Cohomological Hall algebras of 1-dimensional sheaves and Yangians over the Bridgeland's space of stability conditions

(based on

- joint work with D.-E. Diaconescu, M. Porta, O. Schiffmann, and Eric Vasserot, arXiv: 2502.19445;

- joint work with O. Schiffmann and P. Shimpi, arXiv: 2511.08576)

1. Overview

2-dimensional Cohomological Hall algebras (COHAs)

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

Construction of COHAs

Description of the "quantum nature" of a COHA

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

0-dimensional sheaves on smooth surfaces $\longleftarrow$ Examples $\longrightarrow$ 1-dimensional sheaves on smooth surfaces

2. COHAs of surfaces: Construction

$$\boxed{\begin{array}{c}\text{Geometry}\\ \text{Moduli stack}\end{array}} \longrightarrow \text{COHA} \longrightarrow \boxed{\begin{array}{c}\text{Algebra}\\ \text{Associative algebra}\end{array}}$$

S = smooth quasi-projective surface $/\mathbb{C}$

T = (possibly trivial) torus $\curvearrowright S$

$\underline{\mathrm{Coh}}_{ps}(S)$ = moduli stack of properly supported coherent sheaves on S

Remark

We can also define

- $\underline{\mathrm{Coh}}_0(S) \subset \underline{\mathrm{Coh}}_{ps}(S) \longleftrightarrow$ 0-dim. sheaves
- $\underline{\mathrm{Coh}}_{\leq 1}(S) \subset \underline{\mathrm{Coh}}_{ps}(S) \longleftrightarrow$ sheaves of dim ≤ 1

Example

$S = T^*X$, X = smooth projective curve$/\mathbb{C}$

$\underline{\mathrm{Coh}}_{ps}(T^*X)$ = moduli stack of Higgs sheaves $(\mathcal{F}, \phi : \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega^1_X)$ on X

Schiffmann-Vasserot: $S = \mathbb{C}^2$, S.-Schiffmann: $S = T^*X$, Kapranov-Vasserot, Yu Zhao in dim = 0, Porta-S.:

$$\boxed{\exists\ \mathrm{COHA}_S^{(T)} = \text{(T-equivariant) COHA associated to properly supported sheaves on } S}$$

= unital associative algebra structure on

$$\underline{H_*^{(T)}}\left(\mathbb{R}\underline{\mathrm{Coh}}_{ps}(S)\right) = \underline{H_*^{(T)}}\left(\underline{\mathrm{Coh}}_{ps}(S)\right)$$

(equivariant Borel-Moore homology)

where the multiplication $(\mathbb{R}p)_* \circ (\mathbb{R}q)^!$ is induced by

$$\underline{\mathbb{R}\mathrm{Coh}}_{ps}(S)\times\underline{\mathbb{R}\mathrm{Coh}}_{ps}(S)\xleftarrow{\ \mathbb{R}q\ }\underline{\mathbb{R}\mathrm{Coh}}^{ext}_{ps}(S)\xrightarrow{\ \mathbb{R}p\ }\underline{\mathbb{R}\mathrm{Coh}}_{ps}(S)$$

(stack of extensions)

Here,

$$p:\ 0\to E_2\to E\to E_1\to 0\ \longmapsto E$$
$$q:\ \qquad\qquad \text{''} \qquad\qquad \longmapsto (E_2, E_1)$$

Remark

1. p is a proper representable map $\Longrightarrow \exists\ \mathbb{R}p_*$

2. $\mathbb{R}q$ is quasi-smooth $\Longrightarrow \exists\ (\mathbb{R}q)^!$

Remark

- $\exists\ \mathrm{COHA}^{(T)}_{S,0\text{-dim}}$ associated to $\underline{\mathrm{Coh}}_0(S)$
- $\exists\ \mathrm{COHA}^{(T)}_{S,\leq 1}$ associated to $\underline{\mathrm{Coh}}_{\leq 1}(S)$

Now, we introduce a "nilpotent" COHA.

Fix a T-invariant closed reduced subscheme $C\subset S$

Consider

$$\boxed{\begin{array}{l}\underline{\mathrm{Coh}}_C(S) = \text{moduli stack of coherent sheaves on } S \\ \qquad\qquad \text{set-theoretically supported on } C\end{array}}$$

Example

$S = T^*X$, X = smooth projective curve$/\mathbb{C}$

$C = X \subset T^*X$ zero section

$$\Longrightarrow \underline{\mathrm{Coh}}_X(T^*X) = \text{moduli stack of Higgs sheaves } (\mathcal{F}, \phi: \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega^1_X) \text{ on } X, \text{ such that } \phi \text{ is nilpotent}$$

S-Schiffmann: $S = T^*X$; Diaconescu-Porta-S-Schiffmann-Vasserot:

1. $\exists$ a unital associative algebra structure on

$$H_*^{(T)}(\mathbb{R}\underline{\mathrm{Coh}}_C(S))$$

$$\Longrightarrow \mathrm{COHA}^{(T)}_{S,C} =: \mathrm{HA}^{(T)}_{S,C}$$

2. $\mathbf{HA}^{(T)}_{S,C}$ depends "locally" on C, i.e., given (S_1, C_1) and (S_2, C_2) such that the formal completions $\widehat{(S_1)}_{C_1} \simeq \widehat{(S_2)}_{C_2}$ we have:

$$\mathbf{HA}^{(T)}_{S_1,C_1} \simeq \mathbf{HA}^{(T)}_{S_2,C_2}$$

Now, the questions I would like to address are:

- Q1: Do $\mathbf{COHA}^{(T)}_{S}$, $\mathbf{COHA}^{(T)}_{S,0\text{-dim}}$, and $\mathbf{COHA}^{(T)}_{S,\leq 1}$ admit a presentation by gen.s and rel.s?

- Q2: Do $\mathbf{COHA}^{T}_{S}$, $\mathbf{COHA}^{T}_{S,0\text{-dim}}$, and $\mathbf{COHA}^{T}_{S,\leq 1}$ geometrically realize halves of Yangians?

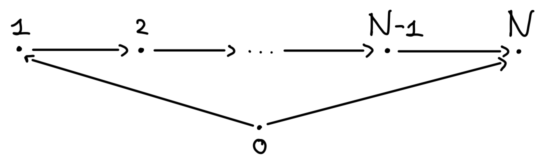
We can also formulate Q1 and Q2 for $\mathbf{HA}^{(T)}_{S,C}$.

Before addressing these questions, let me introduce the relevant quantum group to consider.

2. Intermezzo: Yangians

Fix a quiver $Q = (I = \{\text{vertices}\}, \Omega = \{\text{edges}\})$, i.e., an oriented graph.

Examples

- Q = one-loop quiver : 
- $Q = A_N$-quiver : $\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \cdots \longrightarrow \overset{N-1}{\bullet} \longrightarrow \overset{N}{\bullet}$
- $Q = A_N^{(1)}$-quiver : 

Fix a subgroup $T \subseteq T_{max} = (\mathbb{C}^*)^{\Omega} \times \mathbb{C}^*$.

The relevant quantum group to consider is :

$$\boxed{Y_{Q,T}^{MO} = \text{Maulik-Okounkov Yangian of } Q}$$

Key facts:

- $\mathbb{Y}^{MO}_{Q,T}$ = Hopf algebra$/H^*_T(pt)$
- $\mathbb{Y}^{MO}_{Q,T} = \mathrm{Sym}\big(\underbrace{\mathfrak{g}^{MO}_{Q,T}[u]}_{\text{current Lie algebra}}\big)$ as graded vector spaces

$$\Longrightarrow \mathbb{Y}^{MO}_{Q,T} = \text{deformation of } U\big(\mathfrak{g}^{MO}_{Q,T}[u]\big)$$

- Botta-Davison: $\mathfrak{g}^{MO}_{Q,T} \simeq \mathfrak{g}^{MO}_{Q} \otimes H^*_T(pt)$

Attention ⚠: $\mathfrak{g}^{MO}_{Q}$ = Maulik-Okounkov Lie algebra
= BPS Lie algebra of Q

Examples

1. Q = finite ADE quivers (e.g. $Q = A_N$)

McBreen: $\mathbb{Y}^{MO}_{Q,\mathbb{C}^*} = \mathbb{Y}^{D}_{Q,\mathbb{C}^*}$ as Hopf algebras

Attention ⚠: ∃ another Yangian:

$$\boxed{\mathbb{Y}^{D}_{Q,\mathbb{C}^*} = \text{Drinfel'd Yangian}}$$

Key facts:

- $\mathbb{Y}^{D}_{Q,\mathbb{C}^*}$ = Hopf algebra$/H^*_{\mathbb{C}^*}(pt)$, given by gen.s and rel.s
- Nakajima, Varagnolo, Guay:

definition of $\mathbb{Y}^{D}_{Q,\mathbb{C}^*}$ for any quiver without ede-loops

2. Q = affine ADE quivers (e.g. $Q = A^{(1)}_N$)

Jindal in Type A, Diaconescu-Porta-S-Schiffmann-Vasserot:

$$\boxed{\mathbb{Y}^{MO}_{Q,(\mathbb{C}^*)^2} = \mathbb{Y}^{D}_{Q,(\mathbb{C}^*)^2}}$$

where

- $\mathbb{Y}^{D}_{Q,(\mathbb{C}^*)^2}$ is given by gen.s and rel.s

(definition given by Kodera, Bershtein-Tsymbaliuk)

- Guay-Regelskis-Wendlandt, DPSSV:

$\exists$ a filtration $F^\cdot \subset \mathbb{Y}^D_{Q,(\mathbb{C}^*)^2}$ such that

$$\mathrm{gr}_{F^\cdot} \mathbb{Y}^D_{Q,(\mathbb{C}^*)^2} \simeq U(\mathfrak{g}_{ell}) \otimes H^*_{(\mathbb{C}^*)^2}(pt)$$

where

$$\begin{cases} \mathfrak{g}_{ell} := \text{u.c.e.}\left(\mathfrak{g}_{Q_{fin}}[s^{\pm 1}, u]\right) = \text{elliptic Lie algebra} \\ \mathfrak{g}_{Q_{fin}} := \text{simple Lie algebra associated to the finite ADE quiver } Q_{fin} \end{cases}$$

3. Q = one-loop quiver.

Maulik-Okounkov, Schiffmann-Vasserot:

$$\mathbb{Y}^{MO}_{1\text{-loop},(\mathbb{C}^*)^2} = SH^c := \text{SV's degenerate spherical DAHA of type } GL_\infty$$

Key facts:

- SH^c is defined by generators and relations
- $\exists$ a filtration $F^\cdot \subset SH^c$ such that

$$\mathrm{gr}_F \cdot SH^c \simeq U(W_{1+\infty}) \otimes H^*_{(\mathbb{C}^*)^2}(pt)$$

where

$W_{1+\infty} :=$ u.c.e.(Lie algebra of regular differential operators on $\mathbb{C}^*$) $= \langle c, z^\ell D^k : \ell \in \mathbb{Z}, k \in \mathbb{N} \rangle / \text{rel.s}$

Attention ⚠: both $\mathfrak{g}_{ell}$ and $W_{1+\infty}$ are examples of affinized BPS Lie algebras.

3. COHAs of surfaces: "Quantum" nature

Let's state again questions Q1 and Q2 for 0-dimensional sheaves:

- Q1: Do $\mathrm{COHA}^{(T)}_{S,0\text{-dim}}$ and $\mathrm{HA}^{(T)}_{S,C,0\text{-dim}}$ admit a presentation by gen.s and rel.s?
- Q2: Do $\mathrm{COHA}^{T}_{S,0\text{-dim}}$ and $\mathrm{HA}^{T}_{S,C,0\text{-dim}}$ geometrically realize halves of Yangians?

For $S=\mathbb{C}^2$, we have the following answer.

Schiffmann-Vasserot, Negut:

$$\mathrm{COHA}^{(\mathbb{C}^*)^2}_{\mathbb{C}^2,0\text{-dim}} \simeq \mathbb{Y}^{MO,+}_{1\text{-loop},(\mathbb{C}^*)^2}$$

$$\mathrm{HA}^{(\mathbb{C}^*)^2}_{\mathbb{C}^2,\mathbb{C},0\text{-dim}} \simeq \mathbb{Y}^{MO,-}_{1\text{-loop},(\mathbb{C}^*)^2}$$

In dim $=0$, we have a complete characterization:

Mellit-Minets-Schiffmann-Vasserot:

Assume that S has pure cohomology. Then, $\mathrm{COHA}_{S,0\text{-dim}}$ can be described explictly in terms of generators and relations.
In particular, if $\omega_S \simeq \mathcal{O}_S$:

$$\mathrm{COHA}_{S,0\text{-dim}} \simeq U\left(W^+_{1+\infty}(S)\right)$$

where $W_{1+\infty}(S) = W_{1+\infty}$-algebra associated to $H^*(S)$.

Remark:

$\exists$ also a T-equivariant version of the above result.

Attention ⚠:

The above explicit description was an important ingredient in the proof of the P=W conjecture by Hausel-Mellit-Minets-Schiffmann.

In general, it is an open problem to answer Q1 and Q2 for $COHA^{(T)}_{S,\leq 1}$ and $HA^{(T)}_{S,C}$.

Now, we will address Q2 for $HA^{(T)}_{S,C}$ when

$$\boxed{S = \text{minimal resolution of ADE singularity}}$$

- $G \subset SL(2,\mathbb{C})$ finite subgroup

$\updownarrow$

Q_{fin} = finite ADE quiver, Q = affine ADE quiver

- $\pi : S \longrightarrow \mathbb{C}^2/G$ Kleinian resolution of singularities
- $C := \pi^{-1}(0)$. Then, $C_{red} = C_1 \cup \cdots \cup C_e$, $C_i \simeq \mathbb{P}^1$

  $(C_i \cdot C_j) = -$ Cartan matrix of Q_{fin}

- torus $T \subset GL(2,\mathbb{C})$ centralizing G
  ($T =$ trivial, $\mathbb{C}^*$, or $\mathbb{C}^* \times \mathbb{C}^*$)

Example

$G = \mathbb{Z}_2 \Longrightarrow Q_{fin} = A_1 = \cdot$, $Q = A_1^{(1)}$: $\bullet \rightleftarrows \bullet$

$\Longrightarrow S = T^*\mathbb{P}^1 \supset C = \mathbb{P}^1 =$ zero section

Recall the derived McKay correspondence:

$$\tau : D^b(\mathrm{Coh}(S)) \xrightarrow{\sim} D^b(\mathrm{Mod}(\Pi_Q))$$

where

- $\Pi_Q =$ preprojective algebra of Q

  $:= \mathrm{End}($tilting bundle on S inducing $\tau)$

- $\mathrm{Mod}(\Pi_Q)$ = finite-dimensional right Π_Q-modules $=: (\Pi_Q\text{-repr.s})$

derived McKay correspondence

$\Downarrow$

$$\begin{cases} K_0(\mathrm{Coh}_c(S)) \xrightarrow{\sim} K_0(\mathrm{nilp}(\Pi_Q)) \simeq \text{root lattice } Y \text{ of } Q \\ \mathrm{Pic}(S) \xrightarrow{\sim} \text{coweight lattice } \check{X}_{fin} \text{ of } Q_{fin} \end{cases}$$

where $\mathrm{nilp}(\Pi_Q) \subset \mathrm{Mod}(\Pi_Q)$ is the subcategory of nilpotent repr.s of Π_Q.

Recall

$\mathfrak{g}_{ell}$:= universal central extension of $\mathfrak{g}_{Q_{fin}}[s^{\pm 1}, u]$

$$\simeq \mathfrak{g}_{Q_{fin}}[s^{\pm 1}, u] \oplus K \qquad \Big(K = \bigoplus_{\ell \in \mathbb{N}} \mathbb{Q}\, c_\ell \oplus \bigoplus_{\substack{k \in \mathbb{Z}_{\neq 0} \\ \ell \in \mathbb{N}_{\geq 1}}} \mathbb{Q}\, c_{k,\ell} \Big)$$

central elements

is graded w.r.t. $Y \times \mathbb{Z}\delta_u$.

Example

For $G = \mathbb{Z}_2$: $\mathfrak{g}_{ell} = \mathfrak{sl}(2)[s^{\pm 1}, u] \oplus K$

Define

$$\mathfrak{g}_{ell}^{+} := \text{Lie subalgebra of } \mathfrak{g}_{ell} \text{ associated to the monoid } K_0(\mathrm{Coh}_c(S))^{+} \subset Y \text{ of effective classes}$$

$$= \mathfrak{n}^{-}_{Q_{fin}}[s^{\pm 1}, u] \oplus s^{-1}\mathfrak{h}_{Q_{fin}}[s^{-1}, u] \oplus \bigoplus_{k<0} \mathbb{Q}\, c_{k,\ell}$$

where $\mathfrak{g}_{Q_{fin}} = \mathfrak{n}^{-}_{Q_{fin}} \oplus \mathfrak{h}_{Q_{fin}} \oplus \mathfrak{n}^{+}_{Q_{fin}}$ is a triangular decomposition.

Theorem (Diaconescu-Porta-S-Schiffmann-Vasserot)

- $\exists$ a canonical algebra isomorphism

$$\mathbb{HA}_{S,C} \simeq \widehat{U}(\mathfrak{g}_{ell}^{+})$$

(completion)

- $\exists$ a canonical algebra isomorphism

$$\mathbb{HA}^{T}_{S,C} \simeq \mathbb{Y}_{S,C}$$

where $\mathbb{Y}_{S,C}$ is a deformation of $\widehat{U}(\mathfrak{g}_{ell}^{+})$, i.e., $\exists$ a filtration $F^{\bullet} \subset \mathbb{Y}_{S,C}$ such that $\mathrm{gr}_{F^{\bullet}} \mathbb{Y}_{S,C} \simeq \widehat{U}(\mathfrak{g}_{ell}^{+}) \otimes H_T^{*}(pt)$

Let us recall the questions I have stated. First:

Q: Does $\mathbb{HA}^{T}_{S,C}$ geometrically realizes a half of a Yangian?

Conjecture

$\mathbb{HA}^{T}_{S,C} \simeq \mathbb{Y}_{S,C}$ realizes a half of a completion of $\mathbb{Y}^{MO}_{Q,T}$

4. Proof of the Theorem

Note that derived McKay correspondence

$$\tau : D^b(\mathrm{Coh}_c(S)) \xrightarrow{\sim} D^b(\mathrm{nilp}(\Pi_Q))$$

is not t-exact w.r.t. the standard t-structures, i.e.,

$$\begin{array}{c} \mathrm{Coh}_c(S) \xrightarrow{\cancel{\ \sim\ }} \mathrm{nilp}(\Pi_Q) \\ \Downarrow \\ \underline{\mathrm{Coh}}_c(S) \not\simeq \Lambda_Q = \text{stack of nilpotent repr.s of } \Pi_Q \end{array}$$

$$\Downarrow$$

$$HA^{T}_{S,C} \;\cancel{\simeq}\; COHA^{T,nil}_{Q}$$

where $COHA^{T,nil}_{Q}$ = Schiffmann-Vasserot's nilpotent 2d COHA of the quiver Q

Attention ⚠: The relation between $Coh_c(S)$ and $nilp(\Pi_Q)$ is more subtle:

$$\boxed{Coh_c(S) = \text{"limit" of } nilp(\Pi_Q)}$$

More precisely,

- hearts of bounded t-structures on $\mathcal{C} = D^b(nilp(\Pi_Q))$ form a partial ordered set:

$$\boxed{H_1 := \mathcal{C}_1^{\geq 0} \cap \mathcal{C}_1^{\leq 0} \leq H_2 := \mathcal{C}_2^{\geq 0} \cap \mathcal{C}_2^{\leq 0} \iff \mathcal{C}_1^{\geq 0} \subseteq \mathcal{C}_2^{\geq 0}}$$

- $\check{\theta} \in \check{X}_{fin}$ strictly dominant (i.e. $\check{\theta}(\alpha_i) > 0, \forall i$) $\rightsquigarrow \mathcal{L}_{\check{\theta}}$ π-ample

Set

$$\tilde{L}_{\check{\Theta}} := \tau \circ (\mathcal{L}_{\check{\Theta}} \otimes -) \circ \tau^{-1} : D^b(\mathrm{nilp}(\Pi_Q)) \longrightarrow D^b(\mathrm{nilp}(\Pi_Q))$$

- Shimpi: $\boxed{\inf_{n \geq 0} \tilde{L}_{\check{\Theta}}^{-n}(\mathrm{nilp}(\Pi_Q)) = \mathrm{Coh}_c(S)}$

Now we translate Shimpi's result in the theory of COHAs.

We will use:

- Bridgeland stability conditions
- braid group actions on bounded derived cat.s

Now, fix a strictly dominant coweight $\check{\Theta} \in \check{X}_{fin} \hookrightarrow \check{X}$.

$\check{\Theta}$ defines a King's stability condition on $\mathrm{nilp}(\Pi_Q)$:

$$Z_{\check{\Theta}} : K_0(\mathrm{nilp}(\Pi_Q)) \simeq \mathbb{Z}I \longrightarrow \mathbb{C}$$
$$\underline{d} \longmapsto -(\check{\Theta}, \underline{d}) + (\check{\rho}, \underline{d})\, i$$

$\check{\rho}(\alpha_i) = 1$

$$\Longrightarrow \boxed{\exists\ (Z_{\check{\theta}}, \mathcal{P}_{\check{\theta}}) = \text{Bridgeland's stability condition on } D^b(\mathrm{nilp}(\Pi_Q))}$$

Here, $\mathcal{P}_{\check{\theta}}$ = slicing = family of full additive subcats

$$\mathcal{P}_{\check{\theta}}(\phi) \subset D^b(\mathrm{nilp}(\Pi_Q)) \quad \forall \phi \in \mathbb{R}$$

satisfying certain conditions.

Set $\mathcal{P}_{\check{\theta}}(I) := \langle \mathcal{P}_{\check{\theta}}(\phi) : \phi \in I \rangle$ $\forall$ interval $I \subset \mathbb{R}$

Lemma

1. $\forall k \in \mathbb{Z}$, $\tilde{L}_{\check{\theta}}^{-2k} : \mathrm{nilp}(\Pi_Q) \xrightarrow{\sim} \mathcal{P}_{\check{\theta}}((\nu_{-k}, \nu_{-k}+1]))$

with $\nu_\ell := \frac{1}{\pi} \arctan(2h\ell)$ Coxeter number

2. $\mathcal{P}_{\check{\theta}}((-\frac{1}{2}, \frac{1}{2}]) \simeq \mathrm{Coh}_c(S)$

For $\ell, k \in \mathbb{N}$, $k \geq \ell$, set

$$\boxed{\Lambda_Q^{\ell,k} := \text{derived moduli stack of objects in } \mathcal{P}_{\check{\Theta}}((\nu_{-\ell}, \nu_{-k}+1])}$$

Attention: $\tilde{L}^{2k}_{\check{\Theta}} : \Lambda_Q^{\ell,k} \xrightarrow{\sim} \Lambda_Q^{\ell-k,0} \simeq \underbrace{\Lambda_Q^{>\nu_{-\ell+k}}}_{\text{Harder-Narasimhan stratum}} \subset \Lambda_Q = \text{usual}$

Lemma

1. The vector space

$$\boxed{HA^{(T)}_{\check{\Theta}} := \lim_{\ell} \operatorname*{colim}_{k \geq \ell} H^{(T)}_{*}(\Lambda_Q^{\ell,k})}$$

has the structure of an unital associative algebra with multiplication induced from that of $COHA_Q^{(T),nil}$

2. $\exists$ an algebra isomorphism $HA^{(T)}_{S,C} \simeq HA^{(T)}_{\check{\Theta}}$

Corollary

$HA^{(T)}_{\check{\Theta}}$ does not depend on the specific choice of strictly dominant finite coweight.

Now, we need to relate $HA^{T}_{\check{\theta}}$ to Yangians.

Recall that

- the extended affine braid group

$$\boxed{B_{ex} \simeq \left(B_{fin} \cup \{L_{\check{\lambda}} : \lambda \in \check{X}_{fin}\}\right) / \text{rels}}$$

Lemma

$\exists$ a group homomorphism

$$\rho : B_{ex} \longrightarrow \operatorname{Aut}(D^b(\text{nilp}(\Pi_Q)))$$

such that $\rho(L_{\check{\lambda}}) = \tilde{L}_{\check{\lambda}} \;\; \forall \lambda \in \check{X}_{fin}$

Furthermore, we have

Schiffmann-Vasserot: $COHA^{T,nil}_{Q} \simeq Y^{-}_{Q} = Y^{MO,-}_{Q,(\mathbb{C}^*)^2}$

Finally, a careful analysis of the compatibility between the action of B_{ex} on Yangians and on $COHA^{T,nil}_{Q}$

yields:

Lemma

$\exists$ an algebra isomorphism $HA^{T}_{\check{\Theta}} \xrightarrow{\sim} \mathbb{Y}_{S,C}$, where

$$\boxed{\mathbb{Y}_{S,C} := \lim_{\ell} T^{2\ell}_{\check{\Theta}}(\mathbb{Y}^{-}_{Q}) \big/ T^{2\ell}_{\check{\Theta}}(J)}$$

where $J := \sum_{\mu_{\check{\Theta}}(\underline{d})>0} \mathbb{Y}^{-}_{Q} \cdot \mathbb{Y}^{-}_{Q,-\underline{d}}$ and $\mu_{\check{\Theta}}(\underline{d}) := \frac{(\check{\Theta}, \underline{d})}{(\check{\rho}, \underline{d})}$.

The proof of the Thm follows from the above lemmas.

$\square$

Let's address what happens when we drop "strictly" from $\check{\Theta}$.

Fix $J \subseteq I_{fin}$ and let $\check{\Theta} \in \check{X}_{fin}$ be any dominant coweight such that

$$\check{\Theta}(\alpha_i) = 0 \quad \forall i \in I_f \setminus J$$

simple root

Shimpi: $\inf_{n \geq 0} L^{-n}_{\check{\Theta}}(\mathrm{nilp}(\Pi_Q)) = P_C(S/S_J)$

where

- contraction of C_i, for $i \in I_{fin} \setminus J$:

$$\begin{tikzcd} S \arrow[rr, "\pi_J"] \arrow[dr, "\pi"'] & & S_J \arrow[dl] \\ & \mathbb{C}^2/G & \end{tikzcd}$$

- $P_c(S/S_J) \subset D^b(\mathrm{Coh}_c(S))$ heart of the tilted t-structure with respect to

$$\begin{cases} \mathcal{T}_J := \left\{ \mathcal{E} \in \mathrm{Coh}_c(S) : \mathbb{R}^1\pi_{J*}(\mathcal{E}) = 0,\ \mathrm{Hom}(\mathcal{E}, \mathcal{F}) = 0 \ \forall \mathcal{F} \text{ s.t. } \mathbb{R}\pi_{J*}(\mathcal{F}) = 0 \right\} \\ \mathcal{F}_J := \left\{ \mathcal{E} \in \mathrm{Coh}_c(S) : \mathbb{R}^0\pi_{J*}(\mathcal{E}) = 0 \right\} \end{cases}$$

($P_c(S/S_J)$ = Van der Bergh's abelian category of perverse coherent sheaves set-theoretically supported on C)

$\Longrightarrow \exists$ COHA$^{(T)}$ of $P_c(S/S_J) := HA_J^{(T)}$

Theorem (S-Schiffmann-Shimpi)

- $\exists$ a canonical algebra isomorphism
$$\mathbb{HA}_J \simeq \hat{U}(\mathfrak{g}_{ell}^J)$$

where $\mathfrak{g}_{ell}^J$ is a positive half of $\mathfrak{g}_{ell}$.

- $\exists$ a canonical algebra isomorphism
$$\mathbb{HA}_J^T \simeq \mathbb{Y}_J$$

where $\mathbb{Y}_J$ is a deformation of $\hat{U}(\mathfrak{g}_{ell}^J)$

Now, let

- $\mathrm{Stab}(S) = \mathrm{Stab}(D^b(\mathrm{Coh}_c(S)))$ = space of stability conds
- $\mathrm{Stab}^\circ(S) \subset \mathrm{Stab}(S)$ be its distinguished connected component.

Then, $\forall\, x \in (Z, P) \in \mathrm{Stab}^\circ(S)$ and $\forall\, I \subset \mathbb{R}$ interval of lenght one, the abelian category $\mathcal{P}(I)$ can be recovered from $\mathcal{P}_c(S/S_J)$ for some $J \subset I_{fin}$, up to shifts and the action of the affine braid group.

Therefore, we get:

Theorem (S-Schiffmann-Shimpi)
$\forall\, x = (\mathcal{P}, Z) \in \mathrm{Stab}^\circ(S)\ \exists\ HA_x^{(T)}$ associated to $\mathcal{P}(0,1]$
Moreover,
$$HA_x^T = \text{deformation of } \widehat{U}^x(\mathfrak{g}_{ell}^x)$$