S)\right)=H^{(T)}_*\left(\underline{\mathrm{Coh}}_{ps}(S)\right)$$

with multiplication $(\mathbb{R}p)_*\circ(\mathbb{R}q)^!$.

Remark

- $\exists\ \mathrm{COHA}^{(T)}_{S,0\text{-dim}}$ associated to $\underline{\mathrm{Coh}}_0(S)$
- $\exists\ \mathrm{COHA}^{(T)}_{S,\leq 1}$ associated to $\underline{\mathrm{Coh}}_{\leq 1}(S)$

Remark (0-dimensional sheaves)

By the work of

- Schiffmann-Vasserot, Neguț for $S=\mathbb{C}^2$,

- Mellit-Minets-Schiffmann-Vasserot in general

$\mathrm{COHA}^{(T)}_{S,0\text{-dim}}$ has an explicit presentation in terms of gen.s and rel.s

Furthermore, when S has a trivial canonical bundle, $\mathrm{COHA}^{T}_{S,0\text{-dim}}$ is a Yangian.

On the other hand, a description of $\mathrm{COHA}^{(T)}_{S,\leq 1}$ in terms of gen.s and rel.s is out of reach at the moment

For this reason, we now introduce a "nilpotent" variant of the construction of COHAs of surfaces.

3. Nilpotent COHAs of surfaces

S = smooth quasi-projective surface $/\mathbb{C}$

$T =$ (possibly trivial) torus $\curvearrowright S$

$C \subset S$ T-invariant closed reduced subscheme

Consider

$$\boxed{\underline{Coh}_C(S) = \text{moduli stack of coherent sheaves on } S \text{ set-theoretically supported on } C}$$

$\longrightarrow$ sheaf analog of nilpotency

Example

$S = T^*X$, $X =$ smooth projective curve $/\mathbb{C}$

$C = X \subset T^*X$ zero section

$\Longrightarrow$ $\underline{Coh}_X(T^*X) =$ moduli stack of Higgs sheaves $(\mathcal{F}, \phi : \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega^1_X)$ on X, such that ϕ is nilpotent

The stack $\underline{\mathbb{R}Coh}_C(S)$ admits two equivalent definitions:

① Let $i : U := S \setminus C \hookrightarrow S$. Then $\underline{\mathbb{R}\mathrm{Coh}}_C(S)$ is

$$\begin{tikzcd}
\underline{\mathbb{R}\mathrm{Coh}}_C(S) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \underline{\mathbb{R}\mathrm{Coh}}(S) \arrow[d, "i^*"] \\
* \arrow[r, "0"] & \underline{\mathbb{R}\mathrm{Coh}}(U)
\end{tikzcd}$$

② Let I_C be the ideal sheaf of C.

Set $C^{(n)} := \mathrm{Spec}_S\left(\mathcal{O}_S / I_C^{n+1}\right)$. Then

$$\underline{\mathbb{R}\mathrm{Coh}}_C(S) = \operatorname*{colim}_n \underline{\mathbb{R}\mathrm{Coh}}(C^{(n)})$$

Theorem (DPSSV-1)

These two definitions are equivalent.

Using these equivalent realizations of $\underline{\mathbb{R}\mathrm{Coh}}_C(S)$, we were able to prove:

Theorem (DPSSV-1)

a. $\exists$ an associative $H_T^*(pt)$-algebra structure on

$$\mathrm{COHA}^{(T)}_{S,C} := H^{(T)}_*(\underline{\mathrm{Coh}}_C(S))$$

b. $\mathrm{COHA}^{(T)}_{S,C}$ depends "locally" on C,

i.e., given (S_1, C_1) and (S_2, C_2) such that the formal completions $\widehat{(S_1)}_{C_1} \simeq \widehat{(S_2)}_{C_2}$, we have:

$$\mathrm{COHA}^{(T)}_{S_1,C_1} \simeq \mathrm{COHA}^{(T)}_{S_2,C_2}$$

Remark

We can define the algebra $\mathrm{COHA}^{(T)}_{S,C,0\text{-dim}}$ associated to 0-dimensional sheaves; it can be described by gen.s and rel.s

4. The derived McKay correspondence

- $G \subset SL(2,\mathbb{C})$ finite subgroup

$\updownarrow$

Q_{fin} = finite ADE quiver, Q = affine ADE quiver

- $\pi : S \longrightarrow \mathbb{C}^2/G$ Kleinian resolution of singularities
- $C := \pi^{-1}(0)$. Then, $C_{red} = C_1 \cup \cdots \cup C_e$, $C_i \simeq \mathbb{P}^1$

  $(C_i \cdot C_j) = -$ Cartan matrix of Q_{fin}
- torus $T \subset GL(2, \mathbb{C})$ centralizing G

  ($T =$ trivial, $\mathbb{C}^*$, or $\mathbb{C}^* \times \mathbb{C}^*$)

Example: $G = \mathbb{Z}_2 \Longrightarrow Q_{fin} = A_1 = \cdot$, $Q = A_1^{(1)}$: 

$\Longrightarrow S = T^*\mathbb{P}^1 \supset C = \mathbb{P}^1 =$ zero section

Now, I need to recall the following notion:

Definition

A vector bundle $\mathcal{P}$ on S is tilting if:

- $\mathrm{Ext}^i(\mathcal{P}, \mathcal{P}) = 0 \quad \forall i \neq 0$,

- $\mathcal{P}$ is a generator of $D^b\mathrm{Coh}(S)$, i.e.,

$$\forall E \in D^b\mathrm{Coh}(S),\ \mathbb{R}\mathrm{Hom}(\mathcal{P}, E) = 0 \implies E = 0$$

Given a tilting bundle $\mathcal{P}$ on S, the functors

$$\boxed{\begin{aligned} \mathbb{R}\mathrm{Hom}(\mathcal{P}, -) &: D^b\mathrm{Coh}(S) \longrightarrow D^b\mathrm{Mod}(\mathrm{End}(\mathcal{P})) \\ (-) \overset{\mathbb{L}}{\otimes}_{\mathrm{End}(\mathcal{P})} \mathcal{P} &: D^b\mathrm{Mod}(\mathrm{End}(\mathcal{P})) \longrightarrow D^b\mathrm{Coh}(S) \end{aligned}}$$

are T-equivariant quasi-inverse equivalences.

Van der Bergh proved that $\exists$ a tilting bundle $\mathcal{P}$ on S s.t.

$$\boxed{\Pi_Q = \mathrm{End}(\mathcal{P})}$$

Then,

$$\boxed{\tau := \mathbb{R}\mathrm{Hom}(\mathcal{P}, -) : D^b\mathrm{Coh}(S) \longrightarrow D^b\mathrm{Mod}(\Pi_Q)}$$

is the derived McKay equivalence.

Remarks

- Kapranov-Vasserot first defined τ using Fourier-Mukai transforms.
- τ restricts to T-equivariant equivalences:

$$D^b\mathrm{Coh}_{ps}(S) \longrightarrow D^b\mathrm{mod}(\Pi_Q)$$

$$D^b\mathrm{Coh}_c(S) \longrightarrow D^b\mathrm{nilp}(\Pi_Q)$$

- τ induces $K_0(\mathrm{Coh}_c(S)) \xrightarrow{\sim} K_0(\mathrm{nilp}(\Pi_Q)) \simeq R =$ root lattice of Q