Hall algebras

and

Hopf algebras

The goal of the morning seminars is to:

$\boxed{\text{(cohomological / K-theoretical) Hall algebras}} \longleftrightarrow \boxed{\text{Moduli spaces in Algebraic Geom.}}$

$\updownarrow$

$\boxed{\text{Hopf algebras (Drinfeld-Jimbo's quantum groups, Yangians, quantum loop algebras, ...)}}$

During the first talk:

- Hall algebras associated to abelian categories
- Example: Hall algebras of quivers

During the second talk: The quantum toroidal algebra of $gl(1)$ as

- the Hall algebra of an elliptic curve
- The 2d K-theoretical Hall algebra of the 1-loop quiver

1. Hall algebras associated to abelian categories

Fix

- $k = \mathbb{F}_q$ finite field with q elements
- $\mathcal{A}$ = abelian category s.t.
  1. $\mathcal{A}$ is k-linear
  2. $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ and $\mathrm{Ext}^1_{\mathcal{A}}(X,Y)$ are finite-dim. k-vector sp.
  3. $\mathrm{Ext}^i_{\mathcal{A}}(X,Y) = 0 \ \forall i \geq 2$ ("hereditary")

Example

$Q = (I = \{\text{vertices}\}, E = \{\text{edges}\})$ = quiver (i.e., oriented graph)

$\forall e : \overset{i}{\bullet} \longrightarrow \overset{j}{\bullet} \in E$, $s(e)$ = source of $e = i$, $t(e)$ = target of $e = j$

$$\left.\begin{array}{ll} A_1 : & \underset{1}{\bullet} \\ A_2 : & \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \\ A_n : & \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet} \ \cdots \ \longrightarrow \underset{n}{\bullet} \end{array}\right\} \text{Type A Dynkin quivers}$$

Def.

A representation of Q over k is a pair $\left(V = \bigoplus_{i \in I} V_i, (f_e)_{e \in E}\right)$ where

- V is a I-graded vector space
- $f_e : V_{s(e)} \longrightarrow V_{t(e)}$ is a linear map $\forall e \in E$

Moreover, a representation is:

- finite-dim. if $\dim_K V_i < \infty \ \forall i \in I$
- nilpotent if $\exists N > 0$ s.t. $f_{e_n} \circ \dots \circ f_{e_1} = 0 \ \forall$ path $e_1 \cdots e_n$ of length $n > N$ of Q.

Example

$A_1: \quad \underset{1}{\bullet} \qquad \Longrightarrow (V_1, 0)$

$A_2: \quad \underset{1}{\bullet} \xrightarrow[e]{} \underset{2}{\bullet} \qquad \Longrightarrow \left(V_1 \oplus V_2, \ f_e : V_1 \longrightarrow V_2\right)$

Rmk:

A representation of Q over K = right module over KQ

path algebra of Q

Set

$$\boxed{\mathcal{A} = \mathrm{nilp}_K(Q) = \text{abelian category of nilpotent finite-dim. repr.s of } Q}$$

Then

$$\dim_K \mathrm{Hom}_{\mathcal{A}}(X, Y) < \infty, \ \dim_K \mathrm{Ext}^1_{\mathcal{A}}(X, Y) < \infty, \ \mathrm{Ext}^i_{\mathcal{A}}(X, Y) = 0 \ \forall i \geq 2$$

Let's go back to the general framework.

Set

- $\mathcal{M}_{\mathcal{A}} := \{\text{iso. classes of objects of } \mathcal{A}\}$

- $K_0(\mathcal{A})$ = Grothendieck group of $\mathcal{A}$

= free ab. group gen. by $Obj(\mathcal{A}) / \langle Y - X - Z : 0 \to X \to Y \to Z \to 0 \rangle$

We introduce the following notation:

- $\forall \alpha \in \mathcal{M}_{\mathcal{A}}$, $\alpha = [V_\alpha]$ with $V_\alpha \in \mathcal{A}$

- $\forall M \in Obj(\mathcal{A}) \rightsquigarrow \overline{M} \in K_0(\mathcal{A})$.

Recall that the Euler form: $\langle \cdot, \cdot \rangle : K_0(\mathcal{A})^{\times 2} \longrightarrow \mathbb{Z}$ such that

$$\langle \overline{M}, \overline{N} \rangle = \dim_k Hom(M,N) - \dim_k Ext^1(M,N) \quad \forall M,N \in Obj(\mathcal{A})$$

Symmetrized Euler form: $(\mu,\nu) := \langle \mu, \nu \rangle + \langle \nu, \mu \rangle \quad \forall \nu, \mu \in K_0(\mathcal{A})$

Example

- Q = quiver without edge-loops. Recall that there exists

$\mathfrak{g}_Q :=$ Kac-Moody algebra of Q $= \langle e_i, f_i, h_i : i \in I \rangle / \text{rels}$

Example: A_n : $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet} \cdots \longrightarrow \underset{n}{\bullet}$ $\rightsquigarrow \mathfrak{g}_{A_n} = \mathfrak{sl}(n+1)$

$\Longrightarrow K_0(\mathrm{nilp}(Q)) \simeq \mathbb{Z}^I \simeq$ root lattice of $\mathfrak{g}_Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$

$$\overline{S_i} \longmapsto \alpha_i$$

where

$$S_i = \left(\bigoplus_{j \in I} V_j , 0 \right), \quad V_j = \begin{cases} k & i = j \\ 0 & i \neq j \end{cases}$$

Rmk $\{S_i : i \in I\}$ corresponds to the set of all simple objects of $\mathrm{nilp}(Q)$

$\Longrightarrow$ Symmetrized Euler form = Cartan form on the root lattice

Let's go back to the general framework.
Define "structure constants"

- $\mathrm{aut}_\alpha := \# \mathrm{Aut}(V_\alpha)$ for $\alpha = [V_\alpha] \in \mathcal{M}_A$
- $g^\gamma_{\alpha,\beta} := \# \{ X \subseteq V_\gamma : \beta = [X] \text{ and } \alpha = [V_\gamma / X] \}$ $\forall \alpha, \beta, \gamma \in \mathcal{M}_A$

Example

$Q = A_1$. Then

- $\exists !$ simple repr. $S = S_1 = (k, 0)$.

- All repr.s $M \in \mathrm{nilp}(A_1)$ are of the form $M \simeq S^{\oplus m}$, $m \in \mathbb{N}$:

$$\boxed{S^{\oplus m} = \left(k^{\oplus m}, 0\right)}$$

- the class $\overline{S^{\oplus m}} \in K_0(\mathrm{nilp}(A_1))$ corresponds to $m \in \mathbb{Z}$

$$\Longrightarrow g^{2}_{1,1} = \#\left\{S \subseteq S^{\oplus 2} : S^{\oplus 2}/S \simeq S\right\} = \#\left\{k \subset k^{\oplus 2}\right\}$$

$$= \#\mathbb{P}^1(\mathbb{F}_q) = \#\left(\mathbb{F}_q \cup \{\infty\}\right) = 1 + q$$

In general:

$$\Longrightarrow g^{i+j}_{i,j} = \#\left\{S^{\oplus j} \subseteq S^{\oplus i+j} : S^{\oplus i+j}/S^{\oplus j} \simeq S^{\oplus i}\right\} = \#Gr(j; i+j)(\mathbb{F}_q)$$

$$= \underbrace{\begin{bmatrix} i+j \\ j \end{bmatrix}_+}_{q\text{-binomial}} := \frac{[i+j]_+ \cdots [i+1]_+}{[2]_+ \cdots [j]_+} \in \mathbb{N}[q] \quad \left([t]_+ := 1 + q + \cdots + q^{t-1}\right)$$

Recall the following definition:

Def.
An abelian category $\mathcal{A}$ is a length category if $\forall M \in \mathcal{A}$ $\exists$ a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

s.t. M_{i+1}/M_i is simple

Rmk: $\mathrm{nilp}(Q)$ is a length category.

Now, I have introduced all the notions needed in the def. of Hall algebras.

Set $\nu = \sqrt{q}$.

Def. (Ringel, Green, Xiao)
The Hall algebra $H_{\mathcal{A}}$ of $\mathcal{A}$ is the associative unital algebra$/\mathbb{C}$ for which

$$\boxed{H_{\mathcal{A}} := \bigoplus_{\alpha \in M_{\mathcal{A}}} \mathbb{C} u_\alpha \,, \quad u_\alpha u_\beta = \sum_{\gamma \in M_{\mathcal{A}}} \nu^{\langle \alpha, \beta \rangle} g^{\gamma}_{\alpha,\beta} u_\gamma \,, \text{ and } 1 = u_0}$$

(Attention ⚠: the product depends on "extensions")

The twisted Hall algebra $HA_{\mathcal{A}}^{tw}$ of $\mathcal{A}$ is the Hopf algebra given as

$$\boxed{HA_{\mathcal{A}}^{tw} := HA_{\mathcal{A}} \otimes_{\mathbb{C}} \mathbb{C}[K_0(\mathcal{A})]}$$

with

- product: $K_\mu K_\nu = K_{\mu+\nu} \quad \forall\, \mu,\nu \in K_0(\mathcal{A})$

$$K_\mu u_\alpha = \nu^{(\mu,\alpha)} u_\alpha K_\mu \quad \forall\, \alpha \in M_{\mathcal{A}},\ \mu \in K_0(\mathcal{A})$$

- Δ coproduct: $\Delta(K_\mu) = K_\mu \otimes K_\mu$

$$\boxed{\Delta(u_\gamma) = \sum_{\alpha,\beta \in M_{\mathcal{A}}} \nu^{\langle\alpha,\beta\rangle} \frac{aut_\alpha\, aut_\beta}{aut_\gamma}\, g^{\gamma}_{\alpha,\beta}\, u_\alpha K_\beta \otimes u_\beta}$$

(Attention ⚠: the coproduct depends on "all possible ways" the repr. V_γ can be obtained as extensions)

- ε counit: $\varepsilon(u_\alpha) = \delta_{\alpha,0}$, $\varepsilon(K_\mu) = 1$

▸ S antipode: $S(K_\mu) = K_{-\mu}$, $S(u_0)=0$,

$$S(u_\gamma) = \sum_{m\geq 1} (-1)^m \sum_{0=M_0\subset\cdots\subset M_n=V_\gamma} (\text{quantity dep. on the filtr.})\, K_{-\gamma} u_{\alpha_n} u_{\alpha_{n-1}} \cdots u_{\alpha_1}$$

$$M_A \ni \alpha_{i+1} = [M_{i+1}/M_i]$$

Rmk

If A is not a length category

1. HA_A^{tw} is a topological bialgebra:

$\Longrightarrow \Delta$ takes value in a completion $HA_A^{tw} \hat{\otimes} HA_A^{tw}$

2. S is not well-defined: one could define S^{-1}.

Given an abelian category A, it is natural to ask:

Question: could we describe explictly HA_A?

When $A = \mathrm{nilp}(Q)$ for a quiver without edge-loops,

$$\boxed{HA_A \simeq \text{quantum group associated to } Q}$$

Let's define such a quantum group:

Def. (Drinfeld-Jimbo quantum group)
Let Q be a quiver without edge-loops and $\mathfrak{g}_Q$ its Kac-Moody Lie algebra.

$U_\nu(\mathfrak{g}_Q)$ is the Hopf algebra generated by $\{E_i, F_i, K_i^{\pm}\}_{i\in I}$ subject to the relations:

- $K_i K_j = K_j K_i$
- $K_i E_j K_i^{-1} = \nu^{a_{ij}} E_j$ and $K_i F_j K_i^{-1} = \nu^{-a_{ij}} F_j$ $\left(A = (a_{ij}) \text{ Cartan mat.}\right)$
- $[E_i, F_j] = \delta_{ij} \dfrac{K_i - K_i^{-1}}{\nu - \nu^{-1}}$
- quantum Serre relations
- $\Delta(K_i) = K_i \otimes K_i$, $\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i$, $\Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i^{-1}$

- $S(K_i)=K_i^{-1}$, $S(E_i)=-K_i^{-1}E_i$, $S(F_i)=-F_iK_i$

We denote by:

- $U_\nu(n_+)$ = Subalgebra generated by $\{E_i\}_{i\in I}$
- $U_\nu(b_+)$ = ——— // ——— $\{E_i, K_i^{\pm 1}\}_{i\in I}$

Here, $g_Q = n_- \oplus h_Q \oplus n_+$ is the triangular decomposition and $b_+ = h_Q \oplus n_+$

The celebrated Ringel-Green thm states:

Thm (Ringel - Green)
The assignment

$$\boxed{E_i \longmapsto u_{[S_i]} \quad \text{and} \quad K_i \longmapsto K_{\overline{S_i}} \quad \forall i \in I}$$

extends to an embedding of Hopf algebras:

$$\boxed{\Psi : U_\nu(b_+) \longrightarrow HA^{tw}_{nilp(Q)}}$$

Ψ is an isomorphism $\Leftrightarrow$ Q = finite ADE Dynkin quiver.

It is natural to wonder if we can also realize the "whole" $U_v(\mathfrak{g}_Q)$.

Let $\widetilde{\mathcal{H}}_{A}^{\mathrm{tw},-}$:= the dual Hopf algebra of $\widetilde{\mathcal{H}}_{A}^{\mathrm{tw}}$ with opposite coproduct

Def (Ringel pairing)

Let

$$\boxed{(\cdot,\cdot) : \widetilde{\mathcal{H}}_{A}^{\mathrm{tw}} \times \widetilde{\mathcal{H}}_{A}^{\mathrm{tw},-} \longrightarrow \mathbb{C}}$$

be the bilinear form given by

$$\boxed{(K_\mu u_\alpha, K_\nu u_\beta^-) = v^{-(\mu,\nu)-(\alpha,\nu)+(\mu,\beta)} (\mathrm{aut}_\alpha)^{-1} \delta_{\alpha,\beta}}$$

Rmk: $(\cdot,\cdot)$ is a skew-Hopf pairing.

$\Longrightarrow$ Drinfeld double $\widetilde{D\mathcal{H}}_{A}^{\mathrm{tw}}$:= Hopf algebra structure on $\widetilde{\mathcal{H}}_{A}^{\mathrm{tw}} \otimes \widetilde{\mathcal{H}}_{A}^{\mathrm{tw},-}$

$\Longrightarrow$ reduced Drinfeld double $D\mathcal{H}_{A}^{\mathrm{tw}}$:

$$\boxed{\mathrm{DHA}_{\mathcal{A}}^{\mathrm{tw}} := \widetilde{\mathrm{DHA}}_{\mathcal{A}}^{\mathrm{tw}} \big/ \langle K_{\mu}\otimes 1 - 1\otimes K_{-\mu} : \mu\in K_0(\mathcal{A}) \rangle}$$

Going back to quivers, Xiao proved:

<u>Thm</u> (Xiao)
The assignment

$$\boxed{E_i \longmapsto u_{[S_i]}, \quad F_i \longmapsto -\nu^{-1} u^{-}_{[S_i]}, \quad K_i \longmapsto K_{S_i} \quad \forall i\in I}$$

extends to an embedding of Hopf algebras:

$$\boxed{\Psi : U_\nu(\mathfrak{g}_Q) \longrightarrow \mathrm{DHA}^{\mathrm{tw}}_{\mathrm{nilp}(Q)}}$$

Ψ is an isomorphism $\Leftrightarrow$ Q = finite ADE Dynkin quiver

Let us summarize the dictionary we have established so far.

Q = quiver without edge-loops.

Abelian category $\mathrm{nilp}(Q)$	Kac-Moody Lie algebra $\mathfrak{g}_Q$
Grothendieck group $K_0(\mathrm{nilp}(Q))$	Root lattice $\bigoplus_{i\in I}\mathbb{Z}\alpha_i$
Symmetrized Euler form	Cartan-Killing form
simple object S_i	Simple root α_i
$\mathbb{HA}_{\mathrm{nilp}(\Pi_Q)}$	$U_\nu(\mathfrak{n}_+)$
$\mathbb{C}[K_0(\mathcal{A})]$	$U_\nu(\mathfrak{h}_Q)$
$\mathbb{HA}^{\mathrm{tw}}_{\mathrm{nilp}(Q)}$	$U_\nu(\mathfrak{b}_+)$

Now, one could ask:

<u>Q. 1</u>: What does it happen for quivers with loops?

<u>Q. 2</u>: Is the whole $\mathbb{HA}^{\mathrm{tw}}_{\mathrm{nilp}(Q)}$ a quantum group?

<u>Q. 3</u>: Why is all of this important?

Answer to Q1:
Q = quiver without edge-loops $\rightsquigarrow$ arbitrary quiver Q

Drinfeld-Jimbo quantum group $\rightsquigarrow$ Quantum Borcherds-Bozec al.
(Kang)

Answer to Q2:
Sevenhant-van den Bergh: $HA^{tw}_{nilp(Q)} \simeq U_\nu(\mathfrak{b}_+^{MO})$, where

$$\boxed{\mathfrak{b}_+^{MO} = \mathfrak{h}_Q^{MO} \oplus \mathfrak{n}_+^{MO} \subset \mathfrak{g}_Q^{MO} = \text{Maulik-Okounkov Lie algebra of } Q}$$

Answer to Q3: constructions in the theory of Quantum Grps
- PBW bases when Q = finite ADE quiver
- Lusztig's theory of canonical bases

The quantum toroidal algebra of $gl(1)$

from two perspectives

One can apply the construction of Hall algebras to other abelian categories, for example:

$\mathrm{Coh}(X)$:= abelian category of coherent sheaves on a smooth projective curve $/\mathbb{F}_{q^{-1}}$ of genus g

(Attention: the choice q^{-1} is made to have "nice" formulas later on)

Recall:

- Any coherent sheaf $\mathcal{F}$ on X is: $\mathcal{F} \simeq T \oplus V$, where V is a vector bundle and T is a 0-dimensional sheaf on X.

(Ex: $\mathcal{O}_x$ = skyscraper sheaf at $x \in X$)

- Any coherent sheaf $\mathcal{F}$ on X has a rank and a degree.

(Ex: $\mathcal{O}_x$ has rank zero and degree 1).

Recall that the zeta function of X:

$$\zeta_X(x) = \exp\Big(\sum_{n\geq 1} \frac{x^n}{n} \underbrace{\# X(\mathbb{F}_{q^{-n}})}_{\text{cardinality of set of pts}/\mathbb{F}_{q^{-n}}}\Big) = \frac{\prod_{i=1}^{g} (1-\sigma_i x)(1-\bar{\sigma}_i x)}{(1-x)(1-q^{-1}x)}$$

where $\sigma_1, \ldots, \sigma_g \in \mathbb{C}$ are such that $\sigma_i \bar{\sigma}_i = q^{-1}$ $\forall i = 1, \ldots, g$.

Attention: $\mathbb{C}(\sigma_1^{-1}, \ldots, \sigma_g^{-1}, q)$ will be the field over which the Hall algebra of $Coh(X)$ will be defined.

Note that, in this case, the role of the "twisted" Hall algebra is played by

$$\boxed{H_X := H_{Coh(X)} \otimes \mathbb{C}[\varkappa^{\pm 1}]} \qquad \big(\mathbb{C}[K_0(A)] \longleftrightarrow \mathbb{C}[\varkappa^{\pm 1}]\big)$$

with

$$\varkappa u_{[\mathcal{E}]} \varkappa^{-1} = q^{rk(\mathcal{E})(g-1)} u_{[\mathcal{E}]}$$

As in the quiver case, it is "easier" to describe a subalgebra of H_X.

Define the following elements of H_X:

$$\boxed{u_{0,d} := \sum_{\substack{T \text{ torsion} \\ \deg(T)=d}} u_{[T]} \quad \text{and} \quad u_{1,n} := \sum_{\substack{L \text{ line bundle} \\ \deg(L)=n}} u_{[L]}}$$

Let

$$H_X^{sph} := \langle u_{0,d}, u_{1,n} : d,n \in \mathbb{Z}, d \geq 1 \rangle \subset H_X$$

Attention ⚠:

H_X is a topological bialgebra with a compatible non-degenerate pairing

H_X^{sph} ——— // ———

$\Longrightarrow \exists$ (reduced) Drinfeld doubles DH_X and DH_X^{sph}

Fact: $\langle u_{0,d} : d \geq 1 \rangle \simeq \mathbb{C}[1_{0,1}, 1_{0,2}, \ldots] \simeq \mathbb{C}[\Theta_{0,1}, \Theta_{0,2}, \ldots]$

free commutative algebra

Define the "modes"

$$\boxed{\begin{gathered} E(z) := \sum_{d\in\mathbb{Z}} u_{1,d} z^{-d}, \quad F(z) := \sum_{d\in\mathbb{Z}} u^-_{1,d} z^{-d}, \text{ and} \\ H^+(z) := \varkappa\Big(1+\sum_{d\geq 1} \Theta_{0,d} z^{-d}\Big) \text{ and } H^-(z) := \varkappa^{-1}\Big(1+\sum_{d\geq 1} \Theta^-_{0,d} z^{-d}\Big) \end{gathered}}$$

Theorem (Schiffmann-Vasserot; Neguț-S.-Schiffmann)

DH_X^{sph} is the unital associative algebra $/\mathbb{C}(q, \sigma_1^{-1}, \dots, \sigma_g^{-1})$ generated by $\varkappa^{\pm 1}, \theta_{0,d}^{\pm}, u_{1,n}^{\pm}$ with $d, n \in \mathbb{Z}, d \geq 1$, subject to:

1. $$E(z)H^{\pm}(w) = H^{\pm}(w)E(z)\tilde{\zeta}_X(z/w)\tilde{\zeta}_X(w/z)^{-1} \quad |w^{\pm}| >> |z^{\pm}|$$
$$F(z)H^{\pm}(w) = H^{\pm}(w)F(z)\tilde{\zeta}_X(z/w)\tilde{\zeta}_X(w/z)^{-1} \quad |w^{\pm}| >> |z^{\pm}|$$

2. $$E(z)E(w)\tilde{\zeta}_X(w/z) = E(w)E(z)\tilde{\zeta}_X(z/w)$$
$$F(w)F(z)\tilde{\zeta}_X(w/z) = F(z)F(w)\tilde{\zeta}_X(z/w)$$

3. Cubic Serre rels for $E(z)$'s and $F(z)$'s, resp.

4. $$[u_{1,d}, u_{1,k}^{-}] = \begin{cases} -\varkappa\theta_{d+k} & \text{if } d+k>0 \\ \varkappa^{-1} - \varkappa & \text{if } d+k=0 \\ \varkappa\theta^{-}_{0,-d-k} & \text{if } d+k<0 \end{cases}$$

Here, $\tilde{\zeta}_X(x) = (1 - q^{-1}x)\zeta_X(x)$

Rmk

1. DH_X^{sph} has the structure of a topological Hopf algebra with a non-degenerate Hopf pairing.
In particular,

$$\Delta(H^{\pm}(z)) = H^{\pm}(z)\otimes H^{\pm}(z) \quad , \quad \Delta(E(z)) = E(z)\otimes 1 + H^{+}(z)\otimes E(z)$$

$$\Delta(F(z)) = 1\otimes F(z) + F(z)\otimes H^{-}(z)$$

2. Neguț-S.-Schiffmann : Also, DH_X admits a description in terms of generators and relations

3. For genus = 1:

$$\boxed{DH^{sph}_{elliptic} = \text{quantum toroidal algebra of } \mathfrak{gl}(1) = U_{\sigma,q}(\hat{\hat{\mathfrak{gl}}}(1))}$$

Before moving to K-theoretical Hall algebras, let me mention "shuffle algebras":

Theorem (Schiffmann-Vasserot)

The assignment $u_{1,d} \longmapsto z^d$ extends to an injective homomorphism of unital associative algebras:

$$\langle u_{1,d} : d\in\mathbb{Z}\rangle \hookrightarrow Sh_X$$

where

- as a vector space, $Sh_X := \bigoplus_{n\geq 0} \mathbb{C}(q,\sigma_1^{-1},\dots,\sigma_g^{-1})[z_1^{\pm},\dots,z_n^{\pm}]^{\mathfrak{S}_n}$
- $f(z_1,\dots,z_n),\ g(z_1,\dots,z_m)\in Sh_X$:

$$f(z_1,\dots,z_n) * g(z_1,\dots,z_m) := \mathrm{Sym}\left(\frac{1}{n!\,m!} f(z_1,\dots,z_n)\,g(z_{1+n},\dots,z_{n+m}) \cdot \prod_{\substack{1\leq i\leq n\\ n+1\leq j\leq n+m}} \tilde{\zeta}_X(z_i/z_j)\right)$$

This homomorphism is an intertwiner:

$$\boxed{\langle T_{0,n} : n\geq 1\rangle \circlearrowright \langle u_{1,d} : d\in\mathbb{Z}\rangle \hookrightarrow Sh_X \circlearrowleft \text{power sums } p_n := \sum z_i^n}$$

Rmk
Neguts: extension to "double" shuffle algebra, etc...

2d K-theoretical Hall algebras

Now, we introduce a completely different framework.
First, we shall work at the level of

$$\boxed{G_0(-) := \text{Grothendieck group of coherent sheaves}}$$

For any $d \in \mathbb{Z}, d \geq 0$, define:

$$\boxed{\begin{gathered} C_d := \{(A,B) \in gl(d;\mathbb{C}) : [A,B]=0\} \\ (\text{commuting variety}) \end{gathered}}$$

It admits an action : $(\mathbb{C}^*)^2 \times GL(d;\mathbb{C}) \curvearrowright C_d$:

$$\boxed{(t_1,t_2)\cdot(A,B) = (t_1A, t_2B) \quad \text{and} \quad G\cdot(A,B) = (GAG^{-1}, GBG^{-1})}$$

Goal: Define a unital associative algebra structure on

$$\boxed{\bigoplus_{d\geq 0} G_0^{(\mathbb{C}^*)^2 \times GL_d}(C_d)}$$

— equivariant G_0

Consider

$$\begin{tikzcd}[column sep=small]
& C_{d_1+d_2} \cap \left(\text{parabolic of } \mathfrak{gl}(d_1+d_2;\mathbb{C}) \text{ corresponding to fixing } \mathbb{C}^{d_1} \subset \mathbb{C}^{d_1+d_2} \right) \arrow[dddl, "q"'] \arrow[dddr, "p"] & \\
& \rotatebox{90}{$\in$} & \\
& \left(A = \left(\begin{array}{c|c} A_1 & * \\ \hline 0 & A_2 \end{array}\right), B = \left(\begin{array}{c|c} B_1 & * \\ \hline 0 & B_2 \end{array}\right) \right) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
C_{d_2} \times C_{d_1} \ni \left((A_2, B_2), (A_1, B_1) \right) & & (A, B) \in C_{d_1+d_2}
\end{tikzcd}$$

Theorem (Schiffmann-Vasserot)

The map

$$\begin{tikzcd}
G_0^{(\mathbb{C}^*)^2 \times GL(d_2)}(C_{d_2}) \otimes G_0^{(\mathbb{C}^*)^2 \times GL(d_1)}(C_{d_1}) \arrow[r, "m_{d_2,d_1} := p_* \circ q^!"'] & G_0^{(\mathbb{C}^*)^2 \times GL(d_1+d_2)}(C_{d_1+d_2})
\end{tikzcd}$$

endows

$$\left(\bigoplus_{d\geq 0} G_0^{(\mathbb{C}^*)^2\times GL(d)}(C_d),\ m := m_{d_2,d_1}\right)$$

of the structure of a unital associative algebra.

At this point, two questions arise naturally:

- What is the relation with the algebra discussed before?
- Why is this some kind of Hall "algebra"?

To answer the first question note that:

- the $T=(\mathbb{C}^*)^2$-fixed locus $\mathcal{C}_d^T$ of $\mathcal{C}_d$ is $\{(0,0)\}$.
- $G_0^{(\mathbb{C}^*)^2\times GL(d)}(\{(0,0)\}) \simeq \mathbb{C}[t_1^\pm, t_2^\pm][z_1^\pm,\dots,z_d^\pm]^{\mathfrak{S}_d}$
- The inclusion $\{(0,0)\} \xrightarrow{i} \mathcal{C}_d$ induces a map

$$G_0^{(\mathbb{C}^*)^2\times GL(d)}(\mathcal{C}_d) \xrightarrow{i^*} G_0^{(\mathbb{C}^*)^2\times GL(d)}(\mathcal{C}_d)$$
$$\longrightarrow G_0^{(\mathbb{C}^*)^2\times GL(d)}(\{(0,0)\}) \otimes_{\mathbb{C}[t_1^\pm,t_2^\pm]} \mathbb{C}(t_1,t_2)$$
$$\simeq \mathbb{C}(t_1,t_2)[z_1^\pm,\dots,z_d^\pm]^{\mathfrak{S}_d}$$

Theorem (Schiffmann-Vasserot)

The above map induces an injective homomorphism of unital associative algebras

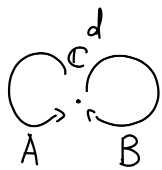
$$\boxed{\left(\bigoplus_{d\geq 0} G_0^{(\mathbb{C}^*)^2\times GL(d)}(\mathcal{C}_d), m\right) \hookrightarrow Sh_{\text{elliptic}}}$$

for which the $\boxed{LHS \simeq H_{\text{elliptic}}}$ w.r.t. $\sigma_1 \mapsto t_1^{-1}$, $\bar{\sigma}_1 \mapsto t_2^{-1}$.

Attention: We have 2 geometric realization of $U^+_{\sigma^{-1},q}(\hat{\mathfrak{gl}}(1))$

They should be related by the Geometric Langlands corr.

Let's move to the second question. Note that

$(A,B)\in \mathfrak{gl}(d;\mathbb{C})$ s.t. $[A,B]=0 \longleftrightarrow$ 

$+ [A,B]=0$

$\longleftrightarrow$ a representation (A,B) of the double-loop quiver

 that satisfies $[A,B]=0$

$\longleftrightarrow$ a right module of $\boxed{\mathbb{C}\langle e, e^* \rangle / [e, e^*]}$

$\|$

Preprojective Algebra $\Pi_{1\text{-loop}}$ of the 1-loop quiver G.

This equivalent description yields:

$$\boxed{\text{quotient stack } \mathcal{C}_d / GL(d) \simeq \text{stack } \underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}; d) \text{ of finite-dim. repr.s of } \Pi_{1\text{-loop}}}$$

Moreover

- $T = (\mathbb{C}^*)^2 \curvearrowright \mathcal{C}_d \longleftrightarrow T \curvearrowright \underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}; d)$
- Set $\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}) := \bigsqcup_{d \geq 0} \underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}; d)$. Then

$$\boxed{\bigoplus_{d \geq 0} G_0^{T \times GL(d)}(\mathcal{C}_d) \simeq G_0^T(\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}))}$$

Finally, the convolution diagram seen before:

$$\begin{tikzcd}[column sep=small]
& C_{d_1+d_2} \cap \left(\text{parabolic of } gl(d_1+d_2;\mathbb{C}) \text{ corresponding to fixing } \mathbb{C}^{d_1} \subset \mathbb{C}^{d_1+d_2}\right) \arrow[dl, "q"'] \arrow[dr, "p"] & \\
C_{d_2} \times C_{d_1} & & C_{d_1+d_2}
\end{tikzcd}$$

$$\left(A = \left(\begin{array}{c|c} A_1 & * \\ \hline 0 & A_2 \end{array}\right), B = \left(\begin{array}{c|c} B_1 & * \\ \hline 0 & B_2 \end{array}\right)\right) \in C_{d_1+d_2} \cap (\ldots)$$

$$C_{d_2} \times C_{d_1} \ni \left((A_2, B_2), (A_1, B_1)\right) \qquad\qquad (A,B) \in C_{d_1+d_2}$$

becomes

$$\underline{Rep}(\Pi_{1\text{-loop}}) \times \underline{Rep}(\Pi_{1\text{-loop}}) \xleftarrow{\ q\ } \underline{Rep}^{ext}(\Pi_{1\text{-loop}}) \xrightarrow{\ p\ } \underline{Rep}(\Pi_{1\text{-loop}})$$

$$q: \ 0 \longrightarrow E_2 \longrightarrow E \longrightarrow E_1 \longrightarrow 0 \ \longmapsto (E_2, E_1)$$

$$p: \ 0 \longrightarrow E_2 \longrightarrow E \longrightarrow E_1 \longrightarrow 0 \ \longmapsto E$$

$\Longrightarrow$ $\boxed{\left(G_0\left(\underline{Rep}(\Pi_{1\text{-loop}})\right), m = p_* \circ q^!\right) = \text{2d K-theoretical Hall algebra of the 1-loop quiver}}$

We are now ready to provide a unified framework for all the constructions we have seen so far.

Set

- $\mathbb{K}$ = field
- $\mathcal{A}$ = $\mathbb{K}$-linear abelian category
- $\mathcal{M}_{\mathcal{A}}$ = moduli stack of objects of $\mathcal{A}$

Examples:

- $\mathcal{A} = \mathrm{Rep}(Q)$ = category of (f.-d.) repr.s of Q – 1d category

  $\rotatebox{90}{$\subseteq$}$

  $\mathcal{A}^{nil} = \mathrm{nilp}(Q)$ = —————— " —— nilp. ———— " ————

- $\mathcal{A} = \mathrm{Rep}(\Pi_Q)$ = category of (f.-d.) repr.s of the preproj. alg. Π_Q – 2d category

  $\rotatebox{90}{$\subseteq$}$

  $\mathcal{A}^{nilp} = \mathrm{nilp}(\Pi_Q)$ = —————— " —— nilp. ———— " ————

We always have the convolution diagram:

$$\boxed{\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} \xleftarrow{\ q\ } \mathcal{M}_{\mathcal{A}}^{ext} \xrightarrow{\ p\ } \mathcal{M}_{\mathcal{A}}}$$

$$q: 0 \to E_2 \to E \to E_1 \to 0 \longmapsto (E_2, E_1)$$
$$p: 0 \to E_2 \to E \to E_1 \to 0 \longmapsto E$$

Therefore

1. $\left(\mathrm{Fun}_{cs}\left(\mathcal{M}_{\mathcal{A}}(\mathbb{F}_q)/_{\mathrm{iso}}, \mathbb{C}\right), p_* \circ q^!\right)$ is a unital associative algebra

compactly supported

= the Hall algebra of $\mathcal{A}$

- $\mathcal{A}$ of dimension one:

$$\left(\mathrm{Fun}_{cs}\left(\mathcal{M}_{\mathcal{A}}(\mathbb{F}_q)/_{\mathrm{iso}}, \mathbb{C}\right), p_* \circ q^!, q_* \circ p^!\right) = (\text{"twisted"}) \text{ bialgebra}$$

Ex:

For $\mathcal{A} = \mathrm{nilp}(Q)$ $\left(\text{resp. } \mathrm{Coh}(X)\right)$: we recover $H_{\mathrm{Rep}(Q)}$ $\left(\text{resp. } H_{\mathrm{Coh}(X)}\right)$

2. $\mathcal{A}$ of dimension one:

$$\left(H^*(\mathcal{M}_{\mathcal{A}}), p_* \circ q^!, q_* \circ p^!\right) = (\text{"twisted"}) \text{ bialgebra}$$

singular cohomology

= 1d Kontsevich-Soibelman cohomological Hall algebra

3. $\mathcal{A}$ of dimension two, T = algebraic torus:

$$\left(H_*^T(\mathcal{M}_{\mathcal{A}}), p_* \circ q^!\right) = \mathrm{COHA}_Q^T = \text{2d cohomological Hall algebra of } Q$$

$$\left(G_0^T(\mathcal{M}_{\mathcal{A}}), p_* \circ q^!\right) = \mathrm{KHA}_Q^T = \text{2d K-theoretical Hall algebra of } Q$$

(Here, $H_*^T(-)$ = T-equivariant Borel-Moore homology)

Let me conclude by:

Theorem

Botta-Davison, Schiffmann-Vasserot: $\mathrm{COHA}_Q^T \simeq Y^{MO,+}(\mathfrak{g}_Q^{MO})$

where $\begin{cases} \mathfrak{g}_Q^{MO} = \text{Maulik-Okounkov } \mathbb{Z}\text{-graded Lie algebra of } Q \\ Y(\mathfrak{g}_Q^{MO}) = \text{filtered deformation of } U(\mathfrak{g}_Q^{MO}[z]) \end{cases}$

Varagnolo-Vasserot: $\mathrm{KHA}_Q^T \simeq U_\nu^+(\mathfrak{g}_Q^{KM}[z^{\pm 1}])$ for Q = finite or affine ADE

Conjecture: $\mathrm{KHA}_Q^T \simeq U_\nu^+(\mathfrak{g}_Q^{MO}[z^{\pm 1}])$