Cohomological Hall algebras of 1-dimensional sheaves and Yangians

<u>DPSSV-1</u>, arXiv:2502.19445

Diaconescu-Porta-S.-Schiffmann-Vasserot,
Nilpotent cohomological Hall algebras of surfaces

<u>DPSSV-2</u>, arXiv:2603.03386

Diaconescu-Porta-S.-Schiffmann-Vasserot,
Cohomological Hall algebras of one-dimensional sheaves on surfaces and Yangians

Lecture 6

1. COHAs and τ

Notation: we denote by $\mathbf{HA}^{(T)}_{S,C}$ the algebra $\mathbf{COHA}^{(T)}_{S,C}$ with the twisted Hall product

Attention ⚠: τ does not induce

$$
\begin{array}{ccc}
\mathrm{Coh}_C(S) & \cancel{\xrightarrow{\sim}} & \mathrm{nilp}(\Pi_Q) \\
 & \Downarrow & \\
\underline{\mathrm{Coh}}_C(S) & \cancel{\simeq} & \Lambda_Q \\
 & \Downarrow & \\
\mathbf{HA}^T_{S,C} & \cancel{\simeq} & \mathbf{HA}^T_Q \simeq \mathbb{Y}^-
\end{array}
$$

In the remaining lectures, we will establish a subtle relation between $\mathbf{HA}^T_{S,C}$ and $\mathbb{Y}^-$:

Main Theorem (DPSSV-2)

Let $\check{\theta} = \sum_i \check{\theta}_i \check{\omega}_i \in \check{X}$ be such that $\check{\theta}_i > 0 \ \forall i \neq 0$ and $\check{\theta}_0 \doteq -\sum_{i \neq 0} \check{\theta}_i r_i$. Then

1. The topological vector space

$$\boxed{\mathbb{Y}_{S,C} := \lim_{\ell} \overline{L}^{-2\ell}_{\check{\theta}_f}(\mathbb{Y}^-) / \overline{L}^{-2\ell}_{\check{\theta}_f}(J)}$$

where $J := \sum_{(\check{\theta}, \underline{d}) \leq 0} \mathbb{Y}^-_{-\underline{d}} \cdot \mathbb{Y}^-$ and $\check{\theta}_f := \sum_{i \neq 0} \check{\theta}_i \check{\omega}_i$, has the structure of an associative graded $H^*_T(pt)$-algebra.

2. $\exists$ a canonical $H^*_T(pt)$-algebra isomorphism

$$\mathbb{HA}^T_{S,C} \simeq \mathbb{Y}_{S,C}$$

3. $\exists$ a canonical algebra isomorphism

$$\mathbb{HA}_{S,C} \simeq \widehat{U}(\mathfrak{n}^{S,C}_{ell})$$

where $\mathfrak{n}^{S,C}_{ell} :=$ Lie subalgebra of $\mathfrak{g}_{ell}$ corresp. to the submonoid $K_0(Coh_C(S))^+$ of R consisting of effective classes, i.e.,

$$\mathfrak{n}^{S,C}_{ell} = \mathfrak{n}_{fin}[s^{\pm 1}, t] \oplus s^{-1}\mathfrak{h}_{fin}[s^{-1}, t] \oplus \bigoplus_{k<0} \mathbb{Q}\, c_{k,\ell}$$

Here, $h_{fin} :=$ Cartan subalgebra of g_{fin} and $\forall$ Lie algebra $g = \bigoplus_{\underline{d} \in \mathbb{N}I} g_{\underline{d}}$, $\hat{U}^{\check{\theta}}(g) := \lim_{k} U(g) / \left(\bigoplus_{(\check{\theta}, \underline{d}) < k(\check{\rho}, \underline{d})} g_{\underline{d}} \right) \cdot U(g)$

Now, we introduce the following tools:

- limiting COHAs
- Bridgeland's stability conditions
- B_{ex}-action and stability conditions

2. Limiting COHAs

The first notion we need is that of a slicing. We will justify this notion starting from t-structures.

Let $\mathcal{C}$ be a triangulated category.

Definition

A t-structure on $\mathcal{C}$ is a full subcategory $D \subset \mathcal{C}$, satisfying $D[1] \subset D$, such that for any $E \in \mathcal{C}$ $\exists$ a triangle

$$\begin{tikzcd} F \arrow[rr] & & E \arrow[dl] \\ & G \arrow[ul, dashed, "{[1]}"] & \end{tikzcd}$$

with $F \in \mathcal{D}$ and $G \in \mathcal{D}^{\perp}$.

Here, $\mathcal{D}^{\perp} := \{ M \in \mathcal{C} : \operatorname{Hom}_{\mathcal{C}}(F, M) = 0 \ \forall F \in \mathcal{D} \}$.

The heart of the t-structure $\mathcal{D} \subset \mathcal{C}$ is

$$\mathcal{C}^{\heartsuit} := \mathcal{D} \cap \left(\mathcal{D}^{\perp}[1] \right)$$

A t-structure $\mathcal{D} \subset \mathcal{C}$ is bounded if

$$\mathcal{C} = \bigcup_{i,j \in \mathbb{Z}} \left(\mathcal{D}[i] \right) \cap \left(\mathcal{D}^{\perp}[j] \right)$$

Remark

$\mathcal{C}^{\heartsuit}$ is always an abelian category.

Note that bounded t-structures are uniquely determined by their hearts:

Lemma

Let $\mathcal{A} \subset \mathcal{C}$ be a full subcategory.
Then, $\mathcal{A}$ is the heart of a bounded t-structure $\iff$ the following two conditions hold:

1. $\forall\, k_1 > k_2$ integers, $\mathrm{Hom}_{\mathcal{C}}(A_1, A_2) = 0$ for $A_i \in \mathcal{A}[k_i]$

2. $\forall\, 0 \neq E \in \mathcal{C}$, $\exists\, k_1 > k_2 > \cdots > k_n$ integers and

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[rr] & & E_2 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& \mathcal{A}[k_1] \ni A_1 \arrow[ul, dashed, "{[1]}"] & & A_2 \in \mathcal{A}[k_2] \arrow[ul, dashed, "{[1]}"] & & & & A_n \in \mathcal{A}[k_n] \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

(called the Harder-Narasimhan filtration of E).

The previous lemma justifies the following:

Definition

A slicing $\mathcal{P}$ of $\mathcal{C}$ consists of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{C}$ for any $\phi \in \mathbb{R}$ such that:

a. $\forall\, \phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,

b. property (1) of the lemma holds after replacing
$K_i \rightsquigarrow \phi_i \in \mathbb{R}$ and $\mathcal{A}[K_i] \rightsquigarrow \mathcal{P}(\phi_i)$

c. property (2) of the lemma holds after replacing
$K_i \rightsquigarrow \phi_i \in \mathbb{R}$ and $\mathcal{A}[K_i] \rightsquigarrow \mathcal{P}(\phi_i)$

Notation: $\forall\, I \subset \mathbb{R}$ interval, $\mathcal{P}(I) := \langle \mathcal{P}(\phi) : \phi \in I \rangle$.

Remarks

- If I has length 1, $\mathcal{P}(I)$ is abelian.

- $\forall\, \phi \in \mathbb{R}$, $\mathcal{P}(\phi, +\infty)$ and $\mathcal{P}[\phi, +\infty)$ define t-structures on $\mathcal{C}$ with hearts $\mathcal{P}(\phi, \phi+1]$ and $\mathcal{P}[\phi, \phi+1)$, resp.

Notation: We denote by τ_ϕ the t-structure $\mathcal{P}(\phi, +\infty)$ $\forall\, \phi \in \mathbb{R}$.

Now, we introduce limiting COHAs.

Let $\mathcal{C} = D^b \mathrm{nilp}(\Pi_Q)$ with Q = affine ADE quiver.
Fix

- a decreasing sequence $\{a_k\}$ of real numbers such that $a_0 = 1$ and $\lim_k a_k = a_\infty \in (0,1)$,
- a slicing $\mathcal{P}$ of $\mathcal{C}$.

Set $\tau_k := \tau_{a_{k-1}}$ $\forall\, k \in \mathbb{N} \cup \{\infty\}$.

Let

- $\underline{\mathbb{R}\mathrm{Perf}^{nil}}(\Pi_Q) :=$ Toën-Vaquié's derived moduli stack of complexes of nilpotent Π_Q-reprs, which are perfect as complexes of $\mathbb{C}$-modules.
- $\underline{\mathbb{R}\Lambda^k_Q} :=$ derived moduli stack of complexes belonging to the heart $\mathcal{P}(a_{k-1}, a_k]$ of τ_k $\forall\, k \in \mathbb{N} \cup \{\infty\}$

We make the following assumption:

A. $\underline{\mathbb{R}\Lambda^k_Q}$ is an open substack of $\underline{\mathbb{R}\mathrm{Perf}^{nil}}(\Pi_Q)$

Remark: Assumption A implies that $(\underline{\mathbb{R}\Lambda^k_Q})_{red}$ is an Artin derived stack locally of finite type$/\mathbb{C}$

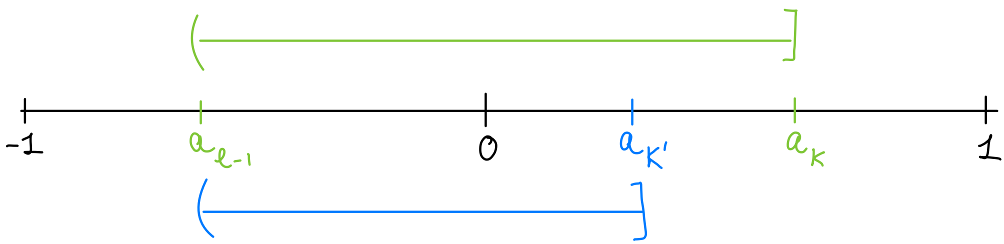
Notation: $\forall\, k \geq \ell \in \mathbb{N}$

- $I_{\ell,k} := (a_{\ell-1}, a_k]$. When $\ell = k$, $I_k := I_{k,k}$.

- $\underline{\mathbb{R}\Lambda}_Q^{\ell,k} := \underline{\mathbb{R}\Lambda}_Q^{\ell} \cap \underline{\mathbb{R}\Lambda}_Q^{k}$

  = derived moduli stack of complexes belonging to $I_{\ell,k}$

Now, fix ℓ and take $k' \geq k$.

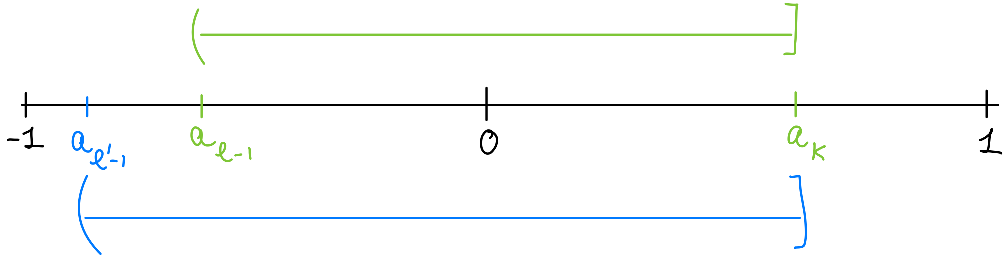


Since $I_{\ell,k'} \subset I_{\ell,k}$, $\exists$ canonical open immersion

$$\underline{\mathbb{R}\Lambda}_Q^{\ell,k'} \longrightarrow \underline{\mathbb{R}\Lambda}_Q^{\ell,k}$$

$\Longrightarrow$ $\underline{\mathbb{R}\Lambda}_Q^{\ell,+\infty} := \text{"}\lim_{k \geq \ell}\text{"}\ \underline{\mathbb{R}\Lambda}_Q^{\ell,k}$ is a pro-derived stack

Now, for $\ell' \geq \ell$,



$\exists$ canonical maps $\underline{\mathbb{R}\Lambda}_Q^{\ell,+\infty} \longrightarrow \underline{\mathbb{R}\Lambda}_Q^{\ell',+\infty}$

$\Longrightarrow \underline{\mathbb{R}\Lambda}_Q^{+\infty} := \text{"}\operatorname{colim}_{\ell}\text{"}\ \underline{\mathbb{R}\Lambda}_Q^{\ell,+\infty}$ is a ind-pro-derived stack

Now, we define a COHA associated to $\underline{\mathbb{R}\Lambda}_Q^{+\infty}$. We need to make two extra assumptions.

Notation:

- $\underline{\mathbb{R}\Lambda}_Q^{k,ext}$:= derived stack of "extensions"

$$\begin{tikzcd} F_1 \arrow[r] & F \arrow[dl] \\ F_2 \arrow[u, dashed, "{[1]}"] & \end{tikzcd}$$

of complexes belonging to $\mathcal{P}(I_k)$

- for $k \geq \ell_2 \geq \ell_1$, $\underline{\mathbb{R}\Lambda}_Q^{\ell_1,\ell_2,k} \subset \underline{\mathbb{R}\Lambda}_Q^{k,ext}$ open substack consisting of those extensions for which

$F_1, F_2 \in \mathcal{P}(I_{\ell_2})$ and $F \in \mathcal{P}(I_{\ell_1})$

We make the following assumption:

B. For any $\ell_1 \in \mathbb{N}$ and $\underline{d}_1, \underline{d}_2 \in \mathbb{Z}I$, $\exists$ integer $N \geq \ell_1$ such that $\forall\, k' \geq k \geq \ell_2 \geq N$, the square

$$\begin{tikzcd} \underline{\mathbb{R}\Lambda}^{\ell_1, \ell_2, k'}_{\underline{d}_1, \underline{d}_2} \arrow[r, "p"] \arrow[d] & \underline{\mathbb{R}\Lambda}^{\ell_1, k'}_{\underline{d}_1 + \underline{d}_2} \arrow[d] \\ \underline{\mathbb{R}\Lambda}^{\ell_1, \ell_2, k}_{\underline{d}_1, \underline{d}_2} \arrow[r, "p"] & \underline{\mathbb{R}\Lambda}^{\ell_1, k}_{\underline{d}_1 + \underline{d}_2} \end{tikzcd}$$

is a pullback and p is proper representable.

Here, p is the map sending an extension to its middle term.

<u>Theorem</u> (DPSSV-2)

Under Assumptions A and B, the topological vector space

$$\boxed{\mathrm{COHA}_Q^{+\infty,(T)} := H_*^{(T)}\left(\underline{\mathbb{R}\Lambda}_Q^{+\infty}\right) := \bigoplus_{\underline{d} \in \mathbb{Z}I} \text{"}\lim_{\ell}\text{"} \operatorname*{colim}_{k \geq \ell} H_*^{(T)}\left(\underline{\mathbb{R}\Lambda}^{\ell, k}_{\underline{d}}\right)}$$

carries a canonical $\mathbb{Z}\times\mathbb{Z}I$-graded $H_T^*(pt)$-algebra structure.

Assume that $\exists$ a COHA associated to $\underline{\mathbb{R}\Lambda_Q^{+\infty}}$.
Now we would like to compare it with $\mathrm{COHA}_Q^{+\infty,(T)}$

To do so, we need to introduce Bridgeland's stability conditions.

Definition

A stability condition in the sense of Bridgeland on $\mathcal{C}$ is a pair $(Z,\mathcal{P})$ consisting of

- a slicing $\mathcal{P}$ on $\mathcal{C}$ and
- a group homomorphism $Z:\mathbb{Z}I \longrightarrow \mathbb{C}$

such that $\forall\, 0\neq E\in\mathcal{P}(\phi)$, we have $Z(E)\in\mathbb{R}_{>0}e^{i\pi\phi}$

Fix a stability condition $(Z,\mathcal{P})$. Fix $\underline{d},\underline{d}_1,\dots,\underline{d}_s\in\mathbb{Z}I$ such that $\underline{d}=\sum_k \underline{d}_k$. Let

$\underline{\mathbb{R}\Lambda}^{k,HN}_{\underline{d}}(\underline{d}_1,\dots,\underline{d}_s)$:= derived moduli stack of complexes $E\in\mathcal{P}(\mathcal{I}_k)$ of K-theory class $\underline{d}$ whose Harder-Narasimhan filtration

$$\begin{tikzcd}
0=E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed, "{[1]}"] & & A_2 \arrow[ul, dashed, "{[1]}"] & & & & A_n \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

is such that each A_i has K-theory class $\underline{d}_i$.

We make the following assumption:

C1. $\underline{\mathbb{R}\Lambda}^{\ell,k}_{\underline{d}}$ is quasi-compact $\forall\, \underline{d}\in\mathbb{Z}I$ and $\forall\, k\geq \ell$,

C2. $\underline{\mathbb{R}\Lambda}^{k,HN}_{\underline{d}}(\underline{d}_1,\dots,\underline{d}_s)$ is a locally closed substack of $\underline{\mathbb{R}\Lambda}^{k}_{\underline{d}}$ $\forall\, \underline{d},\underline{d}_1,\dots,\underline{d}_s\in\mathbb{Z}I$ s.t. $\underline{d}_1+\cdots+\underline{d}_s=\underline{d}$ and $\forall\, k\in\mathbb{N}$.

Theorem (DPSSV-2)

Assume that $H^{(T)}_*(\underline{\mathbb{R}\Lambda}^{+\infty}_Q)$ carries the structure of a cohomological Hall algebra, which we denote by

$$\mathbf{COHA}^{(T)}_{+\infty}$$

Under Assumptions A, B, and C, $\exists$ a canonical algebra isomorphism

$$\mathrm{COHA}_Q^{+\infty,(T)} \simeq \mathrm{COHA}_{+\infty}^{(T)}$$