Geometry and Topology Seminar

2-dimensional Cohomological Hall algebras

1. Motivation

Let X be a smooth (quasi-) projective variety $/\mathbb{C}$.

The "simplest" moduli space of objects on X is:

$$\boxed{\begin{aligned} Sym^n(X) &:= \overbrace{X \times \dots \times X}^{n\text{-th times}} / \mathfrak{S}_n \quad - \text{symmetric group} \\ &= \text{Symmetric Product of } X \end{aligned}}$$

$$= \left\{ \underset{\substack{\| \\ \sum_{i=1}^{n} x_i}}{[x_1, \dots, x_n]} : x_i \in X, i = 1, \dots n \right\}$$

= moduli space of non-ordered n-tuples of points of X

(it is a (quasi-) projective variety)

Example

- $n = 1 \Rightarrow Sym^1(X) \simeq X$

- X = curve $\Rightarrow Sym^n(X)$ is smooth

From now on, X = surface S.

- $n=2 \Rightarrow Sym^2(S) = \{ x+y \mid x,y \in S \} \supset \Delta = \{ 2x \mid x \in S \}$ = diagonal

Attention ⚠ : $Sym^2(S)$ is singular along Δ

In general, $Sym^n(S)$ is singular along the locus of tuples that correspond to non-distinct points.

Definition

Let V be a complex variety.
A resolution of singularities of V is a proper morphism $\varphi : U \longrightarrow V$ from a smooth variety such that $U \setminus \varphi^{-1}(U_{sing}) \simeq V \setminus V_{sing}$.

Consider the resolution of singularities of $Sym^n(S)$:

$$\pi : Hilb^n(S) \longrightarrow Sym^n(S)$$

Example

- $n=1$: $Hilb^1(S) \simeq Sym^1(S) \simeq S$

▸ $n=2$: $\mathrm{Hilb}^2(S) \simeq \underset{\text{blow-up of } \mathrm{Sym}^2(S) \text{ along the diagonal}}{\underbrace{\mathrm{Blow}_\Delta(\mathrm{Sym}^2(S))}} \longrightarrow \mathrm{Sym}^2(S)$

Important ⚠:

1. $\mathrm{Hilb}^n(S)$ = smooth (quasi-) projective variety (of dimension $2n$)

2. $\mathrm{Hilb}^n(S)$ is also a moduli space:

$\mathrm{Hilb}^n(S)$ = Hilbert scheme of n-points on S

= moduli space parametrizing zero-dimensional subschemes $Z \subset S$ such that $\dim_{\mathbb{C}} H^0(Z, \mathcal{O}_Z) = n$

Example ($S = \mathbb{C}^2$)

$$\mathrm{Hilb}^n(\mathbb{C}^2) \simeq \left\{ Z \subset \mathbb{C}^2 : \dim_{\mathbb{C}} H^0(Z, \mathcal{O}_Z) = n \right\}$$

$$= \left\{ I \subset \mathbb{C}[x,y] \text{ ideal} : \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n \right\}$$

Example ($n=2$ revisited)

Let $x,y\in S$, $x\neq y$. Then

with the reduced closed structure

$$Z_{x,y} := \{x,y\} \in \mathrm{Hilb}^2(S) \quad \text{and} \quad \pi(Z) = x+y \in \mathrm{Sym}^2(S)$$

If y "collides" to x : $x+y \rightsquigarrow 2x \in \Delta$ - diagonal

$Z_{x,y} \rightsquigarrow Z_x$ is topologically $\{x\}$, but Z_x "encodes" the direction of collision

In an affine neighborhood of $x\in S$:

$$\boxed{\begin{array}{l} Z_x \longleftrightarrow \text{ideal } I \subset \mathbb{C}[x,y] \text{ such that } m^2 \subset I \subset m = (x,y) \\ \qquad\qquad\qquad\qquad \Downarrow \\ \qquad\qquad I/m^2 = 1\text{-dimensional subspace} \subset m/m^2 \simeq T_0^*\mathbb{C}^2 \end{array}}$$

For simplicity, assume S projective.

Goal: Characterize the cohomology $H^*(\mathrm{Hilb}^n(S)) = H^*(\mathrm{Hilb}^n(S);\mathbb{C})$

Problem ⚠: For $n \geq 3$, $\mathrm{Hilb}^n(S)$ has not an explicit description as the one seen for $n=1$, $n=2$

$$\Downarrow$$

Challenging to describe $H^*(\mathrm{Hilb}^n(S))$

Solution: Consider

$$\boxed{\mathrm{Hilb}(S) := \bigsqcup_{n \geq 0} \mathrm{Hilb}^n(S)}$$

$\Longrightarrow H^*(\mathrm{Hilb}(S)) \simeq \bigoplus_{n\geq 0} H^*(\mathrm{Hilb}^n(S))$ "easy" to describe

Göttsche: computation of the Betti numbers in a generating series:

$$\sum_{n\geq 0} \sum_{i=0}^{4n} \underbrace{\dim_{\mathbb{C}} H^i(\mathrm{Hilb}^n(S))}_{\substack{\shortparallel \\ b_i(\mathrm{Hilb}^n(S)) = \text{Betti number}}} t^{i-2n} q^n = \prod_{m=1}^{\infty} \prod_{j=0}^{4} \left(1-(-1)^j t^{j-2} q^m\right)^{-(-1)^j b_j(S)}$$

Important ⚠:

1. LHS of Göttsche's formula encodes the graded dimension of $H^*(\mathrm{Hilb}(S))$ ($H^*(\mathrm{Hilb}(S))$ bigraded w.r.t. the number n of points and the coh. degree)

2. RHS = character (Poincaré series) of the Fock space representation $\mathbb{V}$ of an infinite-dimensional Heisenberg algebra $\mathcal{H}eis_S$ depending on $H^*(S) := H^*(S, \mathbb{C})$:

$$\boxed{\mathcal{H}eis_S := H^*(S)[t, t^{-1}] \oplus \mathbb{C}c}$$

super skew-symmetric: $[x,y] = (-1)^{\deg(x)\cdot\deg(y)}[y,x]$

with Lie bracket: $\boxed{[\alpha_n, \beta_m] = n\,\delta_{n+m,0}\langle\alpha,\beta\rangle c \;,\; [\alpha_n, c] = 0}$

with $\alpha_n := \alpha t^n$, $\alpha \in H^*(S)$, and $\langle\cdot,\cdot\rangle$ = cup product pairing = $\int_X \alpha \cup \beta$

Remark

Göttsche's formula

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equality between dimensions of the two bigraded vector spaces $H^*(\mathrm{Hilb}(S))$ and $\mathbb{V}$

Theorem (Nakajima, Grojnowski)

$\exists$ an action of $\mathcal{H}eis_S$ on $H^*(\mathrm{Hilb}(S))$ such that

$$\boxed{H^*(\mathrm{Hilb}(S)) \simeq \mathbb{V} \quad \text{as representations of } \mathcal{H}eis_S}$$

Corollary

Elements of the form ${}_1\alpha_{-n_1} \cdots {}_s\alpha_{-n_s} \cdot 1$ where ${}_1\alpha, \dots, {}_s\alpha \in H^*(S)$, $n_1, \dots, n_s \in \mathbb{Z}_{>0}$, and $1 \in H^*(\mathrm{Hilb}^0(S)) \simeq \mathbb{C}$, generate the whole $H^*(\mathrm{Hilb}(S))$.

Idea of the proof: geometric definition of the operators α_d, $d \in \mathbb{Z}$, $\alpha \in H^*(S)$

- $\alpha_0 = 0 \quad \forall \alpha \in H^*(S)$
- $d > 0$. Consider

Hecke correspondence

$$\mathrm{Hecke}_S(n, n+d) = \{ (Z, Z', x) \mid Z \subset Z', \ \pi(Z') = \pi(Z) + d x \}$$

closed subvariety of dimension $2n+d+1$

$$\begin{tikzcd}
 & \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^{n+d}(S) \times S \arrow[dl, "pr_n"'] \arrow[dr, "pr_{n+d}"] \arrow[rr, "supp"] & & S \\
\mathrm{Hilb}^n(S) & & \mathrm{Hilb}^{n+d}(S) &
\end{tikzcd}$$

Set

$$\alpha_{-d} : H^*(\mathrm{Hilb}^n(S)) \longrightarrow H^*(\mathrm{Hilb}^{n+d}(S))$$

$$\boxed{\alpha_{-d}(-) := PD^{-1}\left((pr_{n+d})_* \left((pr_n)^*(-) \cup supp^*(\alpha) \right) \cap [\mathrm{Hecke}_S(n, n+d)] \right)}$$

One defines similarly α_d. $\square$

Natural questions:

1. Is it possible to generalize this result to other "theories"?

For example:

$K_0(-)$ = Grothendieck group of coherent sheaves

$D^b(Coh(-))$ = bounded derived category of coherent sheaves

2. Is it possible to generalize this result to other moduli spaces?

For example:

$\mathrm{Hilb}^n(S) \rightsquigarrow M^{st}(S; r, c_1, ch_2)$ = moduli space of (Gieseker-)stable sheaves on S of rk $= r$, first Chern class c_1 and second Chern character ch_2

$\left(M^{st}(S; 1, 0, n) \simeq \mathrm{Hilb}^n(S)\right)$

First Answer: Readapt Nakajima-Grojnowski's construction

Baranovsky: $\exists$ an action of $\mathcal{H}eis_S$ on

$$\bigoplus_n H^*(M_S^{st}(r, c_1, n))$$

given by operators $\alpha_{\pm d}$ depending on $[\mathrm{Hecke}_S(r, c_1, \underbrace{n, n+d}_{\text{only } ch_2 \text{ varies!}})]$

Problem ⚠: $\underbrace{\bigoplus_n H^*(M_S^{st}(r, c_1, n))}_{(*)} \not\simeq \mathbb{V}$-Fock representation of $\mathcal{H}eis_S$

This means that $\mathcal{H}eis_S$ is "too" small to be used to span the whole (*)

Second Answer: Use the whole "topology" of $\mathrm{Hecke}_S(n, n+d)$

1. Replace operators $\alpha_{\pm d}$ depending on $[\mathrm{Hecke}_S(n, n+d)]$ by operators depending by:

$$\gamma \in H_*(\mathrm{Hecke}_S(n, n+d)), \text{ or } \gamma \in K_0(\mathrm{Hecke}_S(n, n+d)), \text{ or } \gamma \in D^b(\mathrm{Coh}(\mathrm{Hecke}_S(n, n+d)))$$

2. $\mathrm{Hecke}_S(n,n+d) \rightsquigarrow \mathrm{Hecke}_S(r,r',c_1,c_1',ch_2,ch_2')$

Problem ⚠: $\mathrm{Hecke}_S(n,n+l)$ and $\mathrm{Hecke}_S(r,r',c_1,c_1',ch_2,ch_2')$ have an "ugly" geometry: for example:

$$\mathrm{Hecke}_S(n,n+d) \text{ is smooth } \Leftrightarrow d=0,\pm 1$$

Third Answer: We need to use the theory of cohomological Hall algebras

Let us explain it when $S=\mathbb{C}^2$ and using again $\mathrm{Hilb}^n(S)$:

$$\mathrm{Hilb}^n(\mathbb{C}^2) \simeq \left\{ I \subset \mathbb{C}[x,y] : \dim_{\mathbb{C}} \mathbb{C}[x,y]/_I = n \right\}$$

(quiver description) $= \left\{ (A,B,v) : \begin{array}{l} A,B \in \mathrm{Mat}(n,\mathbb{C}),\ [A,B]=0, \\ v \in \mathbb{C}^n \text{ s.t. } \mathbb{C}^n \text{ is generated by } A^k B^l v \end{array} \right\}$

First Step: we "replace" $\mathrm{Hecke}(n,n+l)$ by a more "manageable" space:

$$\mathrm{Hecke}_{\mathbb{C}^2}(n,n+d) = \{ (Z,Z',x) : Z \subset Z',\ \pi(Z') = \pi(Z) + d x \}$$

$$\simeq \left\{ (I_Z, I_{Z'}, x) : I_{Z'} \subset I_Z,\ \mathrm{supp}\left(I_Z/_{I_{Z'}}\right) = \{x\} \right\}$$

"Space parametrizing quotients $I_Z/I_{Z'}$"

$\Big\|$ — when $S=\mathbb{C}^2$

$$\boxed{\text{quotient stack } \Big[\underbrace{\{(A,B)\in \mathrm{Mat}(d,\mathbb{C}) : [A,B]=0\}}_{\text{commuting variety } \mathcal{C}_d}\Big/GL(d)\Big]}$$

Theorem

I will come back later

Schiffmann-Vasserot: $\exists$ an associative algebra structure $\overbrace{(\text{à la Hall})}$ on

$$\boxed{COHA_{\mathbb{C}^2} := \bigoplus_d H_*^{BM}\big([\mathcal{C}_d/GL(d)]\big) \simeq \bigoplus_d H_*^{GL(d)}(\mathcal{C}_d)}$$

Kapranov-Vasserot: $COHA_{\mathbb{C}^2} \simeq \underset{\cap}{\mathrm{Sym}\big(q^{-1}z\,\mathbb{C}[q,z]\big)}$

a positive part of the universal enveloping algebra $U\big(\underset{\mathbb{C}[q,q^{-1}]}{\underset{\parallel}{\hat{\mathfrak{gl}}(1)}}[z]\big)$

Schiffmann-Vasserot: $\exists$ an action of $COHA_{\mathbb{C}^2}$ on $H_*^{BM}(\mathrm{Hilb}(\mathbb{C}^2))$, which induces an action of

$$U(\hat{\mathfrak{gl}}(1)[z]) \curvearrowright H_*^{BM}(\mathrm{Hilb}(\mathbb{C}^2)) \simeq H^*(\mathrm{Hilb}(\mathbb{C}^2))$$

Remark

1. $\exists$ an equivariant version: $\exists\ \mathbb{C}^* \times \mathbb{C}^* \curvearrowright \mathbb{C}^2 \Longrightarrow \mathbb{C}^* \times \mathbb{C}^* \curvearrowright \mathrm{Hilb}^n(\mathbb{C}^2)$

$\exists\ \mathbb{C}^* \times \mathbb{C}^* \curvearrowright \mathcal{C}_d$

$$\Downarrow$$

$$\mathrm{COHA}_{\mathbb{C}^2} \rightsquigarrow \mathrm{COHA}_{\mathbb{C}^2}^{equiv}$$

$U(\hat{gl}(1)[z]) \rightsquigarrow$ Maulik-Okounkov affine Yangian $Y(\hat{gl}(1))$

2. $\exists$ a higher-rank version:

$\mathrm{Hilb}(\mathbb{C}^2) \rightsquigarrow \mathcal{M}_{\mathbb{C}^2}(r) =$ "moduli space of rank r torsion-free sheaves on $\mathbb{C}^2$"

(= moduli space of rank r framed sheaves on $\mathbb{P}^2_{\mathbb{C}}$)

Important ⚠: $\exists\ Y(\hat{gl}(1)) \curvearrowright H_*^{equiv.}(\mathcal{M}(r))$ induces $\mathcal{W}(gl(r)) \curvearrowright H_*^{equiv.}(\mathcal{M}(r))$

$\mathcal{W}(gl(r))$ = affine $\mathcal{W}$-algebra of $gl(r)$

3. $\exists\ K_0$ - version of these results

Now, we are ready to go more into the theory of cohomological Hall algebras

Heuristics about COHAs

- $\mathcal{A}$ = (nice) abelian category
- $\mathcal{M}_{\mathcal{A}}$ = moduli stack of objects of $\mathcal{A}$ (ex: $\bigsqcup_d [\mathcal{C}_d / GL(d)]$)
- $\mathcal{M}_{\mathcal{A}}^{ext}$ = moduli stack of extensions of objects of $\mathcal{A}$

We have a "convolution diagram" à la Hall:

$$\begin{tikzcd}
& \mathcal{M}_{\mathcal{A}}^{ext} \arrow[dl, "ev_3 \times ev_1 = p"'] \arrow[dr, "q = ev_2"] & \\
\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} & & \mathcal{M}_{\mathcal{A}}
\end{tikzcd}$$

where:

- $p: (0 \to \mathcal{E}_2 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_1 \longrightarrow 0) \longmapsto (\mathcal{E}_1, \mathcal{E}_2)$
- $q: (0 \to \mathcal{E}_2 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_1 \longrightarrow 0) \longmapsto \mathcal{E}$

Fix a homology theory with "nice" functorial properties: $A_*(-)$

Examples:

- $A_*(-) = H_*^{BM}(-)$ = Borel-Moore homology
- $A_*(-) = K_0(-)$ = Grothendieck group of coherent sheaves
- ($A_*(-)$ = oriented Borel-Moore homology theory)

We would like to define the Cohomological Hall algebra of $\mathcal{A}$:

$\mathrm{COHA}_{\mathcal{A}}$:= associative algebra $(A_*(\mathcal{M}_{\mathcal{A}}), m = \text{product})$:

$$m: A_*(\mathcal{M}_{\mathcal{A}}) \otimes A_*(\mathcal{M}_{\mathcal{A}}) \xrightarrow{\boxtimes} A_*(\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}}) \xrightarrow{q_* \circ p^*} A_*(\mathcal{M}_{\mathcal{A}})$$

⚠: this definition works only if $\mathrm{gl.dim.}(\mathcal{A}) \leq 2$, indeed:

- q is proper representable $\Rightarrow \exists\, q_*$ = proper pushforward,

but:

- If $\mathrm{gl.dim.}(\mathcal{A}) = 1 \Rightarrow p$ is smooth $\Rightarrow \exists\, p^*$ = pull back
- If $\mathrm{gl.dim.}(\mathcal{A}) = 2 \Rightarrow p^*$ has to be defined carefully

Remark: if $\mathrm{gl.dim.}(\mathcal{A}) = 3$, one has to use Kontsevich-Soibelman's theory of COHAs which I am not going to introduce today.

2-dim. COHAs of quivers

Note that $\mathrm{COHA}_{\mathbb{C}^2}$ = example of a COHA of quivers, indeed:

$$[\mathcal{C}_d / GL(d)] \simeq \underline{\mathrm{Rep}}(\overset{\text{preprojective algebra of 1-loop quiver}}{\Pi_{\substack{\text{1-loop}\\ \text{quiver}}}}, d)$$

For an arbitrary quiver Q:

- $\mathcal{A}$ = category of f.d. representations of Π_Q (representations = right modules)

- $\mathcal{M}_{\mathcal{A}} = \underline{\mathrm{Rep}}(\Pi_Q)$

Theorem (Schiffmann-Vasserot, Yang-Zhao)

$\exists$ associative algebra structure (à la Hall) on

$$\boxed{\mathrm{COHA}_Q^{(\mathbb{C}^*)} = A_*^{(\mathbb{C}^*)}(\underline{\mathrm{Rep}}(\Pi_Q))}$$

Conjecture $\mathrm{COHA}_Q^{\mathbb{C}^*} \simeq$ (positive part of) Maulik-Okounkov affine Yangian of Q

2-dimensional COHAs of surfaces

Let S be a smooth (quasi-)projective surface $/\mathbb{C}$

- $\mathcal{A}_{S,\leq d}$ = category of (properly supported) coherent sheaves on S of dimension $\leq d$.

Theorem ($\mathcal{A}_{S,\leq d}$ - Kapranov-Vasserot, Yu Zhao for $d=0$)

Set $\underline{\mathrm{Coh}}_{\leq d}(S) = \mathcal{M}_{\mathcal{A}_{S,\leq d}}$.

$\exists$ an associative algebra structure à la Hall on

$$\boxed{\mathrm{COHA}_{S,\leq d} := A_*(\underline{\mathrm{Coh}}_{\leq d}(S))}$$

Remark $\mathrm{COHA}_{S,\leq 0} \simeq \mathrm{Sym}(H_*^{BM}(S) \otimes V)$

$\hookrightarrow$ a certain 1-dim. vector space

2-dimensional COHAs of curves

Let X be a smooth projective curve$/\mathbb{C}$, $\omega_X \simeq \Omega^1_X$ sheaf of 1-forms on X

- $\mathcal{A}_X^{Dol}$ = category of Higgs sheaves on X:

$$(\mathcal{F}, \phi : \mathcal{F} \longrightarrow \mathcal{F} \otimes \omega_X) \quad - \quad \phi = \text{morphism of } \mathcal{O}_X\text{-modules}$$

- $\mathcal{A}_X^{dR}$ = category of vector bundles with flat connections on X:

$$(\mathcal{E}, \nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \omega_X) \quad - \quad \nabla = \text{morphism of } \mathbb{C}_X\text{-modules}$$
$$+ \text{ Leibniz rule} + \nabla^2 = 0$$

- $\mathcal{A}_X^{B}$ = category of f.d. representations of $\pi_1(X)$:

$$\rho : \pi_1(X) \longrightarrow GL(n) \quad \text{for some } n$$

Theorem (Dol: S.-Schiffmann, Minets for $rk=0$; dR, B: Porta-S.)
Set $\underline{Coh}^{Dol}(X) = \mathcal{M}_{\mathcal{A}_X^{Dol}}$, $\underline{Coh}^{dR}(X) = \mathcal{M}_{\mathcal{A}_X^{dR}}$, and $\underline{Coh}^{B}(X) = \mathcal{M}_{\mathcal{A}_X^{B}}$.

$\exists$ an associative algebra structure à la Hall on

$$\boxed{\begin{aligned} \mathrm{COHA}_X^{\mathrm{Dol},(\mathbb{C}^*)} &:= A_*^{(\mathbb{C}^*)}\left(\underline{\mathrm{Coh}}^{\mathrm{Dol}}(X)\right) \\ \mathrm{COHA}_X^{\mathrm{dR}} &:= A_*\left(\underline{\mathrm{Coh}}^{\mathrm{dR}}(X)\right) \\ \mathrm{COHA}_X^{\mathrm{B}} &:= A_*\left(\underline{\mathrm{Coh}}^{\mathrm{B}}(X)\right) \end{aligned}}$$

Important ⚠ : by using derived algebraic geometry we can also categorify these algebras

Theorem (Porta–S.)

$\exists$ a derived enhancement $\mathbb{R}\underline{\mathrm{Coh}}_{\leq d}(S)$ of $\underline{\mathrm{Coh}}_{\leq d}(S)$ for which $\exists$ a monoidal structure à la Hall on

$$\boxed{D^b\left(\mathrm{Coh}\left(\mathbb{R}\underline{\mathrm{Coh}}_{\leq d}(S)\right)\right) \text{ - categorified COHA}}$$

which after passing to K_0 induces the algebra structure on

$$K_0\left(\mathbb{R}\underline{\mathrm{Coh}}_{\leq d}(S)\right) \simeq K_0\left(\underline{\mathrm{Coh}}_{\leq d}(S)\right)$$

Similar statement holds for $\underline{\mathrm{Coh}}^{\mathrm{Dol}}(X)$, $\underline{\mathrm{Coh}}^{\mathrm{dR}}(X)$, and $\underline{\mathrm{Coh}}^{\mathrm{B}}(X)$

Attention ⚠ : $D^b(\mathrm{Coh}(\mathbb{R}\underline{\mathrm{Coh}}_{\leq d}(S))) \not\simeq D^b(\mathrm{Coh}(\underline{\mathrm{Coh}}_{\leq d}(S)))$

$\Rightarrow$ $\nexists$ available machinery to construct a monoidal structure on (*)

Remark

1. $\exists$ a version of the Riemann-Hilbert correspondence for these categorified COHAs : $\exists$ an equivalence

$$D^b(\mathrm{AnCoh}(\mathbb{R}\underline{\mathrm{AnCoh}}(X_{dR}))) \simeq D^b(\mathrm{AnCoh}(\mathbb{R}\underline{\mathrm{AnCoh}}(X_{B})))$$

compatible with the Hall monoidal structures.

2. $\exists$ a version of the non-abelian Hodge correspondence for these categorified COHAs

Question: What kind of algebras are these COHAs ?

1. $X = \mathbb{P}^1$: $\boxed{\mathrm{COHA}^{\mathrm{Dol}}_{\mathbb{P}^1} \simeq U^+(\hat{\mathfrak{sl}}(2)[z])}$ (work in progress with Diaconescu-Porta-S-Schiffmann-Vasserot)

2. Resolution of Kleinian singularities

- $G \subset SL(2,\mathbb{C})$ finite group $\longleftrightarrow$ Q = type ADE Dynkin diagram
- $\pi : Y \longrightarrow \mathbb{C}^2/G$ minimal resolution of singularities

$$\boxed{\mathrm{COHA}_{\leq 1}(Y) \simeq U^+(\hat{\mathfrak{g}}_Q[z])}$$

(work in progress with Diaconescu-Porta-S-Schiffmann-Vasserot)