Yangians

and

Cohomological Hall algebras

Plan of the talk:

- affine Yangians and COHAs of quivers
- A conjectural new half of the affine Yangian
- Relation to COHAs of surfaces

Affine Yangians and nilpotent quiver COHA

We start by describing a relation between

$$\begin{tikzcd}
\boxed{Y_Q = \text{affine Yangian}} \arrow[dr, leftrightarrow] & & \boxed{\begin{array}{l} COHA_Q^T = \text{nilpotent} \\ \quad \text{quiver COHA} \end{array}} \arrow[dl, leftrightarrow] \\
& \boxed{\text{Shuffle algebra}} &
\end{tikzcd}$$

Let me start by introducing the shuffle algebra I will consider.

Fix an affine ADE quiver $Q = (I, \Omega)$, with
- I = set of vertices $= \{0, 1, \dots, e\}$
- Ω = set of edges

Let $Q_{fin} := (I_{fin} = \{1, \dots, e\}, \Omega_{fin})$ be the corr. finite ADE quiver.

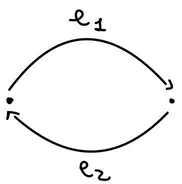
Recall the bilinear form on $\mathbb{C}I$: $\forall\, \underline{d}^1, \underline{d}^2 \in \mathbb{C}I$

$$\boxed{\langle \underline{d}^1, \underline{d}^2 \rangle := \sum_{i \in I} d_i^1 d_i^2 - \sum_{e: i \to j \in \Omega} d_i^1 d_j^2}$$

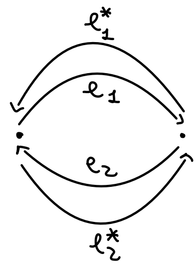
Recall that the double $\overline{Q}$ of Q is:

$$\overline{Q} := \Big(I, \overline{\Omega} := \Omega \sqcup \underset{\{e^* : j \to i \;\mid\; e : i \to j \in \Omega\}}{\underset{\shortparallel}{\Omega^{op}}}\Big)$$

Ex:
$Q = A_1^{(1)}$:



$\rightsquigarrow \overline{Q}$:



Fix the torus $T := \mathbb{C}^* \times \mathbb{C}^*$. Then $H_T := H_T^*(pt) = \mathbb{C}[\varepsilon_1, \varepsilon_2]$
Set $\hbar := \varepsilon_1 + \varepsilon_2$.

We introduce the following H_T-module:

$$\boxed{Sh := \bigoplus_{\underline{d} \in \mathbb{N}I} Sh_{\underline{d}}, \quad Sh_{\underline{d}} := H_T\left[z_{i,\ell} : i \in I, 1 \le \ell \le d_i\right]^{\mathfrak{S}_{\underline{d}}}}$$

where $\mathfrak{S}_{\underline{d}} := \prod_{i \in I} \mathfrak{S}_{d_i}$

$\mathfrak{S}_{d_i}$ = permutation group of $\{z_{i,1}, \dots, z_{i,d_i}\}$

Definition

The shuffle algebra Sh associated with $\overline{Q}$ is Sh_Q endowed with a graded associative algebra structure given by: $\forall\ P_1 \in Sh_{\underline{d}^1},\ P_2 \in Sh_{\underline{d}^2}$

$$(P_1 * P_2)\left(z_{[1,\underline{d}^1+\underline{d}^2]}\right) := (-1)^{\langle \underline{d}^1, \underline{d}^2\rangle + 1} \sum_{\sigma} \left(\prod_{i \in I} \zeta_i\left(z_{[1,\underline{d}^1]}, z_{[\underline{d}^1+1, \underline{d}^1+\underline{d}^2]}\right) \right.$$

$$\left. \prod_{\substack{i,j \in I \\ i \neq j}} \zeta_{i,j}\left(z_{[1,\underline{d}^1]}, z_{[\underline{d}^1+1, \underline{d}^1+\underline{d}^2]}\right) P_1\left(z_{[1,\underline{d}^1]}\right) P_2\left(z_{[\underline{d}^1+1, \underline{d}^1+\underline{d}^2]}\right) \right)$$

where

- σ runs among all I-tuples of (d_i, e_i)-shuffles,
- $$\zeta_i\left(z_{[1,\underline{d}^1]}, z_{[\underline{d}^1+1, \underline{d}^1+\underline{d}^2]}\right) := \prod_{\substack{1 \leq k \leq d_i^1 \\ d_i^1+1 \leq l \leq d_i^1+d_i^2}} \frac{z_{i,k} - z_{i,l} - \hbar}{z_{i,k} - z_{i,l}}$$
- $$\zeta_{i,j}\left(z_{[1,\underline{d}^1]}, z_{[\underline{d}^1+1, \underline{d}^1+\underline{d}^2]}\right) :=$$
$$\prod_{\substack{1 \leq k \leq d_i^1 \\ d_i^1+1 \leq l \leq d_i^1+d_j^2}} \left((z_{i,k} - z_{j,l} + \varepsilon_1)(z_{i,k} - z_{j,l} + \varepsilon_2) \right)^{\#\{e : i \to j \in \Omega\}}$$

- $P_1(z_{[1,d^1]})$ is a function in the variables $z_{i,k}$ with $i \in I$ and $1 \leq k \leq d_i^1$, and similarly for $P_2, \ldots$

Definition

We call **nilpotent quiver cohomological Hall algebra** COHA_Q of Q the subalgebra of Sh:

$$\mathrm{COHA}_Q^T := \langle z_{i,1}^k \in Sh_{\alpha_i} : \forall i \in I,\ k \in \mathbb{N} \rangle$$

where α_i = simple root associated with $i \in I$.

Attention ⚠:

The original definition of COHA_Q^T is within the theory of cohomological Hall algebras.
It makes use of the moduli stack of

$$\boxed{\text{nilpotent finite-dimensional representations of the preprojective algebra } \Pi_Q \text{ of } Q}$$

Theorem (Diaconescu-Porta-S.-Schiffmann-Vasserot)

There is a graded H_T-algebra isomorphism

$$\boxed{\begin{array}{ccc} \mathbb{Y}_Q^- & \xrightarrow{\sim} & \mathrm{COHA}_Q^T \\ x_{i,k}^- & \longmapsto & z_{i,1}^k \end{array}}$$

where $\mathbb{Y}_Q^-$ is the negative part of the 2-parameter version of the Drinfeld's Yangian $\mathbb{Y}_Q$ of Q.

Attention ⚠:
$\mathbb{Y}_Q$ is defined in terms of generators $x_{i,k}^{\pm}, h_{i,k}$ with $i \in I$ and $k \in \mathbb{N}$, and relations.
Since I will not use these relations, I will not introduce them.

By combining with the recent result of Schiffmann-Vasserot (arXiv:2312.15803):

$$\boxed{\mathrm{COHA}_Q^T \xrightarrow{\sim} \mathbb{Y}_Q^{MO,-}}$$

where $\mathbb{Y}_Q^{MO}$ is the Maulik-Okounkov (core) Yangian of Q, we get:

Corollary
There is a graded H_T-algebra isomorphism $\mathbb{Y}_Q^- \xrightarrow{\sim} \mathbb{Y}_Q^{MO,-}$

Attention ⚠:

Given an arbitrary quiver (also with edge-loops), Maulik and Okounkov defined a Yangian $\mathbb{Y}_Q^{MO}$ using

- the theory of stable envelopes,
- T-equivariant cohomology of Nakajima quiver varieties $\mathcal{M}_Q^{\check{\theta}}(\underline{v}, \underline{w})$

$\Longrightarrow$ to define an R-matrix

Recall that

coweight of Q

$$\boxed{\mathcal{M}_Q^{\check{\theta}}(\underline{v}, \underline{w}) = \text{moduli space of } \check{\theta}\text{-stable finite-dimensional representations of } \Pi_{Q^{fr}} \text{ of the framed quiver } Q^{fr} \text{ of } Q}$$

Maulik and Okounkov defines the R-matrix for an arbitrary $\check{\theta}$, BUT all the explicit computations are made for $\underbrace{\check{\theta} > 0}_{\text{i.e., strictly dominant}}$.

Goal of the talk:

- describe a conjectural new half of $\mathbb{Y}_Q$
- mention how it arises from cohomological Hall algebras.

A conjectural new half

As a first thing, I need to recall the action on $\mathcal{Y}_Q$ of

$$\boxed{B_{ex} := \text{extended affine braid group of } Q}$$

Recall that

- $B_{ex} = \left\langle T_w : w \in W_{ex} \right\rangle \Big/ T_v T_w = T_{vw} \text{ if } \ell(v) + \ell(w) = \ell(vw)$

- W_{fin} = Weyl group of Q_{fin}, $\check{X}_{fin}$ = co lattice of Q_{fin}
  W_{aff} = —— " —— of Q, Γ = group of outer automs of Q

$$\Longrightarrow W_{ex} = W_{fin} \ltimes \check{X}_{fin} = \Gamma \ltimes W_{aff} \Longrightarrow B_{ex} = \Gamma \ltimes B_{aff}$$

- $\exists$ an action of B_{aff} on $\mathcal{Y}_Q$ given by

$$T_i \longmapsto \exp(\mathrm{ad}(x^+_{i,0}))\exp(-\mathrm{ad}(x^-_{i,0}))\exp(\mathrm{ad}(x^+_{i,0}))$$

Since Γ acts naturally on $\mathcal{Y}_Q$, we get an action of B_{ex} on $\mathcal{Y}_Q$

Now, $\forall\, w \in W_{ex}$, define

$$\boxed{\overline{T}_w : \mathbb{Y}_Q^- \hookrightarrow \mathbb{Y}_Q \xrightarrow{T_w} \mathbb{Y}_Q \xrightarrow{pr} \mathbb{Y}_Q^-}$$

induced by the triangular decomposition

Proposition

$\exists$ an action B_{ex}^+ on $\mathbb{Y}_Q^-$ given by

$$\boxed{T_w \in B_{ex}^+ \longmapsto \overline{T}_w \in \mathrm{End}(\mathbb{Y}_Q^-)}$$

where $B_{ex}^+ \subset B_{ex}$ is the submonoid generated by T_w with $w \in W_{ex}$

Now, I am ready to define my (conjecturally) "new half".

Fix $\check\Theta \in \check X_{aff}$ such that $(\check\Theta, \alpha_i) > 0 \ \forall i \in I_{fin}$, $(\check\Theta, \delta) = 0$

Imaginary

$\Longrightarrow \exists! \, \check\Theta_{fin} \in \check X_{fin}$ associated with $\check\Theta$, which is strictly dominant.

Define for any $k \in \mathbb{Z}$

$$\boxed{\mathbb{Y}_{\check\Theta,(k)} := \overline{T}_{2k\check\Theta_{fin}}\left(\mathbb{Y}_Q^-\right) \Big/ \overline{T}_{2k\check\Theta_{fin}}\Big(\sum_{\substack{\underline{d} \in \mathbb{N}I \\ \mu_{\check\Theta}(\underline{d}) > 0}} \mathbb{Y}_Q^- \, \mathbb{Y}_{-\underline{d}}^-\Big)}$$

Here :

$$\mu_{\check{\theta}}(\underline{d}) := \frac{(\check{\theta}, \underline{d})}{(\check{\rho}, \underline{d})}$$

$\sum_I$ fund. coweights

is the $\check{\theta}$-slope of the root $\underline{d} \in \mathbb{Z}I$ and $(-,-)$ denotes the canonical pairing between coweights and roots.

Attention ⚠ :

- $\mathbb{Y}_{\check{\theta},(k)}$ is well-defined.
- $\forall\, k,\ \overline{T}_{-2k\check{\theta}_{fin}} : \mathbb{Y}_{\check{\theta},(k)} \xrightarrow{\sim} \mathbb{Y}_{\check{\theta},(0)}$

Note that $\exists$ canonical projection: $\mathbb{Y}_{\check{\theta},(k)} \xrightarrow{\pi_{k,k+1}} \mathbb{Y}_{\check{\theta},(k+1)}$

Definition ("New half"): $\mathbb{Y}_{\check{\theta},(\infty)} := \lim\limits_{k} \mathbb{Y}_{\check{\theta},(k)}$

Theorem (DPSSV)
There exists a canonical graded unital associative algebra structure on $\mathbb{Y}_{\check{\theta},(\infty)}$ induced by the multiplication on $\mathbb{Y}_Q$.

Remark (Explicit description of the multiplication)

Let $(x_k)_k$ and $(y_k)_k$ be elements of $\mathbb{Y}_{\check{\theta},(\infty)}$.
For a fixed k, let $\tilde{x}_k$ and $\tilde{y}_k$ be lifts in $\overline{T}_{2k\check{\theta}_{fin}}(\mathbb{Y}_Q^-)$
Now, for any $n<k$, set

$$\Pi_{n,k} := \Pi_{n,n+1} \circ \Pi_{n+1,n+2} \circ \dots \circ \Pi_{k-1,k} : \mathbb{Y}_{\check{\theta},(n)} \longrightarrow \mathbb{Y}_{\check{\theta},(k)}$$

Then, for a fixed k we have

$$\boxed{(x \cdot y)_k := \Pi_{n,k}(\tilde{x}_n \cdot \tilde{y}_n) \quad \text{for } n \ll k.}$$

Attention ⚠: Since $\mathbb{Y}_{\check{\theta},(k)} \xrightarrow{\sim} \mathbb{Y}_{\check{\theta},(0)}$, we could provide a description of $x \cdot y$ by using lifts:

$$\tilde{x}, \tilde{y} \in \mathbb{Y}_Q^-$$

To understand better $\mathbb{Y}_{\check{\theta},(\infty)}$, let us analyze its classical limit.

Recall that:

Proposition (Guay-Regelskis-Wendlandt for $Q \neq A_1^{(1)}$; DPSSV for $A_1^{(1)}$)

There is a graded H_T-algebra isomorphism

$$\boxed{U(L\mathfrak{g}) \otimes_{\mathbb{C}} H_T \xrightarrow{\sim} \mathrm{gr}\, \mathbb{Y}_Q}$$

that restricts to $U(L\mathfrak{n}) \otimes_{\mathbb{C}} H_T \xrightarrow{\sim} \mathrm{gr}\, \mathbb{Y}_Q^-$,

where

- "gr" is w.r.t. the standard filtration for which

$$\deg(x^{\pm}_{i,k}) = k = \deg(h_{i,k}) \quad \text{and} \quad \deg(\varepsilon_1) = 0 = \deg(\varepsilon_2)$$

- $L\mathfrak{g} := \text{u.c.e.}\left(\mathfrak{g}_{fin}[s^{\pm 1}, t]\right) \simeq \mathfrak{g}_{fin}[s^{\pm 1}, t] \oplus K$

$$\text{with } K := \bigoplus_{\ell \in \mathbb{N}} \mathbb{C} c_\ell \oplus \bigoplus_{\substack{\ell \geq 1 \\ k \in \mathbb{Z}, k \neq 0}} \mathbb{C} c_{k,\ell} \quad \text{central elements}$$

- $L\mathfrak{g}$ is graded by the monoid $\mathbb{Z}I \times \mathbb{N}\delta_t \simeq \mathbb{Z}I_{fin} \times \mathbb{Z}\delta \times \mathbb{N}\delta_t$

Then

$L\mathfrak{n} :=$ Lie algebra associated with the 'half'

$$(\Delta^+_{fin} \times \mathbb{Z}\delta \times \mathbb{N}\delta_t) \cup (\mathbb{Z}_{<0}\delta \times \mathbb{N}\delta_t) \subset \mathbb{Z}I \times \mathbb{N}\delta_t$$

$$\simeq \left(s^{-1}\mathfrak{g}_{fin}[s^{-1}] \oplus \mathfrak{n}_{fin}\right)[t] \oplus \bigoplus_{k<0} \mathbb{C} c_{k,\ell}$$

Corollary

There is an isomorphism $U(Ln_{\check{\theta},(k)}) \otimes_{\mathbb{C}} H_T \xrightarrow{\sim} gr\, \mathbb{Y}_{\check{\theta},(k)}$
where

- $Ln_{\check{\theta},(k)} := \bigoplus_{\underline{d} \in \Delta_{\check{\theta},(k)}} Ln_{-\underline{d}}$

- $\Delta_{\check{\theta},(k)} := \left\{ \underline{d} = \alpha + n\delta \;:\; \alpha \in \Delta^{+}_{fin} \cup \{0\},\; n + k(\check{\theta}_{fin}, \alpha) < 0 \right\}$

Now, we would like to understand what happens in the limit.

First, note that:

Remark

1. $\exists$ surjective morphism $U(Ln_{\check{\theta},(k)}) \longrightarrow U(Ln_{\check{\theta},(k+1)})$

2. $\Delta_{\check{\theta},(0)} \subset \Delta_{\check{\theta},(1)} \subset \Delta_{\check{\theta},(2)} \subset \cdots,\; \bigcup_{k} \Delta_{\check{\theta},(k)} = (\Delta^{+}_{fin} \times \mathbb{Z}\delta) \times (-\mathbb{N}\delta)$

(1) suggests we could take a limit of the $U(Ln_{\check{\theta},(k)})'s$, while (2) should give an indication of the "limit" Lie algebra.

More precisely, we have:

Theorem (DPSSV)

$$\boxed{\operatorname{gr} \mathbb{Y}_{\check{\theta},(\infty)} \simeq \lim_{K} U(Ln_{\check{\theta},(K)}) \otimes_{\mathbb{C}} H_T \simeq \widehat{U}^{\check{\theta}}(Ln') \otimes_{\mathbb{C}} H_T}$$

where

- $Ln' := \bigoplus_{\underline{d} \in \bigcup \Delta_{\check{\theta},(K)}} Ln_{\underline{d}} \simeq n^+_{fin}[s^{\pm 1}, t] \oplus s^{-1} h_{fin}[s^{-1}, t] \oplus \bigoplus_{k<0} \mathbb{C} c_{k,\ell}$

  $\subset Ln$

- $\widehat{U}^{\check{\theta}}(Ln')$ is a certain completion of $U(Ln')$ depending on the fixed $\check{\theta} \in \check{X}$.

Conjecture

$\mathbb{Y}_{\check{\theta},(\infty)}$ is an "half" of a completion $\widehat{\mathbb{Y}}_Q^{\check{\theta}}$ of the affine Yangian $\mathbb{Y}_Q$ with respect to $\check{\theta} \in \check{X}$.

Attention ⚠: $\widehat{\mathbb{Y}}_Q^{\check{\theta}}$ is defined by Maulik-Okounkov, but our ultimate goal is to have a description by generators and relations.

Relation to cohomological Hall algebras of surfaces

Q. How did $\mathbb{Y}_{\check{\theta},(\infty)}$ show up?

A. via COHAs of surfaces.

Recall that

- $Q_{fin} \longleftrightarrow G \subset SL(2,\mathbb{C})$ finite group
- G acts on $\mathbb{C}^2$
- $\pi : X \longrightarrow \mathbb{C}^2/G$ resolution of the singularity at the origin.

Theorem (DPSSV)

$\exists$ a cohomological Hall algebra $\mathrm{COHA}^T_{X,\pi^{-1}(0)}$ associated with the moduli stack of coherent sheaves on X set-theoretically supported on $\pi^{-1}(0)$.

Moreover, $\exists$ a graded algebra isomorphism

$$\mathrm{COHA}^T_{X,\pi^{-1}(0)} \simeq \mathbb{Y}_{\check{\theta},(\infty)}$$

Important: with this geometric viewpoint, we are able to say the following:

1. $\mathbb{Y}_{\check{\theta},(\infty)}$ does not depend on the choice of $\check{\theta} \in \check{X}$ s.t. $(\check{\theta}, \alpha_i) > 0$ $\forall i \in I$ and $(\check{\theta}, \delta) = 0$.

2. Since COHAs always realize 'halves' of a whole quantum group, $\mathbb{Y}_{\check{\theta},(\infty)}$ must be an half.

3. For $Q = A_1^{(1)}$ $\left(\longleftrightarrow G = \mathbb{Z}_2,\ X = T^*\mathbb{P}^1\right)$, $\exists$ a "geometrically defined" set of generators of $\mathbb{Y}_{\check{\theta},(\infty)}^{A_1^{(1)}}$

4. For Q = affine ADE quiver, we get a "geometrically defined" surjective morphism:

$$\boxed{\underbrace{\mathbb{Y}_{\check{\theta},(\infty)}^{A_1^{(1)}} \otimes \mathbb{Y}_{1\text{-loop}}^{MO,-} \ \mathbb{Y}_{\check{\theta},(\infty)}^{A_1^{(1)}} \otimes \mathbb{Y}_{1\text{-loop}}^{MO,-} \cdots \otimes \mathbb{Y}_{1\text{-loop}}^{MO,-} \ \mathbb{Y}_{\check{\theta},(\infty)}^{A_1^{(1)}}}_{\# I_{fin}\text{-times}} \twoheadrightarrow \mathbb{Y}_{\check{\theta},(\infty)}^{Q}}$$

Here, $\mathbb{Y}_{1\text{-loop}}^{MO} = \mathbb{Y}\left(\widehat{gl}(1)\right)$ = so-called affine Yangian of $gl(1)$

$\Longrightarrow$ By combining (3) and (4) we should be able to define a set of generators of $\mathbb{Y}_{\check{\theta},(\infty)}^{Q}$

We need to compute the rels for $\mathbb{Y}^{A_1^{(1)}}_{\check{\theta},(\infty)}$ and $\mathbb{Y}^{Q}_{\check{\theta},(\infty)}$:

this is very challenging!