Cohomological Hall algebras of 1-dimensional sheaves and Yangians

DPSSV-1, arXiv: 2502.19445
Diaconescu-Porta-S.-Schiffmann-Vasserot,
Nilpotent cohomological Hall algebras of surfaces

DPSSV-2, arXiv: 2603.03386
Diaconescu-Porta-S.-Schiffmann-Vasserot,
Cohomological Hall algebras of one-dimensional sheaves on surfaces and Yangians

Lecture 7

1. Proof of the main result

As a first step towards the proof, we need to understand the relation between the derived McKay equivalence and braid group actions.

1.1 Braid group actions and τ

First, note that the derived McKay equivalence

$$\tau : D^b\mathrm{Coh}(S) \longrightarrow D^b\mathrm{Mod}(\Pi_Q)$$

induces

$$K_0(\mathrm{Coh}_c(S)) \xrightarrow{\sim} K_0(\mathrm{nilp}(\Pi_Q)) \simeq \text{root lattice } R \text{ of } Q$$

$$\mathrm{Pic}(S) \xrightarrow{\sim} \text{coweight lattice } \check{X}_{fin} \text{ of } Q_{fin}$$

Moreover,

$$\tau(\mathcal{O}_{C_i}(-1)[1]) = S_i \text{ for } i \neq 0 \quad \text{and} \quad \tau(\mathcal{O}_C) = S_0$$

where the Π_Q-repr.s S_i are the simple objects of $\mathrm{nilp}(\Pi_Q)$. They are also 2-spherical.

Recall that $\exists\, B_{ex} \longrightarrow \mathrm{Aut}(D^b\mathrm{mod}(\Pi_Q))$

It is natural to wonder how to define

$$\boxed{B_{ex} \longrightarrow \mathrm{Aut}(D^b\mathrm{Coh}_{ps}(S))}$$

which is compatible with τ.

Proposition (DPSSV-2)

1. $\exists$ a group homomorphism $B_{ex} \xrightarrow{\rho} \mathrm{Aut}(D^b\mathrm{Coh}_{ps}(S))$ such that

$$\rho(T_i) = T'_{\mathcal{O}_{C_i}(-1)[1]} \quad \text{and} \quad \rho(T_i^{-1}) = T_{\mathcal{O}_{C_i}(-1)[1]} \quad \forall i \neq 0$$

$$\rho(T_0) = T'_{\mathcal{O}_C} \quad \text{and} \quad \rho(T_0^{-1}) = T_{\mathcal{O}_C}$$

2. $\exists$ a group homomorphism $B_{ex} \xrightarrow{\tilde{\rho}} \mathrm{Aut}(D^b\mathrm{Coh}_{ps}(S))$ such that

$$\tilde{\rho}(T_i) = T'_{\mathcal{O}_{C_i}(-1)[1]} \quad \text{and} \quad \tilde{\rho}(T_i^{-1}) = T_{\mathcal{O}_{C_i}(-1)[1]} \quad \forall i \neq 0$$

$$\rho(L_{\check{\lambda}}) = (\mathcal{L}_{\check{\lambda}} \otimes -) =: \tilde{L}_{\check{\lambda}} \qquad \forall \check{\lambda} \in \check{X}_{fin}$$

(w.r.t. $B_{ex} \simeq B_{fin} \cup \{L_{\check{\lambda}} : \check{\lambda} \in \check{X}_{fin}\} / \text{rels}$)

3. $\rho = \tilde{\rho}$.

1.3 Braid group actions and stability conditions

Our next step is to define a suitable stability condition.

By Bridgeland, a stability condition $(Z, \mathcal{P})$ is equivalent to the datum of an abelian category $\mathcal{A}$ and a stability function Z satisfying the Harder-Narasimhan property:

Definition

- A stability function on an abelian category $\mathcal{A}$ is a group homomorphism $Z: K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ such that $\forall\, 0 \neq E \in \mathcal{A}$ we have

$$Z(E) \in \left\{ z \in \mathbb{C} : \operatorname{Im}(z) \geq 0 \text{ and } \atop \text{if } \operatorname{Im}(z) = 0, \text{ then } \operatorname{Re}(z) < 0 \right\}$$

- The phase of an object $0 \neq E \in \mathcal{A}$ is

$$\phi(E) := \frac{1}{\pi} \arg Z(E) \in (0,1]$$

- An object $0 \neq E \in \mathcal{A}$ is semistable if $\forall\, 0 \neq E' \subset E$ $\phi(E') \leq \phi(E)$

- We say that Z satisfies the Harder-Narasimhan property if $\forall\, 0 \neq E \in \mathcal{A}$ $\exists$ a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

such that

- E_i / E_{i-1} is semistable

- $\phi(E_1) > \phi(E_2/E_1) > \cdots > \phi(E/E_{n-1})$

Now, fix a strictly dominant finite coweight $\check{\Theta}_f \in \check{X}_{fin}$, i.e., $(\check{\Theta}_f, \check{\alpha}_i) > 0 \ \forall i \neq 0$.

Since $\check{X}_{fin} \hookrightarrow \check{X}$ with respect to

$$\boxed{\check{\lambda}_i \longmapsto \check{\omega}_i - r_i \check{\omega}_0}$$

at the level of fundamental coweights, we may also regard $\check{\Theta}_f$ as an element of $\check{X}$, and we denote by $\check{\Theta}$ in such a case.

We define the stability function

$$\boxed{\begin{aligned} Z_{\check{\Theta}} : K_0(\mathrm{nilp}(\Pi_Q)) \simeq \mathbb{Z}I &\longrightarrow \mathbb{C} \\ \underline{d} &\longmapsto -(\check{\Theta}, \underline{d}) + i(\check{\rho}, \underline{d}) \end{aligned}}$$

where $\check{\rho} := \sum_i \check{\omega}_i$.

Since $\mathrm{nilp}(\Pi_Q)$ is a Noetherian and Artinian category, one can prove that $Z_{\check{\theta}}$ satisfies the Harder-Narasimhan property.

$\Longrightarrow (Z_{\check{\theta}}, \mathcal{P}_{\check{\theta}})$ corresponding stability condition

Proposition (DPSSV-2)

1. $\forall n \in \mathbb{Z}$,

$$\boxed{\tilde{L}^{-n}_{\check{\theta}} : \mathrm{nilp}(\Pi_Q) \xrightarrow{\sim} \mathcal{P}_{\check{\theta}}\big((\nu_{-n}, \nu_{-n}+1]\big)}$$

with $\nu_n := \frac{1}{\pi}\arctan(nh)$ $\quad$ (h = Coxeter number $= \sum_i r_i$)

2. $\mathcal{P}_{\check{\theta}}\big((-\frac{1}{2}, \frac{1}{2}]\big) \simeq \mathrm{Coh}_c(S)$

Set $a_k := \nu_{-2k} + 1 \;\forall k \in \mathbb{Z}$. Then

- $a_0 = 1$
- $\lim_k a_k = \frac{1}{2}$

By applying the theory of limiting COHAs to we obtain: $\{a_k\}_{k\in\mathbb{N}}$ and $(Z_{\check{\theta}}, P_{\check{\theta}})$

Corollary

The topological vector space

$$HA^{(T)}_{\check{\theta}} := \bigoplus_{\underline{d}\in\mathbb{Z}I} \text{"}\lim_{\ell}\text{"} \operatorname{colim}_{k\geq\ell} H^{(T)}_{*}(\underline{\mathbb{R}\Lambda^{\ell,k}_{d}})$$

has the structure of a graded $H^{*}_{T}(pt)$-algebra with twisted Hall multiplication induced from that of $HA^{(T)}_{Q}$.

Furthermore, $HA^{(T)}_{S,C} \simeq HA^{(T)}_{\check{\theta}}$. In particular, the latter does not depend on the choice of the strictly dominant $\check{\theta}\in\check{X}_{fin}$.

Remark

It is possible to show that

$$\operatorname{colim}_{k\geq\ell} H^{(T)}_{*}(\underline{\mathbb{R}\Lambda^{\ell,k}_{d}}) \simeq H^{(T)}_{*}(\underline{\mathbb{R}^{\ell}\Lambda}_{\underline{d}}) =: HA^{(T)}_{\check{\theta},(\ell),\underline{d}}$$

where $\mathbb{R}^{\ell}\underline{\Lambda}_{\underline{d}}$ is the derived stack parametrizing objects of $\mathcal{P}_{\check{\Theta}}(a_{\ell}-1, \frac{1}{2}]$.

Note that $\mathbb{R}^{\ell}\underline{\Lambda}_{\underline{d}}$, hence $HA^{(T)}_{\check{\Theta},(\ell),\underline{d}}$, can be defined for any $\ell \in \mathbb{Z}$.

Now, it remains to relate $HA^{(T)}_{\check{\Theta}} := HA^{(T)}_{\check{\Theta},(\ell),\underline{d}}$ to Yangians.
For $n \in \mathbb{Z}$, set

$$\boxed{\mathbb{Y}_{(n)} := L^{2n}_{\check{\Theta}_f}(\mathbb{Y}^-) / L^{2n}_{\check{\Theta}_f}(J)}$$

where $J := \sum_{(\check{\Theta},\underline{d}) \leq 0} \mathbb{Y}^-_{-\underline{d}} \cdot \mathbb{Y}^-$

Lemma (DPSSV-2)
$\exists$ isomorphisms

$$\Phi_{(\ell)} : \mathbb{Y}_{(\ell)} \longrightarrow HA^{(T)}_{\check{\Theta},(\ell)}$$

and surjective maps

$$\pi_{\ell-1,\ell} : \mathbb{Y}_{(\ell-1)} \longrightarrow \mathbb{Y}_{(\ell)}$$

that fit into the following commutative diagrams

$$\begin{tikzcd}
\mathbb{Y}_{(\ell)} \arrow[r, "\Phi_{(\ell)}"] \arrow[d, "T^{-n}_{2\check{\theta}_f}"'] & \mathbf{HA}^{(T)}_{\check{\theta},(\ell)} \arrow[d, "\mathbb{RL}^{-n}_{2\check{\theta}_f}"] \\
\mathbb{Y}_{(\ell-n)} \arrow[r, "\Phi_{(\ell-n)}"'] & \mathbf{HA}^{(T)}_{\check{\theta},(\ell-n)}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{Y}_{(\ell-1)} \arrow[r, "\Phi_{(\ell-1)}"] \arrow[d, "\pi_{\ell-1,\ell}"'] & \mathbf{HA}^{(T)}_{\check{\theta},(\ell-1)} \arrow[d, "\rho_{\ell-1}"] \\
\mathbb{Y}_{(\ell)} \arrow[r, "\Phi_{(\ell)}"'] & \mathbf{HA}^{(T)}_{\check{\theta},(\ell)}
\end{tikzcd}$$

<u>Attention</u> ⚠ : The proof of this lemma is based on a compatibility result of braid group actions on quotients of Yangians and quiver COHAs, respectively.

Set

$$\boxed{\mathbb{Y}_{S,C} := \lim_{\ell} \mathbb{Y}_{(-2\ell)}}$$

Theorem (DPSSV-2)

- $\exists$ a canonical associative graded $H_T^*(pt)$-algebra structure on $\mathbb{Y}_{S,C}$ induced by the multiplication on $\mathbb{Y}$.
- $\exists$ a canonical $H_T^*(pt)$-algebra isomorphism

$$\Theta_{S,C} : HA_{S,C} \xrightarrow{\sim} HA^T_{\check{\theta}} \xrightarrow{\sim} \mathbb{Y}_{S,C}$$

induced by $\Phi_{(-2\ell)}$

Remark

$\exists$ a $\mathfrak{S}$-action on $HA^T_{S,C}$ such that $\Theta_{S,C}$ is $\mathfrak{S}$-equivariant

This concludes the proofs of (1) and (2) of the main result.

The last result I want to mention is a generation result for these COHAs.

For $i \in I_{fin}$, $n \in \mathbb{N}$, $d \in \mathbb{Z}$, set

$\mathcal{C}_{i,d}$:= irreducible component containing the sheaf $\mathcal{O}_{C_i}(d-1)$,

$\mathcal{Y}_{i,n}$:= irreducible component containing the 0-dimensional sheaves on S of length n whose support is a single point $x \in C_i$.

We can prove that $\mathbb{H}\mathbb{A}^T_{S,C}$ is an algebra over $\mathbb{S}$.

Theorem (DPSSV-2)

$\mathbb{H}\mathbb{A}^T_{S,C}$ is generated, as an algebra over $\mathbb{S}$, by the fundamental classes of $\mathcal{Y}_{i,n}$ and $\mathcal{C}_{i,d}$ for any $i \in I_{fin}$, $n \in \mathbb{N}$, and $d \in \mathbb{Z}$.