Categorified Hall algebras and their representations

Motives in Moduli and Representation Theory in Nijmegen

Plan

1. Elliptic Hall algebra and the K-theory of Hilbert schemes of pts
2. Cohomological and Categorified Hall algebras
3. Representations of Categorified Hall algebras

Elliptic Hall algebra and the K-theory of Hilbert schemes of pts

S = smooth (quasi-) projective surface $/\mathbb{C}$, $n \in \mathbb{N}$

$\mathrm{Hilb}^n(S)$ = Hilbert scheme of n-pts on S
= moduli space parametrizing zero-dim. subschemes $Z \subset S$ such that $\dim_{\mathbb{C}} H^0(Z, \mathcal{O}_Z) = n$

Remarks

- $\mathrm{Hilb}^n(S)$ is a smooth (quasi-) projective variety $/\mathbb{C}$ of dimension $2n$

- $\mathrm{Hilb}^n(S) \underset{\substack{\uparrow \\ S \text{ projective}}}{=} M_S^{st}(\underset{\substack{\uparrow \\ rk}}{1}, \underset{\substack{\uparrow \\ c_1}}{0}, \underset{\substack{\uparrow \\ ch_2}}{-n})$ = moduli space of Gieseker-stable sheaves on S

Notation: Set $\mathrm{Hilb}(S) := \bigsqcup_n \mathrm{Hilb}^n(S)$

Assumption: $\exists\ T = \text{torus} \curvearrowright S$ such that:

- if S is projective, T could be trivial
- if S is quasi-projective, the fixed locus S^T is proper

Set

- $G_0^T(-)$ = Grothendieck group of T-equivariant coherent sheaves
- $R := G_0^T(pt)$, $K := \mathrm{Frac}(R)$, $M_K := M \otimes_R K$ for any R-mod. M

Thm (Schiffmann-Vasserot for $S = \mathbb{C}^2$, Neguț for arbitrary S)

$\exists$ an action of the Elliptic Hall algebra $\mathcal{E}$ on

$$G_0^T(\mathrm{Hilb}(S))_K$$

Attention ⚠: Elliptic Hall algebra $\mathcal{E} = \langle \text{generators} \rangle / \text{relations}$

The relations depend on a function which resembles the zeta function of an elliptic curve over a finite field

The action is defined explicitly via

Nakajima type explicit operators

i.e., operators depending on tautological bundles over 1-step Hecke correspondences.

Attention ⚠: yesterday, they were mentioned by Schiffmann.

Remarks

- Neguț and Yu Zhao have investigated the categorification of the above result.

- Schiffmann-Vasserot proved a cohomological version of the above result for $S = \mathbb{C}^2 \circlearrowleft T = \mathbb{C}^* \times \mathbb{C}^*$

$\Longrightarrow$ Nakajima-Grojnowski's action of the Heisenberg algebra

Advantages of Nakajima type operators:

- It allows to compute "easily" relations between the generators

- One realize geometric repr.s of the whole elliptic Hall algebra (its categorification, etc) via

$$\mathcal{M}_S(r, c_1) := \bigsqcup_{ch_2} \mathcal{M}_S^{st}(r, c_1, ch_2) \quad \text{(here } S \text{ is projective)}$$

$(r, c_1 \cdot H) = 1$

Gieseker-stable moduli spaces

Problems:

- It is useful to realize repr.s of only one (!) algebra

- The elliptic Hall algebra does NOT contain all possible operators acting on $G_0(-)$, $D^b_{coh}(-)$, etc, of

$$\mathcal{M}_S(r) := \bigsqcup_{c_1} \bigsqcup_{ch_2} \mathcal{M}_S^{st}(r, c_1, ch_2) \quad \text{(here } S \text{ is projective)}$$

In particular, $\mathcal{E}$ does NOT contain operators changing the first Chern class c_1, i.e., which e.g. depend on

$$\left\{ 0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow i_* \mathcal{L}_C \longrightarrow 0 \right\}$$

where $C \overset{i}{\hookrightarrow} S$ is a smooth proj. curve and $\mathcal{L}_C$ line bundle on C.

Different approach:

The theory of 2-dim. cohomological Hall algebras (introduced yesterday by Schiffmann)

$\Longrightarrow$ it allows to define a plethora (!) of associative algebras

A connection with before is given by:

Thm (Schiffmann-Vasserot)

1. $KHA^T(\mathbb{C}^2)_K = \left(G_0^T(\underline{Coh}_{0\text{-}dim}(\mathbb{C}^2))_K , \text{Hall product} \right)$

$\simeq \mathcal{E}^+ =$ positive part of the elliptic Hall algebra

2. $\exists$ an action $KHA^T(\mathbb{C}^2)$ on $G_0^T(\mathrm{Hilb}(\mathbb{C}^2))$ such that

$$\begin{tikzcd}
KHA^T(\mathbb{C}^2)_K \arrow[d, "\wr"'] \arrow[r] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{End}\left(G_0^T(\underline{\mathrm{Coh}}_0(\mathbb{C}^2))_K\right) \\
\mathcal{E}^+ \arrow[ur] &
\end{tikzcd}$$

Similar results hold in BM homology.

Attention $\triangle$! : COHAs realize only (!) halves of the full algebras we care about

Solution (pursued in the past): $\underbrace{\text{doubling algebraically}}_{\text{e.g. "Drinfeld doubles"}}$ the COHAs

Aim of the talk:
Introduce a categorification of COHAs and a geometric procedure of doubling categorified COHAs.

Cohomological and Categorified Hall algebras

Let us give the most general construction of COHAs and their categorified counterpart.

With respect to the construction discussed by Schiffmann:

$$\underline{\mathrm{Coh}}_{0\text{-dim}}(\text{surface}) \rightsquigarrow \text{moduli stack of "flat" objects}$$

$$H_*^{BM}(-),\ G_0(-) \overset{\text{replace}}{\rightsquigarrow} D^b_{coh}(-)$$

Fix

- $\mathscr{C}$ = $\mathbb{C}$-linear stable ∞-category, which is $\underbrace{\text{compactly generated}}_{(\text{i.e., } \mathscr{C} \simeq \mathrm{Ind}(\mathscr{C}^\omega))}$

$\Longrightarrow$ $\mathbb{R}\underline{\mathrm{Perf}}_{ps}(\mathscr{C})$ = Toën-Vaquié's derived moduli stack of pseudo-perfect objects of $\mathscr{C}$, i.e.,

$$\mathbb{R}\underline{\mathrm{Perf}}_{ps}(\mathscr{C})(\underset{\mathrm{dAff}_{\mathbb{C}}}{\underset{\cap}{A}}) := \left\{ E \in \mathscr{C}_A := \mathscr{C} \otimes_{\mathbb{C}} \mathrm{Mod}_A : \forall\, G \in \mathscr{C}^\omega_A,\ \mathrm{Hom}_{\mathscr{C}}(G,E) \in \mathrm{Perf}(A) \right\}^{\simeq}$$

Thm (Toën-Vaquié)
Assume that $\mathcal{C}$ is of finite type (i.e. $\mathrm{Fun}(\mathcal{C},-)$ commutes with filtered colimits)

Then, $\mathbb{R}\underline{\mathrm{Perf}}_{ps}(\mathcal{C})$ is a locally Artin derived stack locally of finite type over $\mathbb{C}$

Examples

- S = smooth (quasi-)projective surface$/\mathbb{C}$, $\mathcal{C} = D_{qcoh}(S)$

  Here, pseudo-perfect $\longleftrightarrow$ properly supported

- Q = quiver, $\Pi_Q :=$ 2CY completion of the path algebra $\mathbb{C}Q$
  ($\simeq$ preprojective algebra of Q if $Q \neq$ finite ADE)

  $\mathcal{C} = \Pi_Q\text{-Mod}$

  Here, pseudo-perfect $\longleftrightarrow$ finite-dimensional

Now, we also fix

- $\tau = (\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ t-structure.

Fact: For any $A \in dAff_{\mathbb{C}}$, under the equivalence

$$\mathcal{C} \otimes_{\mathbb{C}} Mod_A =: \mathcal{C}_A \simeq Fun^R_{\mathbb{C}}(Mod_A^{op}, \mathcal{C})$$

functors which are right adjoint

$\exists$ a t-structure on $\tau_A = (\mathcal{C}_A^{\leq 0}, \mathcal{C}_A^{\geq 0})$ on $\mathcal{C}_A$ induced by τ via the forgetful morphism $\mathcal{C}_A \longrightarrow \mathcal{C}$ (= evaluation at A)

Def. (flat objects)
An object $E \in \mathcal{C}_A$ is τ-flat if $\forall M \in Mod_A^{\heartsuit}$, one gets

$$M \otimes E \in \mathcal{C}_A^{\heartsuit}$$

(= heart w.r.t. τ_A)

Example
S = surface as before, $\tau = \tau_{std}$ = standard t-structure
For $A \in dAff_{\mathbb{C}}$, τ_A = standard t-structure on $\mathcal{C}_A = D_{qcoh}(S \times \operatorname{Spec} A)$
Then

$$E \in \mathcal{C}_A \text{ flat} \iff p^*(M) \otimes E \in \mathcal{C}_A^{\heartsuit} \qquad \forall M \in Mod_A^{\heartsuit}$$

$p : S \times \operatorname{Spec} A \longrightarrow \operatorname{Spec} A$

⚠ if A is classical, it is equivalent the usual notion of flat families

$\exists\ \mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau) \subseteq \mathbb{R}\underline{\mathrm{Perf}}_{ps}(\mathcal{C})$

= derived moduli stack of (τ-) flat pseudo-perfect objects of $\mathcal{C}$

Fact:

$\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau)$ is Artin $\Leftrightarrow$ $\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau) \subseteq \mathbb{R}\underline{\mathrm{Perf}}_{ps}(\mathcal{C})$ is represented by open embeddings

Def. We say that τ satisfies openness of flatness if • holds

From now on, fix: $\mathcal{C}$ of finite type, τ satisfies openness of flatness

$\Longrightarrow$ Goal: define a "Hall algebra" associated to $\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau)$

Consider the "convolution" diagram (the correspondence)

$$\begin{tikzcd}
 & \mathbb{R}\underline{\mathrm{Coh}}^{ext}(\mathcal{C},\tau) \arrow[dl, "q_\tau"'] \arrow[dr, "p_\tau"] & \\
\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau) \times \mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau) & & \mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau)
\end{tikzcd}$$

$$q_\tau : 0 \to \mathcal{E}_2 \to \mathcal{E} \to \mathcal{E}_1 \to 0 \longmapsto (\mathcal{E}_1, \mathcal{E}_2)$$

$$p_\tau : 0 \to \mathcal{E}_2 \to \mathcal{E} \to \mathcal{E}_1 \to 0 \longmapsto \mathcal{E}$$

Facts:

- if $\exists$ a Serre functor $S_{\mathscr{C}}$ s.t. $S_{\mathscr{C}}[-2]$ is τ-exact, then q_τ is quasi-smooth.
- p_τ proper $\Leftrightarrow$ Quot functor for $(\mathscr{C}, \tau)$ represented by proper algebraic spaces (Ex: if $\mathscr{C}$ is proper $\Longrightarrow$ p is proper)

Thm (Porta-S., Diaconescu-Porta-S.)

Assume that

1. $\mathscr{C}$ is of finite type,
2. τ satisfies openness of flatness,
3. Serre functor $S_{\mathscr{C}}$ s.t. $S_{\mathscr{C}}[-2]$ is τ-exact,
4. p_τ proper.

derived enhancement of $D^b_{coh}(-)$

Then, $\exists$ a $\mathbb{E}_1$-monoidal structure on $\mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathscr{C}, \tau))$ induced by

$$\mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathscr{C},\tau)) \otimes \mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathscr{C},\tau)) \xrightarrow{\boxtimes}$$

$$\mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathscr{C},\tau) \times \mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathscr{C},\tau)) \xrightarrow[(p_\tau)_* \circ q_\tau^*]{} \mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathscr{C},\tau))$$

Examples

- $\mathscr{C} = D_{qcoh}(S)$ S = smooth projective surface$/\mathbb{C}$
- $\mathscr{C} = \Pi_Q\text{-Mod}$, Q = quiver
- $\mathscr{C} = D_{qcoh}(X_{Dol}), D_{qcoh}(X_{dR}), D_{qcoh}(X_B)$ - X = smooth proj. curve$/\mathbb{C}$

  Simpson's shapes
- $\mathscr{C}$ = Kuznetsov component (non-commutative K3 surface)

Corollary

- By passing to K-theory, $\exists$ an associative algebra structure on

$$G_0(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathscr{C},\tau)) \simeq G_0(\underline{\mathrm{Coh}}_{ps}(\mathscr{C},\tau)).$$

- By using Khan's theory of BM homology, $\exists$ an associative algebra structure on

$$H_*^{BM}(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathscr{C},\tau)) \simeq H_*^{BM}(\underline{\mathrm{Coh}}_{ps}(\mathscr{C},\tau)).$$

Let $(\mathcal{T}or, \mathcal{F})$ be a torsion pair of $\mathcal{C}^{\heartsuit}_{\tau}$, i.e.

- $\mathrm{Hom}(T,F)=0 \quad \forall\, T \in \mathcal{T}or,\ F \in \mathcal{F}$;
- $\forall\, E \in \mathcal{C}^{\heartsuit}_{\tau}\ \exists\ 0 \longrightarrow \underset{\in \mathcal{T}or}{T} \longrightarrow E \longrightarrow \underset{\in \mathcal{F}}{F} \longrightarrow 0$

Goal: given $(\mathcal{T}or, \mathcal{F})$, there exist:

algebra ass. to $\mathcal{T}or$	and	right and left module ass. to $\mathcal{F}$

Rmk

$v = (\mathcal{T}or, \mathcal{F}) \rightsquigarrow \exists$ new t-structure τ_v s.t.

$$\mathcal{C}^{\heartsuit}_{v} = \{ E \in \mathcal{C} : \mathcal{H}^{-1}(E) \in \mathcal{F},\ \mathcal{H}^{0}(E) \in \mathcal{T}or,\ \mathcal{H}^{i}(E) = 0\ \forall\, i \neq -1, 0 \}$$

$+\ (\mathcal{F}[1], \mathcal{T}or)$ torsion pair of $\mathcal{C}^{\heartsuit}_{v}$

Introduce:

- $\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{T}or}(\mathcal{C},\tau) \subset \mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau),\ \mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau_v)$
- $\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau) \subset \mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau),\ \mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau_v)$

Thm (Diaconescu-Porta-S.)
Assume that:

1. $\mathcal{C}$ is of finite type,
2. τ satisfies openness of flatness,
3. $\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{T}or}(\mathcal{C},\tau)$ and $\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau)$ are open in $\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau)$
4. Serre functor $S_{\mathcal{C}}$ s.t. $S_{\mathcal{C}}[-2]$ is t-exact w.r.t. τ and τ_v,
5. p_τ and p_{τ_v} are proper,
6. $\mathcal{T}or$ is a Serre subcategory.

Then, $\exists$ a $\mathbb{E}_1$-monoidal algebra structure on

$$\mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}_{\tau_{or}}(\mathcal{C},\tau))$$

induced by the one on either

$$\mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau)) \quad \text{or} \quad \mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau_v))$$

equivalently.

7. Furthermore, assume that $\mathbb{R}\underline{\mathrm{Coh}}_{\tau_{or}}(\mathcal{C},\tau)$ is closed in both $\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau)$ and $\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau_v)$

Then, $\mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}_{F}(\mathcal{C},\tau))$ is a left (resp. right) categorical module of $\mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}_{\tau_{or}}(\mathcal{C},\tau))$ induced by the $\mathbb{E}_1$-monoidal structure of $\mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau))$ (resp. $\mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(\mathcal{C},\tau_v))$)

Corollary Similar statements hold for $G_0(-)$ and $H_*^{BM}(-)$.

Example

- S = smooth projective surface$/\mathbb{C}$

$$\mathcal{T}or = Coh_{\le 1}(S) := \{ \mathcal{F} \in Coh(S) : \dim(\mathrm{supp}(\mathcal{F})) \le 1 \}$$

$$\mathcal{F} = Coh_{t.f.}(S) := \{ \text{torsion free sheaves on } S \}$$

Def. (Algebras of "Hecke modifications along curves")

The Yangian of $Coh_{\le 1}(S)$ is the subalgebra of $\mathrm{End}(H_*^{BM}(\underline{Coh}_{t.f.}(S;r)))$ generated by the images of

left action $H_*^{BM}(\underline{Coh}_{\le 1}(S)) \longrightarrow \mathrm{End}(H_*^{BM}(\underline{Coh}_{t.f.}(S;r)))$

right action $H_*^{BM}(\underline{Coh}_{\le 1}(S)) \longrightarrow \mathrm{End}(H_*^{BM}(\underline{Coh}_{t.f.}(S;r)))$

Similarly, we define: (categorified) quantum loop algebra of $Coh_{\le 1}(S)$

Remark

- Similarly, we may define

Algebra of "Hecke modifications" at points $\longleftrightarrow \widetilde{DH}_0(S)$
(ass. to $\mathrm{Coh}_0(S) = \{$0-dim. sheaves on $S\}$)

introduced yesterday by Schiffmann

- If we also replace:

$\underline{\mathrm{Coh}}_{t.f.}(S;1) \rightsquigarrow \mathrm{Hilb}(S) \Longrightarrow$ we recover Neguţ's result in $G_0(-)$ and $D^b_{coh}(-)$

- if instead of $\mathrm{Hilb}(S)$, we consider

$P(S)$:= moduli space of Pandharipande-Thomas stable pairs on S
$\qquad \|$
$(\mathcal{F}, s: \mathcal{O}_S \longrightarrow \mathcal{F})$ with $\mathcal{F}$ pure 1-dimen.
$\mathrm{Coker}(s)$ 0-dimen.

$\Longrightarrow H^{BM}_*(P(S))$ is a right and left module of $H^{BM}_*(\underline{\mathrm{Coh}}_0(S))$
$G_0(\text{---}''\text{---})$ ———— '' ———— $G_0(\text{---}''\text{---})$
$\mathrm{Coh}^b(\mathbb{R}\text{---}''\text{---})$ ———— '' ———— $\mathrm{Coh}^b(\mathbb{R}\text{---}''\text{---})$

- In this case, we can replace $S \rightsquigarrow T^*X$

  X = smooth projective curve$/\mathbb{C}$

$\Longrightarrow$ the above result holds.

In particular,

PT stable pair on T^*X = cyclic Higgs bundle $(\underbrace{\mathcal{E}, \phi}_{\text{Higgs bundle}}, \tau : \mathcal{O}_X \longrightarrow \mathcal{E})$ on X

$\nexists$ saturated Higgs subbundle $(\mathcal{E}', \phi')$ s.t. $\mathrm{Im}(\tau) \subseteq \mathcal{E}'$.

Geometric idea behind the proof:

Consider:

$$\begin{tikzcd}[column sep=small] & \mathbb{R}\underline{\mathrm{Coh}}^{ext}_{\mathcal{T}or,\mathcal{F}}(\mathcal{C},\tau) \arrow[dl, "q_\tau"'] \arrow[dr, "p_\tau"] & \\ \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{T}or}(\mathcal{C},\tau) \times \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau) & & \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau) \end{tikzcd} \qquad (*)$$

$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow T \longrightarrow 0$ in the abelian category $\mathcal{C}^{\heartsuit}_{\tau}$

$$\begin{tikzcd} & \mathcal{E} \arrow[dl, "q_\tau"'] \arrow[dr, "p_\tau"] & \\ (T, \mathcal{E}') & & \mathcal{E} \end{tikzcd}$$

Now, q_τ is quasi-smooth

Fact: $\mathcal{E} \in \mathcal{F} \Longrightarrow \mathcal{E}' \in \mathcal{F}$

$\Rightarrow$ The fiber of p_τ at $\mathcal{E}$ is the Quot scheme parametrizing its torsion quotients

$\Rightarrow$ p_τ is proper $\Rightarrow$ $(*)$ gives rise to the left action.

Now, consider:

$$\begin{tikzcd}[column sep=small] & \mathbb{R}\underline{\mathrm{Coh}}^{\mathrm{ext}}_{\mathcal{F},\mathcal{T}or}(\mathcal{C},\tau) \arrow[dl, "q_\tau"'] \arrow[dr, "p_\tau"] & \\ \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau) \times \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{T}or}(\mathcal{C},\tau) & & \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau) \end{tikzcd}$$

$0 \longrightarrow T \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \longrightarrow 0$ in the abelian category $\mathcal{C}^{\heartsuit}_{\tau}$

$$\begin{tikzcd} & \mathcal{E} \arrow[dl, "q_\tau"'] \arrow[dr, "p_\tau"] & \\ (\mathcal{E}', T) & & \mathcal{E} \end{tikzcd}$$

Attention $\triangle$: $\mathcal{E} \in \mathcal{F} \not\Longrightarrow \mathcal{E}' \in \mathcal{F}$

Thus, the fiber of p_τ at $\mathcal{E}$ is not proper

Consider

$$(**)\quad \begin{tikzcd}[column sep=small] & \mathbb{R}\underline{\mathrm{Coh}}^{\mathrm{ext}}_{\mathcal{T}or,\mathcal{F}}(\mathcal{C},\tau_v) \arrow[dl, "q_{\tau_v}"'] \arrow[dr, "p_{\tau_v}"] & \\ \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau)\times\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{T}or}(\mathcal{C},\tau) & & \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau) \end{tikzcd}$$

$$\mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau) \simeq \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}[1]}(\mathcal{C},\tau_v), \quad \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{T}or}(\mathcal{C},\tau) \simeq \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{T}or}(\mathcal{C},\tau_v), \quad \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau) \simeq \mathbb{R}\underline{\mathrm{Coh}}_{\mathcal{F}[1]}(\mathcal{C},\tau_v)$$

$0 \longrightarrow T \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \longrightarrow 0$ in the abelian category $\mathcal{C}^{\heartsuit}_{\tau_v}$

$$\begin{tikzcd} & \mathcal{E} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ (\mathcal{E}', T) & & \mathcal{E} \end{tikzcd}$$

- $T \simeq \mathcal{H}^0(T) \in \mathcal{T}or \subset \mathcal{C}^{\heartsuit}_{\tau_v}$
- $\mathcal{E} \simeq \mathcal{H}^{-1}(\mathcal{E})[1] \in \mathcal{F} \subset \mathcal{C}^{\heartsuit}_{\tau_v} \Longrightarrow \mathcal{E}' \simeq \mathcal{H}^{-1}(\mathcal{E}')[1] \in \mathcal{F}$

$\Longrightarrow$ as before, p_{τ_v} is proper

$\Longrightarrow$ $(**)$ gives rise to the right action. $\square$