Hecke operators associated to 1-dimensional sheaves

Based on Diaconescu-Porta-S., arXiv: 2207.08926, and
work in progress with Diaconescu-Porta-Y. Zhao

Disclaimer:

The background for this talk is based on Olivier's course. In particular, I will not recall the defs of

- COHA of coherent sheaves on a smooth proj. surface$/\mathbb{C}$
- (2-sided) Hecke patterns

Attention: The results that I will present will be at the level of

$$G_0(-) := \text{Grothendieck group of coh. sheaves}$$

instead of Borel-Moore homology.

Main result

Theorem (Diaconescu-Porta-S.-Y. Zhao)

Let
- C = elliptic curve$/\mathbb{C}$
- N = line bundle on C, $\deg(N) = -d < 0$

Set $S := \mathbb{P}(\mathcal{O}_C \oplus N)$, let $D_\infty \subset S$ be the curve with normal bundle N^{-1}.

Then, $\exists$ a moduli space $\mathcal{M}$ of D_∞-framed vector bundles $\mathcal{F}$ on S, which satisfy a Ginzburg-Kapranov-Vasserot property, for which $\exists$ operators

$$\boxed{\begin{aligned} \mu^{\pm}(z) &: G_0(\mathcal{M}) \longrightarrow G_0(\mathcal{M}\times C)\{z\} \\ h^{\pm}(z,w) &: G_0(\mathcal{M}) \longrightarrow G_0(\mathcal{M}\times C)\{z,w\} \end{aligned}}$$

satisfying the following equalities:

1. $$[\mu^+(z),\mu^-(w)] = \Lambda^{\bullet}(-\sigma^{\vee}_{2,1} w z^{-1})\Big|_{z\sim 0-\infty} \left(h^+(z,w) - h^-(z,w)\right)$$

(Here, $f(z)|_0$ (resp. $f(z)|_\infty$) = Laurent exp. around 0 (resp. ∞))

2. $$h^+(z,w)\mu^+(u) = \mu^+(u)h^+(z,w)\Lambda^{\bullet}(-\sigma^{\vee}_{1,2} w u^{-1})^{-1}\Big|_{w\sim 0} \Lambda^{\bullet}(-\sigma^{\vee}_{2,1} z^{-1} u)\Big|_{z\sim 0}$$

$$h^-(z,w)\mu^+(u) = \mu^+(u)h^-(z,w)\Lambda^{\bullet}(-\sigma^{\vee}_{1,2} w u^{-1})^{-1}\Big|_{w\sim \infty} \Lambda^{\bullet}(-\sigma^{\vee}_{2,1} z^{-1} u)\Big|_{z\sim \infty}$$

3. $$\mu^+(z)\mu^+(w)\Lambda^{\bullet}(-\sigma_{2,1} w z^{-1}) = \mu^+(w)\mu^+(z)\Lambda^{\bullet}(-\sigma_{1,2} z w^{-1})$$

$$\mu^-(w)\mu^-(z)\Lambda^{\bullet}(-\sigma_{2,1} w z^{-1}) = \mu^-(z)\mu^-(w)\Lambda^{\bullet}(-\sigma_{1,2} z w^{-1})$$

(Here, $G_{i,j} := \mathbb{R}\mathcal{H}om(\mathcal{P}^{(i)}, \mathcal{P}^{(j)}) \in K_0(C\times C)$; $\mathcal{P}$ = Poincaré line bundle)

Furthermore, the operators $\mu^{\pm}(z)$ are related to operators of Hecke modifications along $C \subset S$ in K-theory.

Motivation: why is this relevant?

As you noticed from Olivier's talk, $\exists$ two approaches to define "geometrically" interesting algebras.

The 1st one is:

- Cohomological / K-theoretical Hall algebra of ab. category $\mathcal{A}$

Pro:

it is an "intrinsic" approach: it depends only on the moduli stack of objects $\mathfrak{X}_{\mathcal{A}}$.

Cons:

(i) They realize only a nilpotent half of an "interesting" algebra

(ii) The Cartan subalgebra must be defined ad hoc.

(iii) $\nexists$ "intrinsic" way to define the full "interesting algebra"

(If the COHA/KHA extended by the Cartan subalgebra is a (top.) bialgebra $\rightsquigarrow$ consider the Drinfeld double)

The 2nd one is:

- Nakajima-Neguts operators

Pro:

One can define the whole "interesting algebra"

Con:

One needs to find a moduli stack/space $\mathcal{M}$ which is a Hecke pattern for $\mathfrak{X}_A$

$$\rightsquigarrow \text{"interesting algebra"} = \text{subalgebra of } \begin{cases} \mathrm{End}(H_*^{BM}(\mathcal{M})) \\ \mathrm{End}(G_0(\mathcal{M})) \end{cases}$$

Examples

A = category 0-dimensional sheaves on a smooth qproj. surface $S/\mathbb{C}$ $\leftarrow$ already seen in Olivier's course

$\mathcal{M} = \mathrm{Hilb}(S)$ = Hilbert scheme of pts on S

$\mathcal{A}$ = category of properly supported sheaves on S

$\Longrightarrow$ Kapranov-Vasserot: $\exists$ COHA/KHA of $\mathcal{A}$

$\mathcal{A}$ = category of properly supported sheaves on S, set-theoretically supported on a fixed closed subscheme $D \subset S$

$\Longrightarrow$ Diaconescu-Porta-S.-Schiffmann-Vasserot:

1. $\exists$ COHA/KHA of $\mathcal{A}$ (it depends only on $\widehat{S}_D$)

2. $S \xrightarrow{\pi} \mathbb{C}^2/\Gamma$ minimal resolution of ADE singularity.

$D = \pi^{-1}(0) \xhookrightarrow{i} S$, $D_{red} = D_1 \cup \cdots \cup D_e$, $D_i \simeq \mathbb{P}^1$

<u>Theorem</u> (Diaconescu-Porta-S.-Schiffmann-Vasserot)

For this choice of (S, D), the corresponding $\mathrm{COHA}^{\mathrm{equiv}}$ is generated (under the action of tautological classes) by the fundamental classes of

(1) the stacks of 0-dimensional sheaves on S supported at a single point of D_i for $i = 1, \ldots, e$;

(2) The stacks of 1-dimensional sheaves scheme-theoretically supported on D_k, whose K-theory class is $[i_*\mathcal{O}_{D_k}(m)]$ for $k=1,2,\dots,e$ and $m\in\mathbb{Z}$.

Since (1) (and its double) can be explictly described (as Olivier has already explained), this discussion motivates:

Given D = chain of smooth curves $\xhookrightarrow{i} S$.

Question:

Are we able to describe the COHA/KHA of sheaves $\mathcal{E}$ on S such that

- $\mathcal{F}$ is scheme-theoretically supported on D_k
- The K-theory class of $\mathcal{F} = [i_*\mathcal{L}_k]$, where $\mathcal{L}_k$ is a line bundle on D_k, for $k=1,\dots,e$?

(="half" of algebra of Hecke modifications along D)

Answer: The "main result" should give an answer when D = elliptic curve $\subset S$.

Ops, commutator, and Cartan rels

S = smooth projective surface$/\mathbb{C}$

Fix a torsion pair $(\mathrm{Tor}, \mathrm{TF})$ of ab. category $\mathrm{Coh}(S)$, i.e.,

- $\mathrm{Hom}(T, F) = 0 \quad \forall\, T \in \mathrm{Tor},\ F \in \mathrm{TF}$;
- $\forall\, E\ \exists\ 0 \to \underset{\in \mathrm{Tor}}{T} \to E \to \underset{\in \mathrm{TF}}{F} \to 0$

such that

- Tor is a Serre subcategory, and
- the moduli stacks $\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{Tor}}(S)$ and $\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{TF}}(S)$ are open in $\mathbb{R}\underline{\mathrm{Coh}}(S)$.

Examples

- $\mathrm{Tor} = \{\mathcal{E} \in \mathrm{Coh}(S) : \dim(\mathrm{supp}(\mathcal{E})) = 0\}$

  $\mathrm{TF} = \{\mathcal{F} \in \mathrm{Coh}(S) : \mathcal{F} \text{ pure in dimension zero}\}$

  appeared in Olivier's course

- $\mathrm{Tor} = \{\mathcal{E} \in \mathrm{Coh}(S) : \dim(\mathrm{supp}(\mathcal{E})) \leq 1\}$

  $\mathrm{TF} = \{\mathcal{F} \in \mathrm{Coh}(S) : \mathcal{F} \text{ torsion free}\}$

Fix Artin derived stacks

$$\boxed{\mathcal{X} \subset \mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{Tor}}(S) \quad \text{and} \quad \mathcal{M} \subset \mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{TF}}(S)}$$

which "play the role"

$$\mathcal{X} \longleftrightarrow \mathbb{R}\underline{\mathrm{Coh}}_{\delta}(S) \quad \text{and} \quad \mathcal{M} \longleftrightarrow \underline{\mathrm{Hilb}}(S)$$

$\Longrightarrow$ Goal: define operators ass. to $\mathcal{X}$ acting on $G_0(\mathcal{M})$

Following the analogy $\mathcal{X} \leftrightarrow \mathbb{R}\underline{\mathrm{Coh}}_{\delta}(S)$, $\mathcal{M} \leftrightarrow \underline{\mathrm{Hilb}}(S)$ we impose:

Assumptions I

1. $\mathcal{M}$ is a (2-sided) Hecke pattern for $\mathcal{X}$

2. $\mathcal{X}$ is quasi-compact

3. $\mathcal{X} \simeq X \times B\mathbb{G}_m$, where X = quasi-smooth proper derived algebraic space$/\mathbb{C}$

Set

$\mathbb{R}\underline{\mathrm{Coh}}^{ext} :=$ derived moduli stack of s.e.s. $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ such that $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{M}$ and $\mathcal{F}_3 \in \mathfrak{X}$.

$ev_i : \text{s.e.s.} \longmapsto \mathcal{F}_i$ for $i = 1, 2, 3$, and $ev_3^{tw} := p \circ ev_3$, where $p : \mathfrak{X} \longrightarrow X$.

Lemma

1.
$$\begin{tikzcd}
\mathbb{R}\underline{\mathrm{Coh}}^{ext} \arrow[rr, "\sim"] \arrow[dr, "ev_2 \times ev_3"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathrm{Spec}\,\mathrm{Sym}(E_{\mathfrak{M},\mathfrak{X}[-1]}) \setminus \{0\} \arrow[dl] \\
& \mathfrak{M} \times \mathfrak{X} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R}\underline{\mathrm{Coh}}^{ext} \arrow[rr, "\sim"] \arrow[dr, "ev_2 \times ev_3^{tw}"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathbb{P}(E_{\mathfrak{M},X[-1]}) \arrow[dl] \\
& \mathfrak{M} \times X &
\end{tikzcd}$$

where $E_{\mathfrak{M},\mathfrak{X}[-1]} := (\mathbb{R}\mathcal{H}om(\mathcal{F}_{\mathfrak{X}}, \mathcal{F}_{\mathfrak{M}}))^{\vee} \in \mathrm{Perf}(\mathfrak{M} \times \mathfrak{X})$

2.
$$\begin{tikzcd}
\mathbb{R}\underline{\mathrm{Coh}}^{ext} \arrow[rr, "\sim"] \arrow[dr, "ev_3 \times ev_1"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathrm{Spec}\,\mathrm{Sym}(E_{\mathfrak{X},\mathfrak{M}}) \setminus \{0\} \arrow[dl] \\
& \mathfrak{X} \times \mathfrak{M} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R}\underline{\mathrm{Coh}}^{ext} \arrow[rr, "\sim"] \arrow[dr, "ev_3^{tw}\times ev_1"'] & \arrow[d, phantom, "\circlearrowright"] & \mathbb{P}(E_{\mathcal{X},\mathcal{M}}) \arrow[dl] \\
 & \mathcal{X}\times\mathcal{M} &
\end{tikzcd}$$

where $E_{\mathcal{X},\mathcal{M}} := \left(\mathbb{R}\mathcal{H}om(\mathcal{F}_{\mathcal{X}}, \mathcal{F}_{\mathcal{M}})[1]\right)^{\vee} \in \mathrm{Perf}(\mathcal{X}\times\mathcal{M})$

(Here, $\mathcal{F}_{\mathcal{X}}, \mathcal{F}_{\mathcal{M}}$ are the univ. objects of $\mathcal{X}, \mathcal{M}$, resp.)

Next step:

Define ops by

$$\boxed{\begin{array}{l}\text{pushforward}\\ \text{w.r.t. proper maps}\end{array}} \circ \boxed{\begin{array}{l}\text{pullback w.r.t.}\\ \text{quasi-smooth maps}\end{array}}$$

Important: All the complexes $E_{\cdot,\cdot}$ have amplitude $[0,1]$

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$\implies$ $ev_2\times ev_3^{tw}$, $ev_3^{tw}\times ev_1$ are quasi-smooth and proper and ev_2, ev_1 are quasi-smooth and proper

$\implies \exists$ pullbacks and pushforwards w.r.t. these maps

Definition (Nilpotent ops associated to X)

Fix $d \in \mathbb{Z}$.

The **d-th twisted operator**:

$$e_d^{tw} := (ev_2)_* \Big((ev_3^{tw} \times ev_1)^* (-) \cdot \underbrace{\mathcal{L}^{\otimes d}}_{= \mathcal{O}_{\mathbb{P}(E_{X,\mathfrak{M}})}(1)} \Big) : G_0(X) \otimes G_0(\mathfrak{M}) \longrightarrow G_0(\mathfrak{M})$$

The **d-th Nakajima-Negut's operator**:

$$\mu_d^+ := (ev_2 \times ev_3^{tw})_* \Big(ev_1^*(-) \cdot \mathcal{L}^{\otimes d} \Big) : G_0(\mathfrak{M}) \longrightarrow G_0(\mathfrak{M} \times X)$$

By changing the roles of $ev_1 \longleftrightarrow ev_2$, we also define

the **twisted Hall op f_d^{tw}** and the **N-N's op. μ_d^-**

Attention ⚠: we want to define also ops associated to $\mathfrak{X}$

Note that $ev_2 \times ev_3$, $ev_3 \times ev_1$ are quasi-smooth

$\Longrightarrow \exists$ pullbacks w.r.t. these maps

Def. (Nilpotent ops associated to $\mathcal{X}$)
Hall operator:

$$e := (ev_2)_* \circ (ev_3 \times ev_1)^* : G_0(\mathcal{X}) \otimes G_0(\mathcal{M}) \longrightarrow G_0(\mathcal{M})$$

By changing the roles of $ev_1 \longleftrightarrow ev_2$, we also define f.

Attention ⚠: we can define ops associated to the classical truncations ${}^{cl}\mathcal{M}$ and ${}^{cl}\mathcal{X}$ as well

Remark
For ${}^{cl}\mathcal{X} = \underline{Coh}_{\delta}(S) \simeq S \times B\mathbb{G}_m$, one recovers the op.s discussed by Olivier.

Set

$$\boxed{\mu^+(z) := \sum_{d \in \mathbb{Z}} \mu^+_d z^{-d} \quad \text{and} \quad \mu^-(z) := \sum_{d \in \mathbb{Z}} \mu^-_d z^{-d}}$$

Notation: $f(z)|_0$ (resp. $f(z)|_\infty$) = Laurent exp. around 0 (resp. ∞)

Set

$$\boxed{\begin{aligned} h^+(z,w) &:= \Lambda^{\bullet}(-E_{\mathcal{M},X}\, w)\big|_0 \, \Lambda^{\bullet}(-E_{\mathcal{M},X[-1]}\, z^{-1})\big|_0 \\ h^-(z,w) &:= \Lambda^{\bullet}(-E_{\mathcal{M},X}\, w)\big|_\infty \, \Lambda^{\bullet}(-E_{\mathcal{M},X[-1]}\, z^{-1})\big|_\infty \end{aligned}}$$

Denote by $h^{\pm}(z,w) : G_0(\mathcal{M}) \longrightarrow G_0(\mathcal{M} \times X)$ the corresponding operators of multiplication

Proposition

Assume that $\mathcal{M}$ is a separated derived algebraic space. Then

$$\boxed{[\mu^+(z), \mu^-(w)]\big|_{K_0(\mathcal{M})} = \zeta\left(\frac{w}{z}\right)\Big|_{z \sim 0 - \infty} \left(h^+(z,w) - h^-(z,w)\right)}$$

Here, $\zeta(x) = \Lambda^{\bullet}(-\mathcal{O}_{2,1}^{\vee}\, x) \in K_0(X_1 \times X_2)(x)$, $\mathcal{O}_{2,1} := \mathbb{R}\mathcal{H}om(\mathcal{I}_X^{(2)}, \mathcal{I}_X^{(1)})$

Remark

When ${}^{cl}\mathcal{X} := \underline{Coh}_S(S)$, we recover Neguț's lemma (stated by Olivier)

Proposition

We have

$$\boxed{\begin{aligned} h^+(z,w)\,\mu^+(u) &= \mu^+(u)\,h^+(z,w)\,\Lambda^\bullet(-\sigma^\vee_{1,2}\, w u^{-1})^{-1}\Big|_{w\sim 0}\ \Lambda^\bullet(-\sigma^\vee_{2,1}\, z^{-1} u)\Big|_{z\sim 0} \\ h^-(z,w)\,\mu^+(u) &= \mu^+(u)\,h^-(z,w)\,\Lambda^\bullet(-\sigma^\vee_{1,2}\, w u^{-1})^{-1}\Big|_{w\sim \infty}\ \Lambda^\bullet(-\sigma^\vee_{2,1}\, z^{-1} u)\Big|_{z\sim \infty} \end{aligned}}$$

Similar rels hold for $h^\pm(z,w)$ and $\mu^-(u)$.

Attention ⚠:

1. To drop the condition on $\mathfrak{M}$, one needs a further extension of Neguț's theory which is not available at the moment;

2. When ${}^{cl}\mathcal{X} = \underline{\mathrm{Coh}}_\delta(S)$, Neguț's framework is useful also to compute the rels for the $\mu^\pm(z)$'s.

In this generality, it is not possible to use this framework.

$\Longrightarrow$ we shall use another approach

Quadratic rels and semistable KHA

Now, I will define $\mathfrak{X}$ = quasi-compact s.t. $\mathfrak{X} \simeq X \times B\mathbb{G}_m$ and assume that $\exists$ $\mathfrak{M}$ = "(2-sided) Hecke pattern for $\mathfrak{X}$".

At the end of the talk, I will define $\mathfrak{M}$ as well.

$S := \mathbb{P}(\mathcal{O}_C \oplus \mathcal{N})$, where

- C = smooth proj. curve$/\mathbb{C}$
- $\deg(\mathcal{N}) = -d < 0$.

Let $D, D_\infty \subset S$ be the canonical sections $C \longrightarrow S$ with normal bundles $\mathcal{N}$ and $\mathcal{N}^{-1}$, respectively.

Fix an ample class H on S.
Recall that the (H-)slope of a 1-dimensional sheaf $\mathcal{E}$ on S is

$$\boxed{\mu_H(\mathcal{E}) := \frac{\chi(\mathcal{E})}{H \cdot c_1(\mathcal{E})}}$$

$\Longrightarrow$ $\exists$ also $\mu_{H\text{-max}}$ and $\mu_{H\text{-min}}$ w.r.t. Harder-Narasimhan filtrs

Let $s, n \in \mathbb{Z}$ be coprime, with $s \geq 1$

$\mathfrak{X}_D(s,n)$:= derived moduli stack of μ-semistable sheaves $\mathcal{E}$ on S with $ch_1(\mathcal{E}) = s[D]$ and $\chi(\mathcal{E}) = n$.

Proposition

We have

$$\boxed{\mathfrak{X}_D(s,n) \simeq X_D(s,n) \times B\mathbb{G}_m}$$

with

$$X_D(s,n) := \mathrm{Spec}_{M(D;s,n)} \mathrm{Sym}(Ob^{\vee}[1])$$

where

- $M(D;s,n)$:= moduli space of slope-stable vector bundles on D of rank s and Euler characteristic n
- $Ob := \mathbb{R}\mathcal{H}om(\mathcal{V}, \mathcal{V} \otimes p_D^* N)[1]$

Example: $D = \mathbb{P}^1$, $D^2 = -d \leq -2$ + $T^* \times T^* \curvearrowright S$

Set $s=1, n=0$. Then

$$X_{\mathbb{P}^1}(1,0) \simeq \mathrm{Spec}\,\mathrm{Sym}(H^0(D, \iota^*\omega_S)[1]) \text{ and } {}^{cl}X_{\mathbb{P}^1}(1,0) \simeq pt$$

$\Longrightarrow$ The nilpotent subalgebra $\langle e_n : n\in\mathbb{Z}\rangle$ is a subalgebra of the shuffle algebra $\bigoplus_{n\geq 0} \mathbb{C}(q,t)[z_1^{\pm},\dots,z_n^{\pm}]^{\mathfrak{S}_n}$ with the product

$$f(z_1,\dots,z_n) * g(z_1,\dots,z_m) = \frac{1}{n!\,m!}\,\mathrm{Sym}\left(f(z_1,\dots,z_n)\, g(z_{n+1},\dots,z_{n+m}) \prod_{\substack{1\leq l\leq n \\ n+1\leq j\leq n+m}} \left(\frac{\prod_{k=0}^{d-2} (1-q^{-1}t^{-k} z_l z_j^{-1})}{(1-z_l z_j^{-1})} \right) \right)$$

Rmk: For $d=2$, this is the shuffle algebra related to $U_q^+(Lsl(2))$.

Example: D = elliptic curve, $D^2<0$

Set $s=1$, $n=0$. Then

$$^{cl}X_D(1,0) \simeq D$$

We are in the same situation as before, where

$$\langle e_n(\alpha) : \alpha\in K_0(D)\rangle$$

is a subalg. of the Shuffle algebra $\bigoplus_{n\geq 0} \left(K_0(D)^{\times n}[z_1^{\pm 1},\dots,z_n^{\pm 1}]\right)^{\mathfrak{S}_n}$ with the product

$$f(z_1,\dots,z_n) * g(z_1,\dots,z_m) = \frac{1}{n!\,m!}\,\mathrm{Sym}\Bigg(f(z_1,\dots,z_n)\, g(z_{n+1},\dots,z_{n+m})$$

$$\underbrace{\prod_{\substack{1\le i\le n\\ n+1\le j\le n+m}} \underbrace{\left(1+[\mathcal{O}_{\Delta_{i,j}}]\,\frac{z_i z_j^{-1}}{1-z_i z_j^{-1}}\right)}_{\text{Neguț's zeta function of } D} \cdot \left[\mathcal{O}(\Delta_i - \Delta_j)\otimes \mathcal{N}\right]}_{\displaystyle \parallel \atop \displaystyle \Lambda^{\bullet}(-\mathcal{G}_{j,i}\, z_i z_j^{-1}) \text{ seen at the beginning}}\Bigg)$$

Here,

- $\Delta_{i,j} \subset D^{\times n} = (i,j)$-th diagonal
- $\Delta_i := \pi_i^{-1}(\Delta)$ w.r.t. $\pi_i : D^{\times n}\times D \longrightarrow D\times D$ = the projection to the $(i,n+1)$-th fact.

Important: Shuffle algebras will helps us computing quadratic, cubic, ..., rel.s

Let's finish by describing $\mathcal{M}$.

Ginzburg-Kapranov-Vasserot type vector bundles

Set $\alpha := n/s[D]\cdot H$

Def (GKV type vector bundles)

A torsion-free sheaf $\mathcal{F}$ on S is of GKV α-type if

- $\mathcal{F}$ is locally free, and
- $\mu_{H\text{-max}}\left(\iota_* \iota^* \mathcal{F} \otimes \mathcal{O}_S(D)\right) \leq \alpha \leq \mu_{H\text{-min}}\left(\iota^* \iota_* \mathcal{F}\right)$

Set

$\mathfrak{M}_\alpha^{GKV,fr}$ = moduli stack of GKV α-type sheaves $\mathcal{F}$ on S, which are trivial along D_∞ and with a fixed trivialization there.

Theorem

- $\mathfrak{M}_\alpha^{GKV,fr}$ is a Hecke pattern for $\mathcal{X}_D(s,n)$
- $\exists$ Hall ops associated to $\mathcal{X}_D(s,n)$, twisted Hall ops and Nakajima-Negut's ops associated to $X_D(s,n)$.