COHAs of coherent sheaves on smooth surfaces and affine Yangians

based on arXiv:2502.19445, together with Diaconescu-Porta-Schiffmann-Vasserot

We are now ready to provide a unified framework for all the constructions we have seen so far.

Set

- $\mathbb{K}$ = field
- $\mathcal{A}$ = $\mathbb{K}$-linear abelian category
- $\mathcal{M}_{\mathcal{A}}$ = moduli stack of objects of $\mathcal{A}$

Examples:

- $\mathcal{A} = \mathrm{Rep}(Q)$ = category of (f.-d.) repr.s of Q — 1d category

  $\mathcal{A}^{nil} = \mathrm{nilp}(Q)$ = ——— " — nilp. ——— " ———

- $\mathcal{A} = \mathrm{Rep}(\Pi_Q)$ = category of (f.-d.) repr.s of the preproj. alg. Π_Q — 2d category

  $\mathcal{A}^{nilp} = \mathrm{nilp}(\Pi_Q)$ = ——— " — nilp. ——— " ———

We always have the convolution diagram:

$$\boxed{\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} \xleftarrow{\ q\ } \mathcal{M}_{\mathcal{A}}^{ext} \xrightarrow{\ p\ } \mathcal{M}_{\mathcal{A}}}$$

$$q:\ 0 \longrightarrow E_2 \longrightarrow E \longrightarrow E_1 \longrightarrow 0 \ \longmapsto (E_2, E_1)$$
$$p:\ 0 \longrightarrow E_2 \longrightarrow E \longrightarrow E_1 \longrightarrow 0 \ \longmapsto E$$

Therefore

1. $\left(\mathrm{Fun}_{cs}\left(\mathcal{M}_{\mathcal{A}}(\mathbb{F}_q)/_{\mathrm{iso}}, \mathbb{C}\right), p_* \circ q^!\right)$ is a unital associative algebra

compactly supported

= the Hall algebra of $\mathcal{A}$

- $\mathcal{A}$ of dimension one:

$$\left(\mathrm{Fun}_{cs}\left(\mathcal{M}_{\mathcal{A}}(\mathbb{F}_q)/_{\mathrm{iso}}, \mathbb{C}\right), p_* \circ q^!, q_* \circ p^!\right) = (\text{"twisted"}) \text{ bialgebra}$$

Ex:

For $\mathcal{A} = \mathrm{nilp}(Q)$ $(\text{resp. } \mathrm{Coh}(X))$: we recover $H_{\mathrm{Rep}(Q)}$ $(\text{resp. } H_{\mathrm{Coh}(X)})$

2. $\mathcal{A}$ of dimension one:

$$\left(H^*(\mathcal{M}_{\mathcal{A}}), p_* \circ q^!, q_* \circ p^!\right) = (\text{"twisted"}) \text{ bialgebra}$$

singular cohomology

= 1d Kontsevich-Soibelman cohomological Hall algebra

3. $\mathcal{A}$ of dimension two, T = algebraic torus:

$\left(H_*^T(\mathcal{M}_{\mathcal{A}}), p_* \circ q^!\right)$ = $COHA_Q^T$ = 2d cohomological Hall algebra of Q

$\left(G_0^T(\mathcal{M}_{\mathcal{A}}), p_* \circ q^!\right)$ = KHA_Q^T = 2d K-theoretical Hall algebra of Q

(Here, $H_*^T(-)$ = T-equivariant Borel-Moore homology)

Let me conclude by:

Theorem

Botta-Davison, Schiffmann-Vasserot: $COHA_Q^T \simeq Y^{MO,+}(\mathfrak{g}_Q^{MO})$

where
$$\begin{cases} \mathfrak{g}_Q^{MO} = \text{Maulik-Okounkov } \mathbb{Z}\text{-graded Lie algebra of } Q \\ Y(\mathfrak{g}_Q^{MO}) = \text{filtered deformation of } U(\mathfrak{g}_Q^{MO}[z]) \end{cases}$$

Varagnolo-Vasserot: $KHA_Q^T \simeq U_\nu^+(\mathfrak{g}_Q^{KM}[z^{\pm 1}])$ for Q = finite or affine ADE

Conjecture: $KHA_Q^T \simeq U_\nu^+(\mathfrak{g}_Q^{MO}[z^{\pm 1}])$

COHAs of coherent sheaves on a smooth surface

S = smooth quasi-projective surface $/\mathbb{C}$

T = (possibly trivial) torus $\curvearrowright S$

$$\boxed{\underline{\mathrm{Coh}}_{ps}(S) = \text{moduli stack of properly supported coherent sheaves on } S}$$

Remark

We can also define:

- $\underline{\mathrm{Coh}}_0(S) \subset \underline{\mathrm{Coh}}_{ps}(S)$ corresponding to 0-dim. sheaves
- $\underline{\mathrm{Coh}}_{\leq 1}(S) \subset \underline{\mathrm{Coh}}_{ps}(S)$ corresponding to sheaves of dim ≤ 1

Attention ⚠

$\exists$ a derived enhancement

$$\mathbb{R}\underline{\mathrm{Coh}}_{ps}(S) \times \mathbb{R}\underline{\mathrm{Coh}}_{ps}(S) \xleftarrow{\mathbb{R}q} \mathbb{R}\underline{\mathrm{Coh}}^{ext}_{ps}(S) \xrightarrow{\mathbb{R}p} \mathbb{R}\underline{\mathrm{Coh}}_{ps}(S)$$

such that $\mathbb{R}q$ is quasi-smooth $\Longrightarrow \exists\, (\mathbb{R}q)^!$

Kapranov-Vasserot, Yu Zhao (in dim=0):

$$\boxed{\exists\ \mathrm{COHA}_S^{(T)} = (T\text{-equivariant})\ \mathrm{COHA}\ \text{associated to properly supported sheaves on } S}$$

= unital associative algebra structure on

$$H_*^{(T)}\left(\underline{\mathbb{R}\mathrm{Coh}}_{ps}(S)\right) = H_*^{(T)}\left(\underline{\mathrm{Coh}}_{ps}(S)\right)$$

with multiplication $(\mathbb{R}p)_* \circ (\mathbb{R}q)^!$.

Remark

- $\exists\ \mathrm{COHA}^{(T)}_{S,0\text{-dim}}$ associated to $\underline{\mathrm{Coh}}_0(S)$
- $\exists\ \mathrm{COHA}^{(T)}_{S,\leq 1}$ associated to $\underline{\mathrm{Coh}}_{\leq 1}(S)$

Example

$S = \mathbb{C}^2$: $\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}) \xrightarrow{\sim} \underline{\mathrm{Coh}}_0(\mathbb{C}^2)$

$$\begin{array}{c} \rotatebox{90}{$\in$} \\ A_1 \circlearrowright \mathbb{C}^d \circlearrowleft A_2 \end{array} \longmapsto \mathbb{C}^d = \mathbb{C}[A_1, A_2]\text{-module}$$

$$\Longrightarrow \mathrm{COHA}^{(T)}_{\mathbb{C}^2, 0\text{-dim}} \simeq \mathrm{COHA}^{(T)}_{1\text{-loop}}$$

In dim $=0$, we have a complete characterization:

Theorem (Mellit-Minets-Schiffmann-Vasserot)
$\mathrm{COHA}^{(T)}_{S,0\text{-dim}}$ can be described explicitly by generators and relations.
In particular, if S is projective and $\omega_S \simeq \mathcal{O}_S$:

$$\mathrm{COHA}_{S,0\text{-dim}} \simeq U(W_{1+\infty}(S)[t])$$

where $W_{1+\infty}(S)$ is a Lie algebra associated with $H^*(S)$.

$\Longrightarrow$ $\mathrm{COHA}_{S,0\text{-dim}}$ is a "Yangian of $\hat{\mathfrak{gl}}(1)$ twisted by $H^*(S)$"

The questions I would like to address today are:

Question 1: is $\mathrm{COHA}^{T}_{S,\leq 1}$ related to Yangians?

Question 2: can we describe $\mathrm{COHA}^{T}_{S,\leq 1}$ by generators and relations?

COHA of a surface and affine Yangians

First, we introduce a "nilpotent" version of COHA_S.

- S = smooth quasi-projective surface$/\mathbb{C}$
- $C \subset S$ reduced closed subscheme

Consider

$$\boxed{\underline{\mathrm{Coh}}(S, C) = \text{moduli stack of coherent sheaves on } S \text{ set-theoretically supported on } C}$$

sheaf analog of nilpotency

Example: X = smooth projective curve$/\mathbb{C}$

$\underline{\mathrm{Coh}}(T^*X, X) \simeq$ moduli stack of Higgs sheaves $(\mathcal{F}, \phi : \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega^1_X)$ on X such that ϕ is nilpotent

zero section

Theorem 1 (DPSSV)

1. $\exists$ an associative algebra structure on $H_*^{BM}(\underline{Coh}(S,C))$

$$\Longrightarrow \mathrm{COHA}_{S,C} =: \mathrm{HA}_{S,C}$$

If T = torus $\curvearrowright S$, C T-invariant $\Longrightarrow \exists\ \mathrm{COHA}^T_{S,C} =: \mathrm{HA}^T_{S,C}$

2. $\mathrm{HA}^{(T)}_{S,C}$ depends "locally" on C, i.e., given (S_1,C_1) and (S_2,C_2) s.t. the formal completions $\widehat{(S_1)}_{C_1} \simeq \widehat{(S_2)}_{C_2}$, we have:

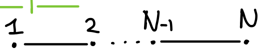
$$\mathrm{HA}^{(T)}_{S_1,C_1} \simeq \mathrm{HA}^{(T)}_{S_2,C_2}$$

The first relation between $\mathrm{HA}^T_{S,C}$ and Yangians is when

$$\boxed{S = \text{minimal resolution of ADE singularity}}$$

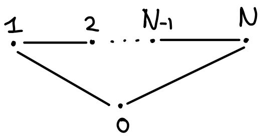
- $G \subset SL(2,\mathbb{C})$ finite group
  $\updownarrow$
  ADE quiver $Q_{fin} \subset$ affine ADE quiver Q

Example

A_N :  $= Q_{fin} \longleftrightarrow \mathbb{Z}_{N+1} \simeq \mu_{N+1} \hookrightarrow SL(2,\mathbb{C})$

group of $(N+1)$-th roots of unity

$\omega \longmapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$

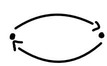
$A_N^{(1)}$:  $= Q$

- $\pi : S \longrightarrow \mathbb{C}^2/G$ Kleinian resolution of singularities

$C_{red} := \pi^{-1}(0)_{red} = C_1 \cup \dots \cup C_\ell \,;\; C_i \simeq \mathbb{P}^1 ;\; (C_i \cdot C_j) = -$ Cartan matrix of Q_{fin}

- torus $T \subset GL(2,\mathbb{C})$ centralizing G ($T =$ trivial or $\mathbb{C}^*$ or $\mathbb{C}^* \times \mathbb{C}^*$)

Example

$G = \mathbb{Z}_2 \implies Q_{fin} = \bullet = A_1 \,,\; Q =$  $= A_1^{(1)}$

$\implies \mathbb{C}^* \times \mathbb{C}^* \curvearrowright S = T^*\mathbb{P}^1 \supset C = \mathbb{P}^1 =$ zero section

Let

$$\boxed{\mathfrak{g}_Q^{MO}[t] = \text{universal central extension of } \mathfrak{g}_{Q_{fin}}[s^{\pm 1}, t] =: \mathfrak{g}_{ell}}$$

$$\simeq \mathfrak{g}_{\mathbb{Q}_{fin}}[s^{\pm 1}, t] \oplus K$$

$$K = \bigoplus_{\ell \in \mathbb{N}} \mathbb{Q} c_\ell \oplus \bigoplus_{\substack{k \in \mathbb{Z}_{\neq 0} \\ \ell \in \mathbb{N}_{\geq 1}}} \mathbb{Q} c_{k,\ell}$$

central elements

Example

$$G = \mathbb{Z}_2 \implies \mathfrak{g}_{ell} \simeq \mathfrak{sl}(2)[s^{\pm 1}, t] \oplus K$$

Define

$$\mathfrak{n}^{s,c}_{ell} := \mathfrak{n}^-_{\mathbb{Q}_{fin}}[s^{\pm 1}, t] \oplus s^{-1}\mathfrak{h}_{\mathbb{Q}_{fin}}[s^{-1}, t] \oplus K_- \quad \left(K_- := \bigoplus_{k<0} \mathbb{Q} c_{k,\ell}\right)$$

Theorem 2 (DPSSV)

completion

- $\exists$ a canonical algebra isomorphism $HA_{S,C} \simeq \hat{U}(\mathfrak{n}^{s,c}_{ell})$
- $\exists$ a canonical algebra isomorphism $HA^T_{S,C} \simeq \mathbb{Y}_{S,C}$

where $\underbrace{\mathbb{Y}_{S,C} \text{ is a filtered deformation}}_{\text{Algebraic definition}}$ of $\hat{U}(\mathfrak{n}^{s,c}_{ell})$

Conjecture

$\mathbb{Y}_{S,C}$ is another 'half' of $\mathbb{Y}^{MO}_{\mathbb{Q}}$.

Rmk (work in progress with Schiffmann-Shimpi)

One can obtain other (conjectural) halves of $\mathbb{Y}_Q^{MO}$ by considering partial resolutions of $\mathbb{C}^2/G$.

Let us summarize what we have so far:

$$\begin{array}{c} G \subset SL(2,\mathbb{C}) \text{ finite group} \\ \updownarrow \\ \text{ADE quiver } Q_{fin} \subset \text{ affine ADE quiver } Q \end{array}$$

Minimal resolution of $\mathbb{C}^2/G$	Partial resolution of $\mathbb{C}^2/G$	$\mathbb{C}^2/G$
$\mathfrak{n}_{ell}^{s,c} = \mathfrak{n}_{Q_{fin}}^{-}[s^{\pm 1}, t] \oplus s^{-1}\mathfrak{h}_{Q_{fin}}[s^{-1}, t] \oplus K_-$	$\mathfrak{n}_{ell}^{\tilde{s},\tilde{c}}$	$\mathfrak{n}_{ell} = s^{-1}\mathfrak{g}_{fin}[s^{-1}, t] \oplus \mathfrak{n}_{fin}[t] \oplus K_-$
$\mathbb{Y}^{s,c}$ = def of $U(\mathfrak{n}_{ell}^{s,c})$	$\mathbb{Y}^{\tilde{s},\tilde{c}}$ = def of $U(\mathfrak{n}_{ell}^{\tilde{s},\tilde{c}})$	$\mathbb{Y}_Q^{MO,+}$ = def of $U(\mathfrak{n}_{ell})$

$\Longrightarrow$ Different 'halves' of the same Yangian $\mathbb{Y}_Q^{MO}$

Question: how do we prove this theorem?

Recall the derived McKay correspondence:

$$\tau : D^b(\mathrm{Coh}(S)) \xrightarrow{\sim} D^b(\mathrm{Mod}(\Pi_Q))$$

τ is not t-exact w.r.t. the standard t-structures

$$\Longrightarrow \underline{\mathrm{Coh}}(Y,C) \cancel{\simeq} \Lambda_Q = \text{stack of nilpotent repr.s of } \Pi_Q$$

$$\Longrightarrow HA^T_{Y,C} \cancel{\simeq} HA^T_Q$$

Attention ⚠: we will "interpolate" between the 2 hearts by using:

- braid group actions on bounded derived cat.s
- Bridgeland stability conditions

finite coweight lattice

Recall that

- extended affine braid group $B_{ex} \simeq \left(B_{fin} \cup \{L_{\check\lambda} : \check\lambda \in \check X_{fin}\}\right) / rels$

- $\check X_{fin} \xrightarrow{\sim} Pic(S)$, $\check\lambda \longmapsto \mathcal{L}_{\check\lambda}$

Lemma

$\exists$ a group homomorphism

abelian category

$$\rho : B_{ex} \longrightarrow Aut\left(D^b\left(Coh_c(S)\right)\right)$$

such that $\rho(L_{\check\lambda}) = (\mathcal{L}_{\check\lambda} \otimes -) =: L_{\check\lambda}$, $\forall \check\lambda \in \check X_{fin}$

$\Longrightarrow$ by the McKay equivalence, $L_{\check\lambda} \in Aut\left(D^b\left(nilp(\Pi_Q)\right)\right)$

Now fix a coweight $\check\Theta = \sum_{i\in I} \check\Theta_i \check\omega_i \in \check X_{aff}$ s.t.

$$\boxed{\check\Theta_i > 0 \ \forall i \neq 0 \ \text{ and } \ \check\Theta_0 = -\sum_{i=1}^{e} \check\Theta_i}$$

$\Longrightarrow \check\Theta_{fin} := \sum_{i\neq 0} \check\Theta_i \check\omega_i \in \check X_{fin}$

Consider the stability function on $\mathrm{nilp}(\Pi_Q)$:

$$Z_{\check{\theta}}: K_0(\mathrm{nilp}(\Pi_Q)) \simeq \mathbb{Z}I \longrightarrow \mathbb{C}$$
$$\underline{d} \longmapsto -(\check{\theta}, \underline{d}) + (\check{\rho}, \underline{d})$$

$$\check{\rho} = \sum_i \check{\omega}_i$$

Attention: This is only a reformulation of King's (semi)$_=$ stability for quiver repr.s.

$\Longrightarrow$ $\exists$ associated $(Z_{\check{\theta}}, \mathcal{P}_{\check{\theta}})$ = Bridgeland's stability condition on $D^b(\mathrm{nilp}(\Pi_Q))$

Here, $\mathcal{P}_{\check{\theta}}$ = slicing = family of full additive subcategories

$$\mathcal{P}_{\check{\theta}}(\phi) \subset D^b(\mathrm{nilp}(\Pi_Q)) \quad \forall \phi \in \mathbb{R}$$

such that

- $\mathcal{P}_{\check{\theta}}(\phi+1) = \mathcal{P}_{\check{\theta}}(\phi)[1] \quad \forall \phi \in \mathbb{R}$;
- $\forall\, \phi_1 > \phi_2$ and $F_j \in \mathcal{P}_{\check{\theta}}(\phi_j)$, $\mathrm{Hom}(F_1, F_2) = 0$;
- $\forall\, 0 \neq E \in D^b(\mathrm{nilp}(\Pi_Q))$, $\exists\, \phi_1 > \cdots > \phi_n$ and triangles

$$\begin{tikzcd}[column sep=small]
0=E_0 \arrow[r] & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[dl] \\
F_1 \in \mathcal{P}_{\check{\Theta}}(\phi_1) \arrow[u, dashed] & & & F_n \in \mathcal{P}_{\check{\Theta}}(\phi_n) \arrow[u, dashed] &
\end{tikzcd}$$

(Harder-Narasimhan type filtration)

Set $\mathcal{P}_{\check{\Theta}}(I) := \langle \mathcal{P}_{\check{\Theta}}(\phi) : \phi \in I \rangle$ $\forall$ interval $I \subset \mathbb{R}$

Lemma

1. $\forall k \in \mathbb{Z}$, $L_{-2k\check{\Theta}_{fin}} : \mathrm{nilp}(\Pi_Q) \xrightarrow{\sim} \mathcal{P}_{\check{\Theta}}((\nu_{-k}, \nu_{-k}+1]))$

with $\nu_\ell := \frac{1}{\pi} \arctan(2h\ell)$

Coxeter number

2. $\mathcal{P}_{\check{\Theta}}((-\frac{1}{2}, \frac{1}{2}]) \simeq \mathrm{Coh}_c(S)$

Remark

- $\nu_k \xrightarrow{k \to +\infty} \frac{1}{2}$
- we have a "sequence" of t-structures $\{\tau_k\}_{k \in \mathbb{N}}$ all equi= valent to $\tau_0 :=$ standard t-structure s.t.

$$\boxed{\tau_k \xrightarrow{k \longrightarrow +\infty} \tau_\infty = t\text{-structure with heart } \mathrm{Coh}_c(S)}$$

For $\ell, k \in \mathbb{N}$, $k \geq \ell$, set

$$\Lambda_Q^{\ell,k} := \text{(derived) moduli stack of objects in } \mathcal{P}_{\check{\theta}}((\nu_{-\ell}, \nu_{-k}+1]))$$

Attention: $L_{2k\check{\theta}_{fin}} : \Lambda_Q^{\ell,k} \xrightarrow{\sim} \Lambda_Q^{\ell-k,0} \simeq \Lambda_Q^{>\nu_{-\ell+k}} = \text{HN stratum}$

Lemma

1. The vector space

$$\boxed{\mathbf{HA}_\infty^{(T)} := \lim_{\ell} \operatorname{colim}_{k \geq \ell} H_*^{(T)}(\Lambda_Q^{\ell,k})}$$

has the structure of an unital associative algebra with multiplication induced from that of $\mathbf{HA}_Q^{(T)}$.

2. $\exists$ an algebra isomorphism $\mathbf{HA}_{S,C}^{(T)} \simeq \mathbf{HA}_\infty^{(T)}$

Finally, a careful analysis of the compatibility between the action of B_{ex} on Yangians and on $\mathbf{HA}_Q^T$ yields:

Lemma

$\exists$ an algebra isomorphism $HA^T_\infty \xrightarrow{\sim} \mathbb{Y}^+_\infty$, where

$$\mathbb{Y}^+_\infty := \lim_{\ell} T_{2\ell\check{\theta}_{fin}}(\mathbb{Y}^-_Q) \Big/ T_{2\ell\check{\theta}_{fin}}(J)$$

where $J := \sum_{\mu_{\check{\theta}}(\underline{d})>0} \mathbb{Y}^-_Q \cdot \mathbb{Y}^-_{Q,-\underline{d}}$.

The proof of Thm 2 follows from the above lemmas.

$\square$

Conjecture $\mathbb{Y}^+_\infty$ is a new half of $\widehat{\mathbb{Y}}^{MO}_Q$.