From Okounkov's conjecture to Schiffmann's conjecture

(work in progress with Hennecart, Porta, and Schiffmann)

1. Kac polynomials for quivers

Fix a field k and a quiver

$$Q = (I = \{\text{vertices}\}, \Omega = \{\text{edges}\})$$

i.e., an oriented graph.

Examples

- $Q = A_1$: $\bullet$, $Q = A_2$: $\underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet}$, $Q = A_N$: $\underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \cdots \underset{N-1}{\bullet} \xrightarrow{\alpha_{N-1}} \underset{N}{\bullet}$
- Q = 1-loop quiver : $\bullet \circlearrowleft^{\alpha}$

We shall recall all the notions needed to define the Kac polynomial.

- kQ = path algebra of Q
  = k-vector space generated by all paths of length $\ell \geq 0$, with multiplication given by concatenation

Examples

- $Q = A_N$: $\underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \cdots \underset{N-1}{\bullet} \xrightarrow{\alpha_{N-1}} \underset{N}{\bullet}$

$$KA_N = K\langle e_i, \alpha_j : i=1,\dots,N,\ j=1,\dots,N-1\rangle$$

path of length zero at the vertex i

$$\xrightarrow{\ \sim\ } \text{algebra of upper triangular matrices of order } N/K$$

(path from i to j =) $p_{ij} \longmapsto E_{ij}$

- Q = 1-loop quiver: $\overset{\alpha}{\circlearrowleft}\bullet \rightsquigarrow KQ = K[\alpha]$

- A **representation** of Q is a left module M of KQ

$$\longleftrightarrow \text{the datum } \Big(\underbrace{\bigoplus_{i\in I} e_i M}_{I\text{-graded v.s.}},\ \big\{ f_\alpha : e_{\underset{\text{source}}{s(\alpha)}} M \longrightarrow e_{\underset{\text{target}}{t(\alpha)}} M \big\}_{\alpha\in\Omega} \Big)$$

Examples

- $Q = A_N$: $\underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \ \cdots\ \underset{N-1}{\bullet} \xrightarrow{\alpha_{N-1}} \underset{N}{\bullet}$

$$\Longrightarrow \quad \begin{array}{c} M_1 \xrightarrow{f_1} M_2 \longrightarrow \cdots \longrightarrow M_{N-1} \xrightarrow{f_{N-1}} M_N \\ \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{N-1}{\bullet} \xrightarrow{\alpha_{N-1}} \underset{N}{\bullet} \end{array}$$

- Q = 1-loop quiver: $\overset{\alpha}{\circlearrowleft}\bullet \rightsquigarrow (M, f\in \mathrm{End}(M))$

- A repr. M is finite-dimensional if K-vector space e_iM is finite-dimensional $\forall i \in I$.

Remark

M = f.d. repr. $\rightsquigarrow$ dimension vector $(\dim_K e_iM)_{i\in I} \in \mathbb{N}I$

- A repr. M is indecomposable if it is nonzero and cannot be written as a direct sum of proper subrep.

- A repr. M is absolutely indecomposable if $M \otimes_K \bar{K}$ is indecomposable over $\bar{K}$ = algebraic closure of K

Example

$$Q = A_N : \quad \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \quad \cdots \quad \underset{N-1}{\bullet} \xrightarrow{\alpha_{N-1}} \underset{N}{\bullet}$$

For $1 \leq i \leq j \leq N$, consider

$$M_{i,j} : \quad \begin{tikzcd} 0 \arrow[r, "0"] & \overset{K}{\overset{\shortparallel}{M_i}} \arrow[r, "id"] & \overset{K}{\overset{\shortparallel}{M_{i+1}}} \arrow[r, "id"] & \cdots \arrow[r, "id"] & \overset{K}{\overset{\shortparallel}{M_{j-1}}} \arrow[r, "id"] & \overset{K}{\overset{\shortparallel}{M_j}} \arrow[r, "0"] & 0 \\ \cdots \underset{i-1}{\bullet} \arrow[r] & \underset{i}{\bullet} \arrow[r, "\alpha_i"'] & \underset{i-1}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{j-1}{\bullet} \arrow[r, "\alpha_{j-1}"'] & \underset{j}{\bullet} \arrow[r] & \underset{j+1}{\bullet} \cdots \end{tikzcd}$$

$\Longrightarrow$ it is absolutely indecomposable.
All abs. indecomposable repr.s are of this form.

Theorem (Kac, 1982)

Fix a finite field $\mathbb{F}_q$ with q elements and $\underline{d} \in \mathbb{N}I$. Then $\exists!$ polynomial $A_{Q,\underline{d}}(t) \in \mathbb{Z}[t]$ such that

$$A_{Q,\underline{d}}(q) = \#\left\{ \begin{array}{l} \text{isom. classes of absolutely} \\ \text{indecomposable repr.s of } Q \\ \text{of dimension } \underline{d} \text{ over } \mathbb{F}_q \end{array} \right\}$$

Moreover, $A_{Q,\underline{d}}(t)$ does not depend on the orientation of Q.

Example

$$Q = A_N : \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \quad \cdots \quad \underset{N-1}{\bullet} \xrightarrow{\alpha_{N-1}} \underset{N}{\bullet}$$

We saw that $\forall$ abs. indecomposables are of the form $M_{i,j}$ for $1 \leq i \leq j \leq N$:

$$A_{A_N,\underline{d}}(t) = \begin{cases} 1 & \text{if } \underline{d} = \dim M_{i,j} \\ 0 & \text{otherwise} \end{cases}$$

Note that by Gabriel's theorem:

$$\dim M_{i,j} = (0, \dots, 0, \overset{i}{1}, \dots, \overset{j}{1}, 0, \dots, 0) \longleftrightarrow \text{positive root of } \mathfrak{sl}(N+1)$$

Recall that $\forall$ simple Lie algebra $\mathfrak{g}$ we have

$$\boxed{\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \quad \text{with } \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \quad \forall h \in \mathfrak{h}\}}$$

where $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra. Then

(root system =) $\Delta = \{\alpha \in \mathfrak{h}^* \setminus \{0\} : \mathfrak{g}_\alpha \neq 0\} = \underbrace{\Delta^+}_{\text{pos.}} \cup \underbrace{\Delta^-}_{\text{neg.}}$

For $\mathfrak{g} = \mathfrak{sl}(N+1)$, $\mathfrak{h} = \{A \in \mathfrak{sl}(N+1) : A \text{ diagonal}\}$:

$$\boxed{\{\dim M_{i,j} : 1 \leq i \leq j \leq N\} \simeq \Delta^+}$$

Attention ⚠:

This relation between $A_{Q,\underline{d}}(t)$ and Lie theory is a general phenomenon.

Examples

- Q = finite ADE quiver ($\longleftrightarrow$ simple Lie algebra)

$$A_{Q,\underline{d}}(t) = \begin{cases} 1 & \forall \underline{d} \in \Delta^+ \\ 0 & \text{otherwise} \end{cases}$$

- Q = affine ADE quiver ($\longleftrightarrow$ affine Lie algebra)

<u>Ex:</u> $Q = A_N^{(1)}$:

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r] \arrow[drr] & \overset{2}{\bullet} & \cdots & \overset{N-1}{\bullet} \arrow[r] & \overset{N}{\bullet} \arrow[dll] \\ & & \underset{0}{\bullet} & & \end{tikzcd}$$

$\left(\longleftrightarrow \widehat{\mathfrak{sl}}(N+1) \overset{\text{v.s.}}{=} \mathfrak{sl}(N+1)[t^{\pm 1}] \oplus \mathbb{C}c\right)$

$$A_{Q,\underline{d}}(t) = \begin{cases} 1 & \forall\, \underline{d} \in \Delta^+_{re} \\ t + \# I - 1 & \forall\, \underline{d} \in \Delta^+_{im} \\ 0 & \text{otherwise} \end{cases}$$

- Q = 1-loop quiver. $A_{Q,d}(t) = t \quad \forall\, d \in \mathbb{N}_{\geq 1}$.

- For arbitrary quivers, Hua (2000) derived an explicit formula for $A_{Q,\underline{d}}(t)$

Now, I want to make the relation between $A_{Q,\underline{d}}(t)$ and Lie theory clearer.

Recall that:

<u>Kac, Moody</u>: Q = quiver without edge-loops $\rightsquigarrow \mathfrak{g}_Q^{KM}$

<u>Ex:</u> Q = finite ADE quiver $\longleftrightarrow \mathfrak{g}_Q^{KM}$ = simple Lie algebra

Q = affine ADE quiver $\longleftrightarrow$ $\mathfrak{g}_Q^{KM}$ = affine Lie algebra

Important ⚠: $\mathfrak{g}_Q^{KM}$ is $\mathbb{Z}I$-graded: $\mathfrak{g}_Q^{KM} = \bigoplus_{\underline{d} \in \mathbb{Z}I} \mathfrak{g}_{Q,\underline{d}}^{KM}$

Conjecture (Kac, 1982); Theorem (Hausel, 2010)

Assume that Q is without edge-loops. Then, we have

$$\boxed{A_{Q,\underline{d}}(0) = \dim \mathfrak{g}_{Q,\underline{d}}^{KM}}$$

Attention ⚠:

The above result provides a Lie-theoretic interpretation only of the constant term of $A_{Q,\underline{d}}(t)$ and only for quivers without edge-loops.

Questions:

What about the other coefficients?

What about the 1-loop quiver?

Recall that

Maulik-Okounkov Lie algebra: Q = arbitrary quiver

$$\boxed{\mathfrak{g}_Q^{\mathrm{MO}} = (\mathbb{Z}I \times \mathbb{Z})\text{-graded Lie algebra} = \bigoplus_{(\underline{d},k)\in \mathbb{Z}I\times\mathbb{Z}} \mathfrak{g}^{\mathrm{MO}}_{Q,(\underline{d},k)}}$$

Ex: $Q = A_N$: $\mathfrak{g}^{\mathrm{MO}}_{A_N} = \mathfrak{sl}(N+1)$

Q = 1-loop quiver : $\mathfrak{g}^{\mathrm{MO}}_{1\text{-loop}}$ = Heisenberg Lie algebra

$$= \langle e_n, c : n \in \mathbb{Z}\setminus\{0\}\rangle \Big/ \begin{matrix}[c, e_n] = 0 \\ [e_n, e_m] = n\,\delta_{n+m,0}\, c\end{matrix}$$

Conjecture (Okounkov, 2013)

Theorem (Botta-Davison, Schiffmann-Vasserot; 2023)

Let Q be an arbitrary quiver. Then, we have:

$$\boxed{A_{Q,\underline{d}}(t) = \sum_{k\in\mathbb{Z}} \left(\dim \mathfrak{g}^{\mathrm{MO}}_{Q,(\underline{d},k)}\right) t^{-k/2}}$$

Corollary

$A_{Q,\underline{d}}(t) \in \mathbb{N}[t]$.

Important ⚠: This result was obtained using the theory of COHAs and BPS Lie algebras

2. Cohomological Hall algebras and BPS Lie algebras

▸ Quivers

$$Q = \text{quiver} = \left(I = \{\text{vertices}\}, \Omega = \{\text{edges}\}\right)$$

$$\rightsquigarrow Q^{db} = \text{double quiver} = \left(I, \Omega \sqcup \Omega^{op} =: \Omega^{db}\right)$$

$$\Omega^{op} = \left\{ e^* : j \longrightarrow i \;\middle|\; e : i \longrightarrow j \in \Omega \right\}$$

$$\rightsquigarrow \mathbb{C}Q^{db} = \text{path algebra of } Q^{db}$$

$$\rightsquigarrow \Pi_Q = \text{preprojective algebra of } Q = \mathbb{C}Q^{db} \Big/ \underbrace{\textstyle\sum_{e \in \Omega} [e, e^*]}_{\text{(preprojective rels)}}$$

Examples

▸ $Q = A_N : \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \cdots \underset{N-1}{\bullet} \xrightarrow{\alpha_{N-1}} \underset{N}{\bullet}$

$$\Longrightarrow \Pi_{A_N} = K\langle e_i, \alpha_i, \alpha_i^* : i = 1, \dots, N-1 \rangle \Big/ \alpha_1 \alpha_1^* + \sum_{i=2}^{N-2} \left(\alpha_i \alpha_i^* - \alpha_{i-1}^* \alpha_{i-1} \right) - \alpha_{N-1}^* \alpha_{N-1}$$

▸ $Q = 1\text{-loop quiver} : \overset{\alpha}{\circlearrowleft}\bullet \;\rightsquigarrow\; \Pi_{1\text{-loop}} \simeq K\langle \alpha, \alpha^* \rangle \big/ [\alpha, \alpha^*] \simeq K[\alpha, \alpha^*]$

Attention ⚠: A representation of Π_Q is a representation of Q^{db} satisfying the preprojective relations, i.e.,

$$\left(\bigoplus_{i\in I} M_i, \{f_e, f_{e^*}\}_{e\in\Omega}\right) \text{ s.t. } \sum_{e\in\Omega}[f_e, f_{e^*}] = 0$$

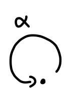
Denote:

$$\boxed{\underline{\mathrm{Rep}}(\Pi_Q) = \text{moduli stack of finite-dimensional representations of } \Pi_Q}$$

Rmk:

- $\underline{\mathrm{Rep}}(\Pi_Q) \simeq T^*\underline{\mathrm{Rep}}(Q)$ (= moduli stack of f.d. reprs of Q)
- $\underline{\mathrm{Rep}}(\Pi_Q) = \bigsqcup_{\underline{d}\in\mathbb{N}I} \underline{\mathrm{Rep}}(\Pi_Q; \underline{d})$ ↳ fixed dimension vector

Example

- Q = 1-loop quiver: 

$$\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}; d) = \left[\underbrace{\{(A_\alpha, A_{\alpha^*}) \in \mathrm{Mat}(\mathbb{C}, d) : [A_\alpha, A_{\alpha^*}] = 0\}}_{\text{commuting variety}} \Big/ GL(d) \right]$$

▶ $\exists$ torus action: $(\mathbb{C}^*)^{\Omega} \times \mathbb{C}^* \curvearrowright \underline{\mathrm{Rep}}(\Pi_Q)$

Schiffmann-Vasserot, Yang-Zhao: fix $T \leq (\mathbb{C}^*)^{\Omega} \times \mathbb{C}^*$ subtorus

$$\boxed{\exists\ \mathrm{COHA}_Q^T = T\text{-equivariant COHA associated to finite-dim. reprs of } \Pi_Q}$$

= unital associative algebra structure on the

T-equiv. Borel-Moore homology of $H_*^T(\underline{\mathrm{Rep}}(\Pi_Q))$

with multiplication $p_* \circ q^!$ induced by:

$$\underline{\mathrm{Rep}}(\Pi_Q) \times \underline{\mathrm{Rep}}(\Pi_Q) \xleftarrow{q} \underbrace{\underline{\mathrm{Rep}}^{ext}(\Pi_Q)}_{=\text{ stack of extensions}} \xrightarrow{p} \underline{\mathrm{Rep}}(\Pi_Q)$$

where:

$$p: 0 \to E_2 \to E \to E_1 \to 0 \longmapsto E \quad \text{(proper)}$$

$$q: \qquad \text{''} \qquad \longmapsto (E_2, E_1)$$

Ex: Q = 1-loop quiver

$$\mathrm{COHA}^{(T)}_{1\text{-loop}} = \text{algebra structure on } \bigoplus_{d \in \mathbb{N}} H_*^{(T)\times GL(d)}(\text{comm. variety}(d))$$

In addition, we have the following:

Davison-Meinhardt: $\exists$ a BPS Lie algebra $\mathfrak{g}^{BPS}_{Q,(T)}$ s.t.

1. $\mathrm{Sym}\left(\mathfrak{g}^{BPS}_{Q,T} \otimes H^*_{\mathbb{C}^*}(pt)\right) \simeq \mathrm{COHA}^T_Q$ (PBW Theorem)

2. $\mathfrak{g}^{BPS}_{Q,T} \simeq \mathfrak{g}^{BPS}_{Q} \otimes_{\mathbb{Q}} H^*_T(pt)$

3. Davison: $A_{Q,\underline{d}}(t) = \sum_{k\in\mathbb{Z}} \left(\dim \mathfrak{g}^{BPS}_{Q,(\underline{d},k)}\right) t^{-k/2}$

As you may notice above, (3) is exactly Okounkov's conjecture for the BPS Lie algebra.

Theorem (Botta-Davison)

$\mathfrak{g}^{MO}_Q = \mathfrak{g}^{BPS}_Q$.

3. Kac-Schiffmann polynomials for curves

Fix a smooth projective curve $X/\mathbb{C}$

Consider the Dolbeault moduli space:

$$\boxed{\begin{array}{l} \mathcal{M}^{(s)s}_{Dol}(X;r,d) = \text{moduli space of (semi)stable Higgs bundles} \\ \qquad\qquad (\mathcal{E}, \phi : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_X) \text{ on } X \\ \qquad\qquad \text{of rank } r \text{ and degree } d \end{array}}$$

Schiffmann, 2016: explicit computation of the Betti numbers of $\mathcal{M}^s_{Dol}(X;r,d)$ for r and d coprime.

Mozgovoy-Schiffmann, 2020: generalization to the noncoprime case

Attention: In both cases, the formulas are given in terms of the Kac-Schiffmann polynomial

To define Kac-Schiffmann polynomial, we need to recall:

▸ The zeta function a smooth projective curve $X/\mathbb{F}_q$ is

$$\boxed{\begin{aligned} Z_X(t) &:= \exp\left(\sum_{n\geq 1} \#X(\mathbb{F}_{q^n})\frac{t^n}{n}\right) \\ &= \frac{\prod_{i=1}^{2g}(1-\sigma_i t)}{(1-t)(1-qt)} \end{aligned}}$$

Here, $\sigma_i \in \mathbb{C}^* =$ Weil numbers of X. They satisfy $\sigma_{2i-1}\sigma_{2i} = q$

▸ Set

$$T_g := \left\{(\alpha_1,\dots,\alpha_{2g}) \in (\mathbb{C}^*)^{2g} : \alpha_{2i-1}\alpha_{2i} = \alpha_{2j-1}\alpha_{2j} \ \forall i,j\right\}$$

$$W_g := \mathfrak{S}_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g$$

$$\Longrightarrow \{\sigma_1,\dots,\sigma_{2g}\} \rightsquigarrow \sigma_X \in T_g/W_g$$

Theorem (Schiffmann, 2016)

Fix $g \in \mathbb{N}$, $(r,d) \in \mathbb{N}\times\mathbb{Z}$. Then $\exists!$

$$\boxed{A_{g,r,d} \in \mathbb{C}[T_g]^{W_g}}$$

such that

$$A_{g,r,d}(\sigma_X) = \#\left\{\begin{array}{l}\text{absolutely indecomposable coherent} \\ \text{sheaves of rank } r \text{ and degree } d \\ \text{on a genus } g \text{ smooth projective} \\ \text{curve}/\mathbb{F}_q \text{ of Weil numbers } \sigma_1,\dots,\sigma_{2g}\end{array}\right\}$$

Moreover, when r,d are coprime, we have

$$\sum_{n\geq 1}(-1)^n \dim H_c^n\left(\mathcal{M}^s_{Dol}(X/\mathbb{C};r,d);\mathbb{Q}\right)t^n = t^{2(1+(g-1)r^2)} A_{g,r,d}(t,\dots,t)$$

Theorem (Mellit, 2020)

$A_{g,r,d}$ is independent of d

Examples

- genus $= 0$: $$A_{0,r,d} = \begin{cases} q+1 & r=0,\ \forall d \\ 1 & r=1,\ \forall d \\ 0 & r\geq 2,\ \forall d \end{cases}$$

- genus $= 1$: $A_{1,r,d}(\sigma_1,\sigma_2) = \sigma_1\sigma_2 + 1 - (\sigma_1+\sigma_2)$

- arbitrary genus: $\exists$ an explicit formula for $A_{g,r,d}$

Now, we formulate a Lie-theoretic interpretation of $A_{g,r,d}$

Note that

- T_g is the maximal torus of $GSp(2g,\mathbb{C})$
- W_g is the Weyl group of $GSp(2g,\mathbb{C})$

$\Longrightarrow \mathbb{C}[T_g]^{W_g}$ is the character ring of $GSp(2g,\mathbb{C})$

Conjecture (Schiffmann, 2018 ICM Talk)

For any $g \in \mathbb{N}$, $\exists$ $\mathfrak{g}_g \in GSp(2g,\mathbb{C})$-mod with

$$\mathfrak{g}_g = \bigoplus_{(r,d)} \mathfrak{g}_{r,d}$$

such that

- $\mathfrak{g}_g$ is a Lie algebra in the category $GSp(2g,\mathbb{C})$-mod
- $ch(\mathfrak{g}_{r,d}) = A_{g,r,d}$

Theorem (Hennecart-Porta-S.-Schiffmann)
Schiffmann's conjecture is true for $g(X) \geq 2$.

Attention ⚠:
The proof is based on the theory of COHAs and BPS Lie algebras.

Let's recall their construction. Consider

$$\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{Dol}}(X) := \text{derived moduli stack of Higgs sheaves on } X \simeq T^*\mathbb{R}\underline{\mathrm{Coh}}(X)$$

$$\mathbb{R}\underline{\mathrm{Coh}}^{(s)s,\nu}_{\mathrm{Dol}}(X) \xrightarrow{\text{open}} \mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{Dol}}(X)$$

$\mathbb{R}\underline{\mathrm{Coh}}^{(s)s,\nu}_{\mathrm{Dol}}(X) :=$ derived moduli stack of (semi)stable Higgs bundles on X of fixed slope $\nu \in \mathbb{Q} \cup \{\infty\}$

Theorem (S.-Schiffmann, 2020; Porta-S., 2023)
There exists a cohomological Hall algebra $\mathrm{COHA}_{\mathrm{Dol}}(X)$, whose underlying vector space is

$$\mathrm{COHA}_{\mathrm{Dol}}(X) := H^{BM}_*(\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{Dol}}(X))$$

and the multiplication is $\mathbb{R}p_* \circ \mathbb{R}q^!$, where

$$\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{Dol}}(X)\times\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{Dol}}(X) \xleftarrow{\mathbb{R}q} \underbrace{\mathbb{R}\underline{\mathrm{Coh}}^{\mathrm{ext}}_{\mathrm{Dol}}(X)}_{\text{stack of extensions}} \xrightarrow{\mathbb{R}p} \mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{Dol}}(X)$$

$p : 0 \to E_1 \to E \to E_2 \to 0 \mapsto E$ - proper

$q :$ —————//————— $\mapsto (E_2, E_1)$ - quasi-sm.

Similarly, $\exists$ $\mathrm{COHA}^{ss,\nu}_{\mathrm{Dol}}(X)$ associated with $\mathbb{R}\underline{\mathrm{Coh}}^{ss,\nu}_{\mathrm{Dol}}(X)$

Using the theory of BPS Lie algebras for CY2 categories:

Theorem (Davison-Hennecart-Schegel-Mejia, 2023)

Assume that $g(X)\geq 2$. For any $\nu\in\mathbb{Q}\cup\{\infty\}$, $\exists$ a Lie algebra

$$\mathfrak{g}_X^{\mathrm{BPS},\nu} = \bigoplus_{\substack{(r,d)\\ d/r=\nu}} \mathfrak{g}_{r,d}^{\mathrm{BPS},\nu}$$

such that

$$\mathrm{COHA}^{ss,\nu}_{\mathrm{Dol}}(X) = \mathrm{Sym}\left(\mathfrak{g}_X^{\mathrm{BPS},\nu}\otimes H^*_{\mathbb{C}^*}(pt)\right) \quad \text{(PBW Thm)}$$

Moreover, we have

$$\mathfrak{g}_X^{\mathrm{BPS},\nu} \simeq \mathrm{FreeLie}\left(\bigoplus_{d/r=\nu} IH^*\left(\mathcal{M}^{ss}_{\mathrm{Dol}}(X;r,d)\right)\right)$$

Attention ⚠:

Each $\mathfrak{g}_{r,d}^{BPS,\nu}$ is $\mathbb{Z}$-graded (w.r.t. the cohomological grading)

For r,d coprime, we have

$$\boxed{\mathrm{gr.dim}_{q^{1/2}}\, \mathfrak{g}_{r,d}^{BPS,\nu} = q^{(g-1)r^2+1} A_{g,r,d}(q^{1/2}, \dots, q^{1/2})}$$

$\Longrightarrow$ $\mathfrak{g}_X^{BPS,\nu}$ could be the "right" candidate for solving Schiffmann's conjecture

The previous result has been refined:

<u>Theorem</u> (Davison-Kinjo-Hennecart-Schiffmann-Vasserot)

$\exists$ a Lie algebra of "primitive elements"

$$\hat{\mathfrak{g}}_X^{(ss,\nu)} := \{ x \in \mathrm{COHA}_{Dol}^{(ss,\nu)}(X) : \underbrace{\Delta^{(ss,\nu)}}_{\text{coproduct}}(x) = x\otimes 1 + 1 \otimes x \}$$

such that

- $\hat{\mathfrak{g}}_X^{(ss,\nu)} = \bigoplus_{(r,d)} \hat{\mathfrak{g}}_{r,d}^{(ss,\nu)}$
- $\forall \nu \in \mathbb{Q} \cup \{\infty\}$, $\exists$ $\hat{\mathfrak{g}}_X^{\nu} := \bigoplus_{\substack{(r,d) \\ d/r=\nu}} \hat{\mathfrak{g}}_{r,d} \xrightarrow{\sim} \hat{\mathfrak{g}}_X^{ss,\nu}$

- $\mathrm{COHA}^{ss,\nu}_{Dol}(X) \simeq U(\hat{\mathfrak{g}}^{\nu}_X)$
- $U(\hat{\mathfrak{g}}_X) \longrightarrow \mathrm{COHA}_{Dol}(X)$ is a dense embedding
- $\forall (r,d) \in \mathbb{N} \times \mathbb{Z}$, $\hat{\mathfrak{g}}_{r,d} \xrightarrow{\sim} \hat{\mathfrak{g}}_{r,d+1}$ (χ-independence)
- As graded vector spaces, $\hat{\mathfrak{g}}^{ss,\nu}_X \simeq \mathfrak{g}^{BPS,\nu}_X \otimes H^*_{\mathbb{C}^*}(pt)$

The natural question at this point is:

Question:

how do we endow the Lie algebra $\mathfrak{g}^{BPS,\nu}_X$ of the structure of a $GSp(2g,\mathbb{C})$-repr. ?

Let $\mathcal{M}_g$ be the moduli stack of smooth projective curves$/\mathbb{C}$. Then, recall:

$$0 \longrightarrow \text{Torelli subgroup} \longrightarrow \pi_1(\mathcal{M}_g) \longrightarrow Sp(2g,\mathbb{Z}) \longrightarrow 0$$

$\Longrightarrow$ As a first step, we could construct

$$\boxed{\pi_1(\mathcal{M}_g)\text{-repr.} \longleftrightarrow \text{local system over } \mathcal{M}_g}$$

Consider

$$\begin{array}{c} \mathcal{C}_g = \text{universal curve} \\ \big\downarrow \\ \mathcal{M}_g \end{array}$$

$$\Longrightarrow \begin{array}{c} \underline{\mathbb{R}\mathrm{Coh}}_{\mathrm{Dol}/\mathcal{M}_g}(\mathcal{C}_g) \\ \pi \big\downarrow \\ \mathcal{M}_g \end{array} \quad \text{and} \quad \begin{array}{c} \underline{\mathrm{RCoh}}^{ss,\nu}_{\mathrm{Dol}/\mathcal{M}_g}(\mathcal{C}_g) \\ \pi^\nu \big\downarrow \\ \mathcal{M}_g \end{array}$$

$\Longrightarrow$ Define $\pi_*\pi^!\mathbb{Q}_{\mathcal{M}_g}$ and $\pi^\nu_*(\pi^\nu)^!\mathbb{Q}_{\mathcal{M}_g}$

<u>Theorem</u> (HPSS)

1. $\pi_*\pi^!\mathbb{Q}_{\mathcal{M}_g}$ and $\pi^\nu_*(\pi^\nu)^!\mathbb{Q}_{\mathcal{M}_g}$ are sheaves of associative algebras
2. $\forall pt \xrightarrow{i_x} \mathcal{M}_g$ and correspondingly $X \xrightarrow{j_x} \mathcal{C}_g$

$$\begin{cases} \iota_X^* \pi_* \pi^! \mathbb{Q}_{\mathcal{M}_g} \simeq \pi_*^X \jmath_X^* \pi^! \mathbb{Q}_{\mathcal{M}_g} = \mathrm{COHA}_{\mathrm{Dol}}(X) \\ \iota_X^* \pi_*^\nu (\pi^\nu)^! \mathbb{Q}_{\mathcal{M}_g} \simeq \pi_*^{\nu,X} \jmath_X^* (\pi^\nu)^! \mathbb{Q}_{\mathcal{M}_g} = \mathrm{COHA}_{\mathrm{Dol}}^{ss,\nu}(X) \end{cases}$$

3. $\exists$ a sheaf $\widehat{\mathcal{BPS}}_g^{(\nu)}$ of Lie algebras such that

$$\begin{cases} \mathrm{Sym}\left(\mathcal{BPS}_g^\nu \otimes H^*_{\mathbb{C}^*}(pt)\right) \simeq \pi_*^\nu (\pi^\nu)^! \mathbb{Q}_{\mathcal{M}_g} \\ \widehat{\mathcal{BPS}}_g^\nu \simeq \mathcal{BPS}_g^\nu \otimes H^*_{\mathbb{C}^*}(pt) \text{ as sheaves of vector spaces} \end{cases}$$

<u>Attention:</u>

(1) and (3) are obtained by generalizing to the "relative" setting what we saw before.

The proof of (2) is highly nontrivial since π and π^ν are not proper.

<u>Theorem</u> (HPSS)

1. $\pi_*^\nu (\pi^\nu)^! \mathbb{Q}_{\mathcal{M}_g}$ and $\widehat{\mathcal{BPS}}_g^\nu$ are local systems on $\mathcal{M}_g$

2. $\pi_* \pi^! \mathbb{Q}_{\mathcal{M}_g}$ is a local system on $\mathcal{M}_g$

Note that we can also define:

- Porta-S.:

de Rham COHA = COHA of flat bundles on X

Betti COHA = COHA of f.d. repr.s of $\pi_1(X)$

- Davison, Hennecart:

de Rham and Betti (affinized) BPS Lie algebras

- Davison, Hennecart proved that

$$\boxed{\begin{gathered}\text{Dolbeault COHA} \simeq \text{de Rham COHA} \simeq \text{Betti COHA} \\ \text{Dolbeault } \widehat{\text{BPS}}\text{ L.} \simeq \text{de Rham } \widehat{\text{BPS}}\text{ L.} \simeq \text{Betti } \widehat{\text{BPS}}\text{ L.}\end{gathered}}$$

$\Longrightarrow$ This proves (1)

The proof of (2) follows from (1) using Harder-Narasimhan strata

Corollary

$\forall$ smooth curve X, $\hat{g}_X$ is a $\pi_1(\mathcal{M}_g)$-representation

Theorem (HPSS)

1. The action of $\pi_1(\mathcal{M}_g)$ on $\hat{\mathfrak{g}}_X$ descends to an action of $Sp(2g,\mathbb{Z})$.

2. The action of $Sp(2g,\mathbb{Z})$ on $\hat{\mathfrak{g}}_X$ extends to an action of $GSp(2g,\mathbb{C})$

3. $$ch_{GSp(2g,\mathbb{C})}(\mathfrak{g}_{r,d}) = q^{(g-1)r^2+1} A_{g,r,d}(q^{1/2},\dots,q^{1/2})$$

$\Longrightarrow$ Schiffmann's conjecture is true.

Let me finish with:

Q: How do we extend this framework to stable curves?