Cohomological Hall algebras of 1-dimensional sheaves and Yangians over the Bridgeland's space of stability conditions

(based on
- SSS: joint work with O. Schiffmann and P. Shimpi, arXiv: 2511.08576;
- DPSSV: joint work with D.-E. Diaconescu, M. Porta, O. Schiffmann, and E. Vasserot, arXiv: 2603.03386)

Fix

- $\mathscr{C}$ = 2-Calabi-Yau triangulated category,
- $\mathrm{Stab}(\mathscr{C})$ = space of Bridgeland stability conditions.

The goal of this talk is to provide positive answers to the following questions:

Q: $\forall\, x = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathscr{C})$ $\exists$ a (T-equivariant) COHA $\mathbf{HA}_x^{(T)}$ associated to the abelian category $\mathcal{P}(0,1]$?

Q: $\forall\, x = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathscr{C})$, $\mathbf{HA}_x^T$ is "related" to Yangians?

when $\mathscr{C} = D^b\mathrm{Coh}_C(S)$ for

- S = minimal resolution of an ADE singularity,
- C = exceptional curve

Now, let's introduce the framework needed to state the main result precisely.

Minimal resolution of an ADE singularity:

- $G \subset SL(2,\mathbb{C})$ finite subgroup

$\updownarrow$

Q_{fin} = finite ADE quiver, Q = affine ADE quiver

$\updownarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\updownarrow$

$\mathfrak{g}_{fin}$ = simple Lie algebra, $\mathfrak{g}_{aff}$ = affine Lie algebra

- $\pi : S \longrightarrow \mathbb{C}^2/G$ Kleinian resolution of singularities

- $C := \pi^{-1}(0)$. Then, $C_{red} = C_1 \cup \cdots \cup C_\ell$, $C_i \simeq \mathbb{P}^1$

$(C_i \cdot C_j) = -$ Cartan matrix of Q_{fin}

- torus $T \subset GL(2,\mathbb{C})$ centralizing G
  (T = trivial, $\mathbb{C}^*$, or $\mathbb{C}^* \times \mathbb{C}^*$)

Example: $G = \mathbb{Z}_2 \implies Q_{fin} = A_1 = \bullet$, $Q = A_1^{(1)} : \bullet \rightleftarrows \bullet$

$\implies S = T^*\mathbb{P}^1 \supset C = \mathbb{P}^1$ = zero section

Recall the derived McKay correspondence:

$$\tau : D^b Coh(S) \xrightarrow{\sim} D^b Mod(\Pi_Q)$$

where

- Π_Q = the preprojective algebra of Q

  := End(tilting bundle on S inducing τ)

Remark

τ restricts to a derived equivalence

$$\tau : D^b\mathrm{Coh}_C(S) \xrightarrow{\sim} D^b\mathrm{nilp}(\Pi_Q)$$

where

- $\mathrm{Coh}_C(S)$ = abelian category of coherent sheaves on S set-theoretically supported on C

- $\mathrm{nilp}(\Pi_Q)$ = abelian category of nilpotent finite-dim. Π_Q-repr.s

Now, let's recall:

Definition (Bridgeland stability conditions)

Let $\mathcal{C}$ be a triangulated category. A stability condition on $\mathcal{C}$ is a pair $(Z, \mathcal{P})$, where

- $\mathcal{P}$ = slicing of $\mathcal{C}$ = collection of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{C}$ for $\phi \in \mathbb{R}$ satisfying
  - compatibility with the shift, i.e., $\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$
  - orthogonality, i.e., $\forall\, \phi_1 > \phi_2 \quad \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$
  - existence of Harder-Narasimhan filtrations
- $Z \in \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{C}), \mathbb{C})$ such that $\forall\, 0 \neq E \in \mathcal{P}(\phi)$

$$Z(E) \in \mathbb{R}_{>0}\, e^{i\pi\phi}$$

  = central charge

Notation: $\forall\, I \subset \mathbb{R}$ interval, $\mathcal{P}(I) := \langle \mathcal{P}(\phi) : \phi \in I \rangle$

Remarks (Bridgeland)

- $\forall\, I \subset \mathbb{R}$ interval of length 1, $\mathcal{P}(I)$ is abelian.

▸ $$\{(Z,\mathcal{P})\} \xleftrightarrow{1:1} \left\{\begin{array}{l}(\tau = \text{a bounded t-str. on } \mathcal{C},\\ Z\in \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{C}^{\heartsuit_\tau}),\mathbb{C}) \text{ stab. fun.}\\ \text{satisfying the HN property}\end{array}\right\}$$

$$(Z,\mathcal{P}) \longmapsto (\tau = (\mathcal{P}(>0),\mathcal{P}(\leq 0)), Z)$$

$$\mathcal{C}^{\heartsuit_\tau} = \mathcal{P}(0,1] \qquad K_0(\mathcal{C}^{\heartsuit_\tau}) = K_0(\mathcal{C})$$

▸ The set $\mathrm{Stab}(D^b\mathrm{Coh}_C(S))$ of (locally finite) stability conditions is a complex manifold.

Following Bridgeland, we introduce:

Definition

$\mathrm{Stab}(S)$:= connected component of $\mathrm{Stab}(D^b\mathrm{Coh}_C(S))$ that contains

$$\boxed{U = \{(Z,\mathcal{P}) \in \mathrm{Stab}(D^b\mathrm{Coh}_C(S)) : \mathcal{P}(0,1] = \mathrm{nilp}(\Pi_Q)\}}$$

Remark

Only in type A we know that $\mathrm{Stab}(D^b\mathrm{Coh}_C(S))$ is connected.

Attention ⚠ : $\forall x = (Z, P) \in U$, we already know that the questions I posed at the beginning of the talk have positive answers.

Indeed,

- by Schiffmann-Vasserot $\exists$ a (T-equivariant) cohomological Hall algebra $HA_Q^{(T)}$ associated to $\mathrm{nilp}(\Pi_Q)$ such that

$$HA_Q^T \simeq \mathbb{Y}_{Q,T}^{MO,-} := \text{negative half of the Maulik-Okounkov Yangian of } Q$$

- by DPSSV, $\mathbb{Y}_{Q,T}^{MO,-} \simeq \mathbb{Y}_{Q,T}^{-}$:= negative part of the affine Yangian $\mathbb{Y}_{Q,T}$ of Q

Facts (due to Guay-Regelskis-Wendlandt, Guay-Nakajima-Wendlandt, Kodera, Bershtein-Tsymbaliuk, DPSSV)

1. $\mathbb{Y}_{Q,T}$ has a presentation in terms of gen.s and rel.s

2. $\mathbb{Y}_{Q,T}$ = a "deformation" of $U(\mathfrak{g}_{ell})$, i.e.,

$\mathbb{Y}_{Q,T} \otimes_R \mathbb{Q} \simeq U(\mathfrak{g}_{ell})$ and $\exists$ a filtration $F^\bullet$ of $\mathbb{Y}_{Q,T}$ s.t.

$$\mathrm{gr}_{F^\bullet} \mathbb{Y}_{Q,T} \simeq U(\mathfrak{g}_{ell}) \otimes R$$

Here, $R := H_T^*(pt)$ and $\mathfrak{g}_{ell}$:= elliptic Lie algebra:

$\boxed{\mathfrak{g}_{ell} := \text{universal central extension of the Lie algebra } \mathfrak{g}_{fin} \otimes \mathbb{Q}[s^{\pm 1}, t]}$

As v.s.:

$$\mathfrak{g}_{ell} = \left(\mathfrak{g}_{fin} \otimes \mathbb{Q}[s^{\pm 1}, t]\right) \oplus \left(\bigoplus_{\ell \in \mathbb{N}} \mathbb{Q} c_\ell \oplus \bigoplus_{\substack{\ell \in \mathbb{N}, \ell \geq 1 \\ k \in \mathbb{Z}, k \neq 0}} \mathbb{Q} c_{k,\ell} \right)$$

central

3. $\mathbb{Y}^-_{Q,T}$ = a "deformation" of $U(\mathfrak{n}_{ell})$,

where

- $\mathfrak{n}_{ell} := \mathfrak{n}[t] \oplus \bigoplus_{k<0, \ell \geq 1} c_{k,\ell}$
- $\mathfrak{n}$:= 'standard' negative half of $\mathfrak{g}_{aff}$

Remarks

▸ Nakajima provided a description of the whole $\mathbb{Y}^{MO}_{Q,T}$ as an "affine" version of the extended Yangian à la Wendlandt (for $Q \neq A_1^{(1)}$).

▸ In type A, Jindal provided a presentation of $\mathbb{Y}^{MO}_{Q,T}$ in terms of gen.s and rel.s.
In particular, he proved that $\mathbb{Y}^{MO}_{Q,T} \supset \mathbb{Y}_{Q,T}$.

Summarizing:
$\forall\, x = (Z, P) \in U\ \exists\ HA_x^{(T)}$ such that HA_x^T is a deformation of U(current Lie algebra).

Q: What about all $x = (Z, P) \in \mathrm{Stab}(S) \setminus U$?

For our purposes, it is enough to characterize $\mathcal{P}(0,1]$ for all points $x = (Z, P) \in \mathrm{Stab}(S) \setminus U$!

We need to introduce the notion of "perverse coherent sheaves".

Fix a subset $\phi \subseteq J \subseteq I_{fin}$:= set of vertices of Q_{fin}.
Consider

$$\begin{tikzcd} \text{minimal res. of singularities} = S \arrow[rr, "\pi_J"] \arrow[dr, "\pi"'] & & S_J \arrow[dl] \\ & \mathbb{C}^2/G & \end{tikzcd}$$

where π_J is obtained by contracting all curves C_i with $i \in I_{fin} \setminus J$

Definition

The abelian category $P_c(S/S_J)$ of (Van der Bergh's) perverse coherent sheaves that are set-theoretically supported on C is:

$$P_c(S/S_J) := \left\{ E \in D^b Coh_c(S) : \begin{array}{l} \mathcal{H}^0(E) \in \mathcal{T}_J, \mathcal{H}^{-1}(E) \in \mathcal{F}_J \\ \mathcal{H}^i(E) = 0 \ \forall i \neq -1, 0 \end{array} \right\}$$

Here,

$$\mathcal{T}_J := \left\{ \mathcal{E} \in Coh_c(S) : \begin{array}{l} \mathbb{R}^1 \pi_{J*}(\mathcal{E}) = 0, \\ Hom(\mathcal{E}, \mathcal{F}) = 0 \ \forall \mathcal{F} \text{ s.t. } \mathbb{R}\pi_{J*}(\mathcal{F}) = 0 \end{array} \right\}$$

$$\mathcal{F}_J := \{ \mathcal{E} \in \mathrm{Coh}_C(S) : \mathbb{R}^0\pi_{J*}(\mathcal{E}) = 0 \}$$

Remark:

$\mathcal{P}_C(S/S_\phi) \xrightarrow[\tau]{\sim} \mathrm{nilp}(\Pi_Q)$, while $\mathcal{P}_C(S/S_{I_{fin}}) = \mathrm{Coh}_C(S)$

Now, note that

$$\mathcal{O}_C \quad \text{and} \quad \mathcal{O}_{C_i}(-1)[1] \;\; \forall i \in I_{fin}$$

are 2-spherical objects; the corresponding Seidel-Thomas' twist functors give rise

$$\text{affine braid group} = B_{aff} \curvearrowright D^b\mathrm{Coh}_C(S)$$

Theorem (SSS)

$\forall x = (Z, P) \in \mathrm{Stab}(S)\ \exists m \in \mathbb{Z}, b_x \in B_{aff}$, and $\phi \subseteq J \subseteq I_{fin}$ such that

$$\boxed{P(0,1] = b_x \cdot \mathcal{P}_C(S/S_J)[m]}$$

$\Longrightarrow$ It would be enough to construct COHAs associated to $\mathcal{P}_c(S/S_J)$!

The other COHAs will be obtained via the B_{aff}-action.

From now on, fix $\emptyset \neq J \subseteq I_{fin}$. The COHA of $\mathcal{P}_c(S/S_J)$ will be constructed via the theory of limiting COHAs (in DPSSV)

First, we show that $\mathcal{P}_c(S/S_J)$ is a "limit" of $\mathrm{nilp}(\Pi_Q)$. More precisely,

- hearts of bounded t-structures on a triangulated category form a partially ordered set:

$$\boxed{H_1 := \mathcal{C}_1^{\geq 0} \cap \mathcal{C}_1^{\leq 0} \leq H_2 := \mathcal{C}_2^{\geq 0} \cap \mathcal{C}_2^{\leq 0} \iff \mathcal{C}_1^{\geq 0} \subseteq \mathcal{C}_2^{\geq 0}}$$

- let $\check{\theta} \in \check{X}_{fin}$ ($:=$ finite coweight lattice) be such that:

$$\boxed{\check{\theta}(\alpha_i) = 0 \ \forall\, i \in I_f \setminus J \quad \text{and} \quad \check{\theta}(\alpha_i) > 0 \ \forall\, i \in J}$$

$\rightsquigarrow \mathcal{L}_{\check{\theta}} \in \mathrm{Pic}(S)$

via McKay

Shimpi proved that

$$\left\{ \mathcal{L}_{\check{\theta}}^{\otimes -n} \otimes \mathrm{nilp}(\Pi_Q) \right\}_{n \in \mathbb{N}}$$

is a decreasing sequence of hearts and

$$\lim_n \mathcal{L}_{\check{\theta}}^{\otimes -n} \otimes \mathrm{nilp}(\Pi_Q) = \inf_n \mathcal{L}_{\check{\theta}}^{\otimes -n} \otimes \mathrm{nilp}(\Pi_Q) = \mathcal{P}_c(S/S_J)$$

A translation of this result within COHA theory is:

Theorem (DPSSV for $J = I_{fin}$; SSS)

1. $\exists$ a cohomological Hall algebra $\mathbf{HA}_J^{(T)}$ associated to $\mathcal{P}_c(S/S_J)$, which is a $(\mathbb{Z} \times \mathbb{Z}I)$-graded $(R\text{-})$algebra

2. $\exists$ a canonical isomorphism of graded algebras

$$\mathbf{HA}_J^T \simeq \mathbb{Y}_{\check{\theta}} := \lim_{\ell} T_{\check{\theta}}^{-2\ell}(\mathbb{Y}_{Q,T}^-) \big/ T_{\check{\theta}}^{-2\ell}(\mathcal{J}_{\check{\theta}})$$

where

- $J_{\check{\Theta}} := \sum_{(\check{\Theta}, \underline{d}) \leq 0} \mathbb{Y}^-_{-\underline{d}} \cdot \mathbb{Y}^-$,
- $T_{\check{\Theta}} \in B_{ex} :=$ extended affine braid group,
- the multiplication of $\mathbb{Y}_{\check{\Theta}}$ is induced by the one of $\mathbb{Y}_{Q,T}$

3. $\exists$ a canonical isomorphism of graded algebras

$$HA_J \simeq \hat{U}^{\check{\Theta}}(n^J_{ell})$$

where

- $n^J_{ell} := n^J[t] \oplus \bigoplus_{\substack{k<0 \\ \ell \geq 1}} c_{k,\ell} \subset g_{ell}$
- $n^J :=$ Lie subalgebra of the affine Lie algebra g_{aff}

corresp. to J, following Jakobsen-Kac's classification

- $\forall$ Lie algebra $m := \bigoplus_{\underline{d}} m_{\underline{d}}$,

$$\widehat{U}^{\check{\theta}}(m) := \lim_{k} U(m) \Big/ \Big(\bigoplus_{(\check{\theta}, \underline{d}) < k(\Sigma \check{\omega}_i, \underline{d})} m_{\underline{d}} \Big) \cdot U(m)$$

Remark

From a geometric viewpoint, the appearance of completions is evident.
Indeed, the moduli stack of objects in $P_C(S/S_J)$ is a disjoint union of stacks locally of finite type/$\mathbb{C}$

$\Longrightarrow HA_J^{(T)}$ = the (T-equiv.) Borel-Moore homology of this stack

$\Longrightarrow HA_J^{(T)}$ = topological vector space

Before giving an idea of the proof of theorem, let me speculate:

Conjecture

HA_J^T is a half of a completion $\widehat{\mathbb{Y}}_{Q,T}^J$ of the affine Yangian of Q.

Attention ⚠

At the moment it is not clear how to define $\widehat{\mathbb{Y}}_{Q,T}^J$

Furthermore, we have defined a pointwise family of algebras.

Q: would it be possible to define a sheaf of algebras over Stab(S)?

Now, I will give an idea of how to prove (1) and (2)

First, we reformulate "$\lim_n \mathcal{L}_{\check{\theta}}^{\otimes -n} \otimes \mathrm{nilp}(\Pi_Q) = P_c(S/S_J)$" as a limit in Stab(S).

Recall that $\exists\ \mathbb{R} \curvearrowright \mathrm{Stab}(S)$ by:

$$\boxed{t * (Z, \mathcal{P}) = (Z', \mathcal{P}')}$$

where $Z' := e^{-i\pi t} Z$ and $\mathcal{P}'(\phi) := \mathcal{P}(\phi + t) \quad \forall \phi \in \mathbb{R}$.

Now, $\check{\theta} \in \check{X}_{fin} \hookrightarrow \check{X} :=$ affine coweight lattice.

$\check{\theta}$ defines a stability function on $\mathrm{nilp}(\Pi_Q)$:

$$Z_{\check{\theta}} : K_0(\mathrm{nilp}(\Pi_Q)) \simeq \mathbb{Z}I \longrightarrow \mathbb{C}$$
$$\underline{d} \longmapsto -(\check{\theta}, \underline{d}) + (\sum \check{\omega}_i, \underline{d})\, i$$

$\Longrightarrow$ $\boxed{\exists\ (Z_{\check{\theta}}, \mathcal{P}_{\check{\theta}}) = \text{Bridgeland stability condition on } D^b\mathrm{Coh}_c(S)}$

Proposition

- $\forall n \in \mathbb{N}$, $\mathcal{L}_{\check{\theta}}^{\otimes -n} \otimes : \mathrm{nilp}(\Pi_Q) \xrightarrow{\sim} \mathcal{P}_{\check{\theta}}(t_{-n}, 1 + t_{-n}]$

where $t_n := \frac{1}{\pi} \arctan(n h)$

Coxeter number

- $\mathcal{P}_{\check{\theta}}(-\frac{1}{2}, \frac{1}{2}] = \mathcal{P}_c(S/S_J)$

Remark

Note that $\mathcal{P}_{\check{\theta}}(t_{-n}, 1 + t_{-n}] = t_{-n} * \mathcal{P}_{\check{\theta}}(0, 1] = t_{-n} * \mathrm{nilp}(\Pi_Q)$

Then

$$\lim_n \mathcal{L}_{\check{\theta}}^{\otimes -n} \otimes \mathrm{nilp}(\Pi_Q) = \mathcal{P}_c(S/S_J)$$

corresponds to:

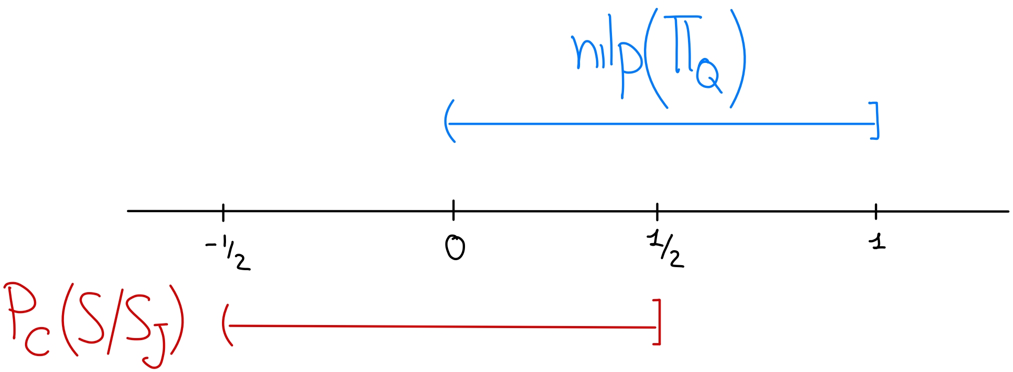
$$\lim_n t_{-n} * (Z_{\check{\Theta}}, \mathcal{P}_{\check{\Theta}}) = -\tfrac{1}{2} * (Z_{\check{\Theta}}, \mathcal{P}_{\check{\Theta}})$$

Now, given

$$\boxed{\text{the slicing } \mathcal{P}_{\check{\Theta}} \text{ and the sequence } \{a_n := t_{-2n} + 1\}_{n \in \mathbb{N}}}$$

we can define the corresponding limiting COHA following DPSSV.

The previous description shows the following:

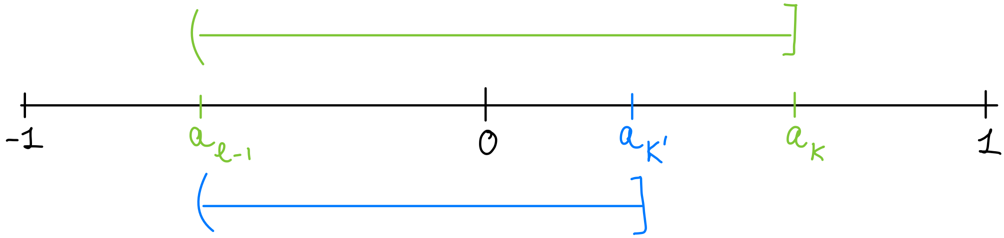


and we would like to "interpolate".

$\forall$ positive integers $k \geq \ell$, set $I_{\ell,k} := (a_{\ell-1}, a_k]$ and let

$\underline{\Lambda}_Q^{\ell,k}$:= moduli stack of objects belonging to $P_{\check{\Theta}}(I_{\ell,k})$

Now, fix ℓ and take $k' \geq k$.

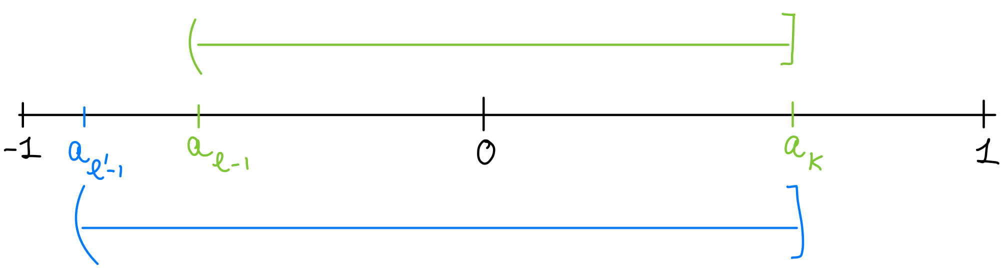


Since $I_{\ell,k'} \subset I_{\ell,k}$, $\exists$ canonical open immersion

$$\underline{\Lambda}_Q^{\ell,k'} \longrightarrow \underline{\Lambda}_Q^{\ell,k}$$

$\Longrightarrow$ $\underline{\Lambda}_Q^{\ell,+\infty} := \text{"}\lim_{k \geq \ell}\text{"}\ \underline{\Lambda}_Q^{\ell,k}$ is a pro-stack

Now, for $\ell' \geq \ell$,



Since $I_{\ell,k} \subset I_{\ell',k}$, $\exists$ canonical maps

$$\underline{\Lambda}_{Q}^{\ell,+\infty} \longrightarrow \underline{\Lambda}_{Q}^{\ell',+\infty}$$

$\Longrightarrow$ $\underline{\Lambda}_{Q}^{+\infty} := \text{"}\operatorname{colim}_{\ell}\text{"}\, \underline{\Lambda}_{Q}^{\ell,+\infty}$ is a ind-pro-stack

Theorem (DPSSV for $J = I_{fin}$, SSS)
The topological vector space

$$\boxed{\mathbf{HA}_{J}^{(T)} := H_{*}^{(T)}\left(\underline{\Lambda}_{Q}^{+\infty}\right) := \text{"}\lim_{\ell}\text{"} \operatorname{colim}_{k \geq \ell} H_{*}^{(T)}\left(\underline{\Lambda}_{Q}^{\ell,k}\right)}$$

carries a canonical $\mathbb{Z} \times \mathbb{Z}I$-graded $(R\text{-})$ algebra structure induced by that on $\mathbf{HA}_{Q}^{(T)}$

Remark
When $J = I_{fin}$, we have an "intrinsic" definition of the COHA of $\mathrm{Coh}_c(S)$.
We proved in DPSSV that it coincides with $\mathbf{HA}_{I_{fin}}^{(T)}$

Remark
Note that $\forall\, n \geq \ell$

$$\mathcal{L}_{\Theta}^{\vee \otimes n} \otimes - : \mathcal{P}_{\Theta}^{\vee}(a_{\ell}-1, a_n] \xrightarrow{\sim} \mathcal{P}_{\Theta}^{\vee}(a_{\ell-n}-1, 1] \subset \mathcal{P}_{\Theta}^{\vee}(0,1]$$

$$\mathcal{L}^{\otimes n}_{\check{\theta}} \otimes - : \quad \underline{\Lambda}^{\ell,n}_{Q} \xrightarrow[\sim]{} \underline{\Lambda}^{\ell-n,0}_{Q} \subset \underline{\Lambda}_{Q}$$

Here,

- $\underline{\Lambda}_Q$ = usual moduli stack of nilpotent Π_Q-repr.s
- $\underline{\Lambda}^{\ell-n,0}_Q \simeq \underline{\Lambda}^{>t_{2(n-\ell)}}_Q$ = usual HN stratum

Proposition

1. As vector spaces

$$\operatorname*{colim}_{k\geq \ell} H^T_*\left(\underline{\Lambda}^{\ell,k}_Q\right) \simeq T^{-2\ell}_{\check{\theta}}\left(\mathbb{Y}^-_{Q,T}\right) \big/ T^{-2\ell}_{\check{\theta}}\left(J_{\check{\theta}}\right)$$

Here, $(\mathcal{L}_{\check{\theta}} \otimes -)$ corresponds to $T_{\check{\theta}} \in B_{ex}$

2. $\mathbb{Y}_{\check{\theta}} := \lim_{\ell} T^{-2\ell}_{\check{\theta}}\left(\mathbb{Y}^-_{Q,T}\right) \big/ T^{-2\ell}_{\check{\theta}}\left(J_{\check{\theta}}\right)$ has the structure of a graded (R-) algebra, whose multiplication is induced by that of $\mathbb{Y}_{Q,T}$

3. $\exists$ a canonical graded algebras iso. $HA^T_J \simeq \mathbb{Y}_{\check{\theta}}$