From Hilbert schemes of pts on a smooth surface to cohomological Hall algebras

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Plan

1. Motivation: study of Hilbert schemes of pts via repr. theory

2. Hall algebras

3. Representations via torsion pairs and "doubling" Hall algebras

Motivation: study of Hilbert schemes of pts via repr. theory

S = smooth (quasi-) projective surface$/\mathbb{C}$
$n \in \mathbb{Z}, n \geq 0$

$\mathrm{Hilb}^n(S)$ = Hilbert scheme of n-pts on S
= moduli space parametrizing zero-dim. subschemes $Z \subset S$ such that $\dim_{\mathbb{C}} H^0(Z, \mathcal{O}_Z) = n$

Remarks

- $\mathrm{Hilb}^n(S)$ is a smooth (quasi-) projective variety$/\mathbb{C}$ of dimension $2n$

- $Z \subset S$ subscheme $\longleftrightarrow$ $I_Z \subset \mathcal{O}_S$ ideal sheaf

$$\Longrightarrow \mathrm{Hilb}^n(S) \text{ "="} \left\{ I \subset \mathcal{O}_S : \dim \operatorname{supp}(\mathcal{O}_S / I) = 0, \dim H^0(\mathcal{O}_S / I) = n \right\}$$

$$\Longrightarrow \mathrm{Hilb}^n(S) = M_S^{st}(\underset{\substack{\uparrow \\ rk}}{1}, \underset{\substack{\uparrow \\ c_1}}{0}, \underset{\substack{\uparrow \\ ch_2}}{-n})$$ = moduli space of Gieseker-stable sheaves on S

- $\pi : \mathrm{Hilb}^n(S) \longrightarrow \mathrm{Sym}^n(S) := \overbrace{S \times \cdots \times S}^{\text{n-th copies}} \big/ \underbrace{\mathfrak{S}_n}_{\text{symmetric group of n letters}}$

is a resolution of singularities, i.e., π is a proper morphism which is an iso over the smooth locus of $\mathrm{Sym}^n(S)$.

Examples

- $n=1$: $\mathrm{Sym}^1(S) = S \simeq \mathrm{Hilb}^1(S)$

- $n=2$: $Z \in \mathrm{Hilb}^2(S) \rightsquigarrow \begin{cases} Z = \{x,y\},\ x \neq y & \Longrightarrow \pi(Z) = x+y \\ Z_{red} = \{x\} & \Longrightarrow \pi(Z) = x+x = 2x \end{cases}$

$$\Longrightarrow \mathrm{Hilb}^2(S) \simeq \mathrm{Blow}_{\underbrace{\Delta}_{\text{diagonal}}}(S \times S)/\mathfrak{S}_2 \xrightarrow{\pi} \mathrm{Sym}^2(S)$$

"Bridge" between $\mathrm{Hilb}^n(S)$ and representation theory

Goal: define an associative algebra which acts on

G_0-theory of $\mathrm{Hilb}^n(S)$ / cohomology of $\mathrm{Hilb}^n(S)$

$\parallel$

(Grothendieck group of coh. sh.s)

such that all classes are determined by the action on one (!) fixed class

- First ingredient = Hecke correspondence

for $k>0$: $\mathrm{Hilb}^{n,n+k}(S) := \{ 0 \longrightarrow J \longrightarrow I \longrightarrow Q_x \longrightarrow 0 :$

- $(I,J) \in \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^{n+k}(S)$
- $\mathrm{supp}(Q_x) = \{x\}$ $\}$

$$\Big\downarrow \quad \text{reduced closed subscheme}$$

$\mathrm{Hilb}^n(S) \times \mathrm{Hilb}^{n+k}(S) \times S$

for $k>0$: $\mathrm{Hilb}^{n+k,n}(S) \hookrightarrow \mathrm{Hilb}^{n+k}(S) \times \mathrm{Hilb}^{n}(S) \times S$

for $k=0$: $\mathrm{Hilb}^{n,n}(S)$ = diagonal $\hookrightarrow \mathrm{Hilb}^{n}(S) \times \mathrm{Hilb}^{n}(S)$

Attention ⚠ : $\mathrm{Hilb}^{n,n'}(S)$ is smooth $\Leftrightarrow |n-n'| \leq 1$

- Second ingredient = Tautological bundles

$\mathcal{Z}_n \subset \mathrm{Hilb}^n(S) \times S$ universal family

$p : \mathrm{Hilb}^n(S) \times S \longrightarrow \mathrm{Hilb}^n(S)$ projection

Fact: $\tau_n := p_*(\mathcal{O}_{\mathcal{Z}_n})$ is a vector bundle of rank n.

Def. (Tautological bundles on Hecke correspondences)

$$\tau_{n,n+1} := \mathrm{Ker}\left(pr_{n+1}^*(\tau_{n+1}) \twoheadrightarrow pr_n^*(\tau_n)\right) \text{ on } \mathrm{Hilb}^{n,n+1}(S)$$

($J \subset I$ as before: $0 \to Q_x \to \mathcal{O}_S/J \to \mathcal{O}_S/I \to 0$)

$\tau_{n+1,n} :=$ —————— " —————— on $\mathrm{Hilb}^{n+1,n}(S)$

$\tau_{n,n} := pr_n^*(\tau_n)$ on $\mathrm{Hilb}^{n,n}(S)$

Notation: $\mathrm{Hilb}(S) := \bigsqcup_{n \geq 0} \mathrm{Hilb}^n(S).$

From now on, set $S = \mathbb{C}^2 \curvearrowleft T = \mathbb{C}^* \times \mathbb{C}^*$

Set

$$R := G_0^T(pt) = \mathbb{C}[q^{\pm 1}, t^{\pm 1}], \quad K := \mathbb{C}(q^{1/2}, t^{1/2})$$

- Third ingredient = geom. defined ops

Define:

$$\boxed{f_{-1,\ell} = \prod_n \underbrace{\mathbb{R}(p_{n+1})_* \left([\tau_{n,n+1}]^{\otimes \ell} \otimes pr_n^*(-) \right)}_{: G_0^T(\mathrm{Hilb}^n(\mathbb{C}^2))_K \longrightarrow G_0^T(\mathrm{Hilb}^{n+1}(\mathbb{C}^2))_K} \quad \text{for } \ell \in \mathbb{Z}}$$

$$f_{1,\ell} = \prod_n \mathbb{R}(p_n)_*\left([\tau_{n+1,n}]^{\otimes \ell} \overset{\mathbb{L}}{\otimes} pr_{n+1}^*(-)\right) \quad \text{for } \ell \in \mathbb{Z}$$

$$e_{0,\ell} = \prod_n \mathbb{R}(p_n)_*\left([\Lambda^\ell \tau_{n,n}] \overset{\mathbb{L}}{\otimes} pr_n^*(-)\right) \quad \text{for } \ell \in \mathbb{Z}, \ell > 0$$

$$e_{0,-\ell} = \prod_n \mathbb{R}(p_n)_*\left([\Lambda^\ell \tau_{n,n}^\vee] \overset{\mathbb{L}}{\otimes} pr_n^*(-)\right) \quad \text{for } \ell \in \mathbb{Z}, \ell > 0$$

$$f_{\pm 1,\ell},\ e_{0,\pm\ell} \in \mathrm{End}\left(G_0^T(\mathrm{Hilb}(\mathbb{C}^2))_K\right)$$

$M_K := M \otimes_R K$

Thm. (Schiffmann-Vasserot)

The algebra generated by $f_{\pm 1,\ell}, e_{0,\pm\ell}$ is isomorphic to the elliptic Hall algebra $\mathcal{E}$ $\left(= \text{quantum toroidal algebra } U(\hat{\hat{gl}}(1)) \text{ of } gl(1)\right)$:

As faithful reprs of $\mathcal{E} = U(\hat{\hat{gl}}(1))$,

$$G_0^T(\mathrm{Hilb}(\mathbb{C}^2))_K \simeq \underbrace{K[x_1, x_2, \dots]^{\mathfrak{S}_\infty}}_{\text{(algebra of symmetric functions in infinitely many variables)}}$$

Attention ⚠ : later on, I will give a "geometric" definition of $\mathcal{E}$.

Note that $\mathcal{E}$ can be defined in terms of
- generators,
- relations which are quadratic and cubic, depending on

zeta function of an elliptic curve over a finite field $\longleftrightarrow$ $\zeta(z) = \dfrac{(1-q^{-1}z)(1-t^{-1}z)}{(1-z)(1-q^{-1}t^{-1}z)} \in G_0^{\mathbb{C}^*\times\mathbb{C}^*}(pt)(z) = \mathbb{C}[q^{\pm 1}, t^{\pm 1}](z)$

$$\Downarrow$$

$$\mathcal{E} \text{ "=" } \langle \text{generators} \rangle \big/ \text{quadratic, cubic rels dep. on } \zeta(z), \text{ Lie theor. rels}$$

In repr. theory, it has been studied also:

$$\langle \text{generators} \rangle \big/ \text{quadratic rels dep. on } \zeta(z), \text{ Lie theor. rels} \twoheadrightarrow \langle \text{generators} \rangle \big/ \text{quadratic and cubic rels dep. on } \zeta(z), \text{ Lie theor. rels}$$

where the left-hand side $=$ DIM (Ding-Iohara-Miki algebra) and the right-hand side $= \mathcal{E}$.

Thm. (Neguț)
Let $S = K3$. The algebra generated by

$$f_{\pm 1, \ell}, e_{0, \pm \ell} \in \mathrm{End}(G_0(\mathrm{Hilb}(K3)))$$

is isomorphic to the DIM associated to $\zeta_{K3}(z) \in G_0(K3)(z)$, where q^{-1}, t^{-1} = Chern roots of Ω^1_{K3}.

Remark
Negut categorified the above result, i.e., he defined functors in

$$\mathrm{End}(D^b_{coh}(\mathrm{Hilb}(K3)))$$

categorifying $f_{\pm 1, \ell}, e_{0, \pm \ell}$

Thm (Schiffmann - Vasserot)
The algebra generated by

$$\underbrace{f_{\pm 1, \ell}, e_{0, \pm \ell}}_{} \in \mathrm{End}\Big(H^*_T(\mathrm{Hilb}(\mathbb{C}^2)) \otimes_{\underbrace{\mathbb{C}[\varepsilon_1, \varepsilon_2]}_{= H^*_T(pt)}} \mathbb{C}(\varepsilon_1, \varepsilon_2)\Big)$$

cohomological version: $\begin{cases} [\tau_{n,n+1}] \longmapsto c_1(\tau_{n,n+1}) \\ [\Lambda^e \tau_{n,n}] \longmapsto c_e(\tau_{n,n}) \end{cases}$

is isomorphic to a "degenerate" version of $\mathcal{E}$ (= affine Yangian $Y(\widehat{gl}(1))$ of $gl(1)$)

This action induces Nakajima-Grojnowski's action of the Heisenberg algebra on $H_T^*(Hilb(\mathbb{C}^2)) \otimes_{\mathbb{C}[\varepsilon_1,\varepsilon_2]} \mathbb{C}(\varepsilon_1,\varepsilon_2) =: H_T^*(Hilb(\mathbb{C}^2))_{loc}$

"Bridge" between moduli spaces and repr. theory based on

$\boxed{\text{explicitly defined operators}}$

Advantages of this approach:

- it allows to compute relations between generators
- it allows to characterize explicitly the "geometric" representation
- it determines that $H_T^*(Hilb(\mathbb{C}^2))_{loc}$ = irreducible highest weight representation (of Heis)

Limit of this approach:

it does NOT realize all possible geometric actions, e.g., the one with Hecke correspondence:

$$\left\{ 0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow i_* \mathcal{L}_C \longrightarrow 0 \right\}$$

where $C \xhookrightarrow{i} S$ is a smooth proj. curve inside a smooth surface, $\mathcal{L}_C$ line bundle on C.

In particular, it does NOT determine that

$$\bigoplus_{c_1, ch_2} H^*\left(\mathcal{M}^{st}_{K3}(r, c_1, ch_2)\right) \simeq \text{irreducible highest weight representation (of some algebra)}$$

Solution: define an even bigger algebra

How: establish "Bridge" between moduli spaces and repr. theory based on

$\boxed{\text{Hall algebras}}$

Hall algebras

S = smooth quasi-projective surface$/\mathbb{C}$.

$(\mathbb{R})\underline{\mathrm{Coh}}_{ps}(S)$ = (derived) moduli stack of properly supported coherent sheaves on S

Consider the Hall convolution diagram

derived stack of short exact sequences

$$\begin{tikzcd} & \mathbb{R}\underline{\mathrm{Coh}}^{ext}_{ps}(S) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{R}\underline{\mathrm{Coh}}_{ps}(S) \times \mathbb{R}\underline{\mathrm{Coh}}_{ps}(S) & & \mathbb{R}\underline{\mathrm{Coh}}_{ps}(S) \end{tikzcd}$$

$$p : 0 \to \mathcal{E}_2 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_1 \longrightarrow 0 \longmapsto (\mathcal{E}_1, \mathcal{E}_2)$$
$$q : \qquad \qquad '' \qquad \qquad \longmapsto \mathcal{E}$$

Facts:

- p is derived l.c.i. (i.e., $\mathbb{L}_p$ is perfect and in Tor-amplitude $[-1,1]$)
- q is representable by proper schemes (which are Quot schemes)

Thm (Porta-S.)

1. $D^b_{coh}(\mathbb{R}\underline{Coh}_{ps}(S))$ has a monoidal structure induced by $q_* \circ p^*$.

2. It descends to an associative algebra structure $KHA(S)$ on $G_0(\underline{Coh}(S))$

Remarks:

- By using the work of Adeel Khan on motivic Borel-Moore homology, (2) holds after replacing $G_0(-)$ with $H_*^{BM}(-) \rightsquigarrow COHA(S)$

- The above Theorem holds also for $\mathbb{R}\underline{Coh}_{\tau_\bullet}(S) \hookrightarrow \mathbb{R}\underline{Coh}_{ps}(S)$, where

$$\tau_{\leq 1} := \{ \mathcal{F} \in Coh_{ps}(S) : \dim(\mathrm{supp}(\mathcal{F})) \leq 1 \}$$

$$\tau_0 := \{ \mathcal{F} \in Coh_{ps}(S) : \dim(\mathrm{supp}(\mathcal{F})) = 0 \}$$

- (2) recovers the constructions by:
  - Kapranov-Vasserot via perfect obstruction theory
  - Schiffmann-Vasserot for $\mathbb{C}^2$ via the Lagrangian formalism
  - S.-Schiffmann for T^*(curve) —— // ——

$= \underline{\mathrm{Coh}}_{\tau_0}(\mathbb{C}^2) \subset \underline{\mathrm{Coh}}_{ps}(\mathbb{C}^2)$

Thm (Schiffmann-Vasserot)

1. $KHA^T(\mathbb{C}^2)_K = \left(G_0^T(\underline{\mathrm{Coh}}_0(\mathbb{C}^2))_K \text{ , Hall product} \right)$

$\simeq \mathcal{E}^+$ = positive part of the elliptic Hall algebra

2. $\exists$ an action $KHA^T(\mathbb{C}^2)$ on $G_0^T(\mathrm{Hilb}(\mathbb{C}^2))$ such that

$$
\begin{tikzcd}
KHA^T(\mathbb{C}^2)_K \arrow[d, "\wr"'] \arrow[drr] \arrow[dr, phantom, "\circlearrowright"] & & \\
\mathcal{E}^+ \arrow[rr] & & \mathrm{End}\left(G_0^T(\underline{\mathrm{Coh}}_0(\mathbb{C}^2))_K\right)
\end{tikzcd}
$$

Similar results hold in BM homology.

Attention: Hall algebras realize only halves of the full algebras we care.

Solution: doubling KHA and COHA

Representations via torsion pairs and doubling Hall algebras

S = smooth projective surface $/\mathbb{C}$.

Consider

$$\mathcal{T}_{\leq 1} := \left\{ \mathcal{F} \in \mathrm{Coh}_{ps}(S) : \dim(\mathrm{supp}(\mathcal{F})) \leq 1 \right\}$$

$$\underline{\mathrm{Coh}}_{\leq 1}(S) := \underline{\mathrm{Coh}}_{\mathcal{T}_{\leq 1}}(S)$$

Introduce : $\mathcal{M}(S;1)$:= moduli space of rank-one torsion-free sheaves on S

<u>Thm</u> (Diaconescu-Porta-S.)
$D^b_{coh}(\mathcal{M}(S;1))$ is a left and right categorical module over $D^b_{coh}(\mathbb{R}\,\underline{\mathrm{Coh}}_{\leq 1}(S))$

In particular,
- $G_0(\mathcal{M}(S;1))$ is a left and right module over $\mathrm{KHA}_{\leq 1}(S)$

- $H^{BM}_*(\mathcal{M}(S;1))$ is a left and right module over $\mathrm{COHA}_{\leq 1}(S)$

Def. (Algebras of "Hecke modifications along curves")
The Yangian of $\underline{\mathrm{Coh}}_{\leq 1}(S)$ is the subalgebra of $\mathrm{End}(H_*^{BM}(\mathcal{M}(S;1)))$ generated by the images of

$$\boxed{\begin{array}{l} \text{left action } a_\ell : \mathrm{COHA}_{\leq 1}(S) \longrightarrow \mathrm{End}(H_*^{BM}(\mathcal{M}(S;1))) \\ \\ \text{right action } a_r : \mathrm{COHA}_{\leq 1}(S) \longrightarrow \mathrm{End}(H_*^{BM}(\mathcal{M}(S;1))) \end{array}}$$

Similarly, we define: quantum loop algebra of $\mathrm{Coh}_{\leq 1}(S)$ and its categorification

Remark

- The Theorem above holds after replacing $\underline{\mathrm{Coh}}_{\leq 1}(S) \rightsquigarrow \underline{\mathrm{Coh}}_0(S)$ and

$$\mathcal{M}(S,1) \rightsquigarrow \text{moduli space of PT stable pairs on } S$$

- $S = K3$.
The Theorem above holds after replacing $\underline{\mathrm{Coh}}_{\leq 1}(K3) \rightsquigarrow \underline{\mathrm{Coh}}_0(K3)$ and $\mathcal{M}(K3,1) \rightsquigarrow \mathrm{Hilb}(K3) \Longrightarrow$ we recover Neguț's construction

The previous theorem is a consequence of the following more general framework.

Thm (Diaconescu-Porta-S.)

Assume that

1. $\mathcal{C}$ is a "nice" triangulated category (e.g. for which Toën-Vaquié's moduli of objects is an Artin derived stack)
2. τ is a t-structure which satisfies openness of flatness
3. $\exists$ a Serre functor $S_{\mathcal{C}}$ such that $S_{\mathcal{C}}[-2]$ is t-exact
4. the Quot functor for $(\mathcal{C},\tau)$ is represented by a proper algebraic space

Then

- $D^b_{coh}(\mathbb{R}\underline{Coh}_{ps}(\mathcal{C},\tau))$ has a monoidal structure induced by $(q_\tau)_* \circ p_\tau^*$.

- It descends to an associative algebra structure $KHA(\mathcal{C},\tau)$ on $G_0(\underline{Coh}(\mathcal{C},\tau))$

  —————— // —————— $COHA(\mathcal{C},\tau)$ on $H_*^{BM}(\underline{Coh}(\mathcal{C},\tau))$

Remark

$(1)+(2) \Longrightarrow \mathbb{R}\underline{Coh}(\mathcal{C},\tau)$ is Artin

$(3) \Longrightarrow p_\tau$ is derived l.c.i. $\Longrightarrow \exists\, p_\tau^*$

$(4) \Longrightarrow q_\tau$ is proper $\Longrightarrow \exists\, (q_\tau)_*$

Consider:

1. $\mathcal{C}$ = "nice" triangulated category (as before)

2. τ = t-structure which satisfies openness of flatness

3. $v = (\mathcal{T}or, \mathcal{F})$ = torsion pair in $\mathcal{C}^\heartsuit$, i.e.,
   - $\mathrm{Hom}(\mathcal{T}or, \mathcal{F}) = 0$
   - $\forall E \in \mathcal{C}^\heartsuit \quad \exists \; 0 \longrightarrow \underset{\mathcal{T}or}{\overset{}{T}} \longrightarrow E \longrightarrow \underset{\in \mathcal{F}}{F} \longrightarrow 0$, $T \in \mathcal{T}or$

$\Longrightarrow \tau_v$ = tilted t-structure on $\mathcal{C}$, whose heart is:

$$\mathcal{C}_v^\heartsuit = \left\{ E \in \mathcal{C} : \mathcal{H}^{-1}_\tau(E) \in \mathcal{F},\ \mathcal{H}^0_\tau(E) \in \mathcal{T}or,\ \mathcal{H}^i_\tau(E) = 0 \ \forall i \neq 0, -1 \right\}$$

4. $\mathbb{R}\,\underline{\mathrm{Coh}}_{\mathcal{T}or}(\mathcal{C},\tau)$, $\mathbb{R}\,\underline{\mathrm{Coh}}_{\mathcal{F}}(\mathcal{C},\tau)$ are open in $\mathbb{R}\,\underline{\mathrm{Coh}}(\mathcal{C},\tau)$

Facts:

(1) + (2) $\Longrightarrow \mathbb{R}\,\underline{\mathrm{Coh}}(\mathcal{C},\tau)$ is Artin

Lieblich shows: (4) $\Longrightarrow \mathbb{R}\,\underline{\mathrm{Coh}}(\mathcal{C},\tau_v)$ is Artin

Thm (Diaconescu-Porta-S.)

Assume that

1. p_τ is derived l.c.i., q_τ is proper
2. p_{τ_v} —— " ——, q_{τ_v} —— " ——
3. $\mathcal{T}or$ is a Serre subcategory

Then

$$D^b_{coh}(\mathbb{R}\underline{Coh}_{\mathcal{T}or}(\mathcal{C}, \tau))$$

has a monoidal structure induced from the one on

$$D^b_{coh}(\mathbb{R}\underline{Coh}(\mathcal{C}, \tau)) \quad \text{or} \quad D^b_{coh}(\mathbb{R}\underline{Coh}(\mathcal{C}, \tau_v))$$

equivalently

Assume furthermore that

4. $\mathbb{R}\underline{Coh}_{\mathcal{T}or}(S, \tau)$ is closed in both $\mathbb{R}\underline{Coh}(\mathcal{C}, \tau)$ and $\mathbb{R}\underline{Coh}(\mathcal{C}, \tau_v)$

Then

- $D^b_{coh}(\mathbb{R}\underline{Coh}_F(\mathcal{C},\tau))$ is a left (resp. right) categorical module of $D^b_{coh}(\mathbb{R}\underline{Coh}_{Tor}(\mathcal{C},\tau))$ induced by the monoidal structure of $D^b_{coh}(\mathbb{R}\underline{Coh}(\mathcal{C},\tau))$ (resp. $D^b_{coh}(\mathbb{R}\underline{Coh}(\mathcal{C},\tau_v))$).

Similar statements hold for $G_0(-)$ and $H^{BM}_*(-)$.

Remark

The first result can be applied to $\mathcal{C}$ = noncommutative K3 surface, i.e., a category with the same properties of $D^b_{coh}(K3)$ (e.g. $\exists$ Serre functor $\simeq$ shift by 2).

A famous example of noncommutative K3 surfaces is

$$\mathcal{C} = Ku(X) = \text{Kuznetsov component of } X = \begin{cases} \text{Fano 3folds of Picard rank one} \\ \text{cubic 4folds} \\ \text{Gushel-Mukai 4folds or 6folds} \end{cases}$$

To apply the second result, one needs "nice" torsion pairs of $\mathcal{C}^{\heartsuit}$:

under investigation.