Cohomological Hall algebras of Higgs sheaves on curves and positivity of Kac-Schiffmann polynomials

(work in progress with Hennecart, Porta, and Schiffmann)

1. Motivation: Kac polynomials for quivers

Fix a quiver $Q = (I = \{\text{vertices}\}, \Omega = \{\text{edges}\})$ and a field k.

The Kac polynomial of Q is an "enumerative invariant".

Goal: Explain a Lie-theoretic interpretation of it.

First, we need to recall:

- kQ = path algebra of Q
  = k-vector space generated by all paths of length $\ell \geq 0$, with multiplication given by concatenation
- A representation of Q is a left module M of kQ
- A repr. M is indecomposable if it is nonzero and cannot be written as a direct sum of proper subrep.
- A repr. M is absolutely indecomposable if $M \otimes_k \bar{k}$ is

Indecomposable over $\bar{K}$ = algebraic closure of K

- A repr. M is finite-dimensional if K-vector space $e_i M$ is finite-dimensional $\forall i \in I$.

  path of length zero at the vertex i

Remark

M = f.d. repr. $\rightsquigarrow$ dimension vector $(\dim_K e_i M)_{i \in I} \in \mathbb{N}I$

Theorem (Kac, 1982)

Fix a finite field $\mathbb{F}_q$ with q elements and $\underline{d} \in \mathbb{N}I$.
Then $\exists!$ polynomial $A_{Q,\underline{d}}(t) \in \mathbb{Z}[t]$ such that

$$A_{Q,\underline{d}}(q) = \#\left\{\begin{array}{l}\text{isom. classes of absolutely} \\ \text{indecomposable repr.s of } Q \\ \text{of dimension } \underline{d} \text{ over } \mathbb{F}_q\end{array}\right\}$$

Moreover, $A_{Q,\underline{d}}(t)$ does not depend on the orientation of Q.

Examples

- Q = finite ADE quiver ($\longleftrightarrow$ simple Lie algebra)

$$A_{Q,\underline{d}}(t) = 1 \iff \underline{d} \in \Delta^+$$

- Q = affine ADE quiver ($\longleftrightarrow$ affine Lie algebra)

$$\Longrightarrow \begin{cases} A_{Q,\underline{d}}(t) = 1 & \forall\, \underline{d} \in \Delta^+_{re} \\ A_{Q,\underline{d}}(t) = t + \#I - 1 & \forall\, \underline{d} \in \Delta^+_{im} \end{cases}$$

- Q = 1-loop quiver. $A_{Q,d}(t) = t \quad \forall\, d \in \mathbb{N}_{\geq 1}$.

- For arbitrary quivers, Hua (2000) derived an explicit formula for $A_{Q,\underline{d}}(t)$

Let's now introduce the Lie algebra associated with Q:

- Q without edge-loops $\rightsquigarrow$ $\mathfrak{g}_Q^{KM}$ = Kac-Moody Lie algebra = $\mathbb{Z}I$-graded Lie algebra

- Q arbitrary $\rightsquigarrow$ $\mathfrak{g}_Q^{MO}$ = Maulik-Okounkov Lie algebra

$= (\mathbb{Z}I \times \mathbb{Z})$-graded Lie algebra

Example: Q = finite ADE quiver

- $\mathfrak{g}_Q^{KM}$ = simple Lie algebra $\mathfrak{g}_{ADE}$

- $\mathfrak{g}_Q^{MO}$ = universal central extension of $\mathfrak{g}_{ADE} \supset \mathfrak{g}_{ADE} = \mathfrak{g}_Q^{KM}$

We are ready to state the Lie-theoretic interpretation of $A_{Q,\underline{d}}(t)$:

Conjecture (Kac, 1982); Theorem (Hausel, 2010)

Assume that Q is without edge-loops.
Then, we have

$$\boxed{A_{Q,\underline{d}}(0) = \dim \mathfrak{g}_{Q,\underline{d}}^{KM}}$$

($\mathfrak{g}_{Q,\underline{d}}^{KM}$ = $\underline{d}$-th graded piece of $\mathfrak{g}_Q^{KM}$)

Attention ⚠: The above result provides a Lie-theoretic interpretation only of the constant term of $A_{Q,\underline{d}}(t)$

Question: What about the other coefficients?

Conjecture (Okounkov, 2013)

Theorem (Botta-Davison, Schiffmann-Vasserot; 2023)

Let Q be an arbitrary quiver. Then, we have:

$$\boxed{A_{Q,\underline{d}}(t) = \sum_{k\in\mathbb{Z}} \Big(\dim \underbrace{\mathfrak{g}^{MO}_{Q,(\underline{d},k)}}_{(\underline{d},k)\text{-th graded piece}}\Big)\, t^{-k/2}}$$

Remark

This result was obtained using the theory of COHAs and BPS Lie algebras

Corollary

$A_{Q,\underline{d}}(t) \in \mathbb{N}[t]$.

Remark

Note that the above result was obtained also by Hausel-Letellier-Rodriguez-Villegas (2013)

Today: Explain a similar framework when Q is replaced by a smooth curve X.

2. Kac-Schiffmann polynomials for curves

Fix a smooth projective curve $X/\mathbb{C}$

Consider the Dolbeault moduli space:

$\mathcal{M}^{(s)s}_{Dol}(X;r,d)$ = moduli space of (semi)stable Higgs bundles on X of rank r and degree d

Schiffmann, 2016: explicit computation of the Betti numbers of $\mathcal{M}^{s}_{Dol}(X;r,d)$ for r and d coprime.

Mozgovoy-Schiffmann, 2020: generalization to the noncoprime case

Attention: In both cases, the formulas are given in terms of the Kac-Schiffmann polynomial

To define Kac-Schiffmann polynomial, we need to recall:

- The zeta function a smooth projective curve $X/\mathbb{F}_q$ is

$$\boxed{Z_X(t) := \exp\left(\sum_{n\geq 1} \#X(\mathbb{F}_{q^n})\frac{t^n}{n}\right) = \frac{\prod_{i=1}^{2g}(1-\sigma_i t)}{(1-t)(1-qt)}}$$

Here, $\sigma_i \in \mathbb{C}^* =$ Weil numbers of X. They satisfy $\sigma_{2i-1}\sigma_{2i} = q$

- Set

$$T_g := \left\{(\alpha_1,\dots,\alpha_{2g}) \in (\mathbb{C}^*)^{2g} : \alpha_{2i-1}\alpha_{2i} = \alpha_{2j-1}\alpha_{2j} \ \forall i,j\right\}$$

$$W_g := \mathfrak{S}_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g$$

$$\Longrightarrow \{\sigma_1,\dots,\sigma_{2g}\} \rightsquigarrow \sigma_X \in T_g/W_g$$

<u>Theorem</u> (Schiffmann, 2016)

Fix $g \in \mathbb{N}$, $(r,d) \in \mathbb{N}_{\geq 1} \times \mathbb{Z}$. Then $\exists!$

$$\boxed{A_{g,r,d} \in \mathbb{C}[T_g]^{W_g}}$$

such that

$$\boxed{A_{g,r,d}(\sigma_X) = \#\left\{\begin{array}{l}\text{absolutely indecomposable vector bundles of rank } r \text{ and degree } d \\ \text{on a genus } g \text{ smooth projective curve}/\mathbb{F}_q \text{ of Weil numbers } \sigma_1,\dots,\sigma_{2g}\end{array}\right\}}$$

Moreover, when r,d are coprime, we have

$$\boxed{\sum_{n\geq 1}(-1)^n \dim H_c^n\left(\mathcal{M}^s_{Dol}(X/\mathbb{C};r,d);\mathbb{Q}\right)t^n = t^{2(1+(g-1)r^2)}A_{g,r,d}(t,\dots,t)}$$

Theorem (Mellit, 2020)

$A_{g,r,d}$ is independent of d

Examples

- genus $= 0$:

$$A_{0,r,d} = \begin{cases} q+1 & r=0,\ \forall d \\ 1 & r=1,\ \forall d \\ 0 & r\geq 2,\ \forall d \end{cases}$$

- genus $= 1$: $A_{1,r,d}(\sigma_1,\sigma_2) = \sigma_1\sigma_2 + 1 - (\sigma_1+\sigma_2)$

- arbitrary genus: $\exists$ an explicit formula for $A_{g,r,d}$

Now, we formulate a Lie-theoretic interpretation of $A_{g,r,d}$
Note that

- T_g is the maximal torus of $GSp(2g,\mathbb{C})$
- W_g is the Weyl group of $GSp(2g,\mathbb{C})$

$\Longrightarrow \mathbb{C}[T_g]^{W_g}$ is the character ring of $GSp(2g,\mathbb{C})$

Conjecture (Schiffmann, 2018 ICM Talk)
For any $g \in \mathbb{N}_{\geq 2}$, $\exists$ a Lie algebra

$$\mathfrak{g} = \bigoplus_{(r,d)} \mathfrak{g}_{r,d}$$

such that $\mathfrak{g} \in GSp(2g,\mathbb{C})$-mod for which

$$ch(\mathfrak{g}_{r,d}) = A_{g,r,d}$$

Theorem (Hennecart-Porta-S.-Schiffmann)
Schiffmann's conjecture is true.

Attention ⚠: The proof is based on the theory of COHAs and BPS Lie algebras.

Let's recall their construction.
Consider

$$\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{Dol}}(X) := \text{derived moduli stack of Higgs sheaves on } X$$
$$\uparrow \text{open}$$
$$\mathbb{R}\underline{\mathrm{Coh}}^{(s)s,\nu}_{\mathrm{Dol}}(X) := \text{derived moduli stack of (semi)stable Higgs bundles on } X \text{ of fixed slope } \nu \in \mathbb{Q} \cup \{\infty\}$$

Theorem (S.-Schiffmann, 2020; Porta-S., 2023)
There exists a cohomological Hall algebra $\mathrm{COHA}_{\mathrm{Dol}}(X)$, whose underlying vector space is

$$\mathrm{COHA}_{\mathrm{Dol}}(X) := H^{BM}_*(\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{Dol}}(X))$$

and the multiplication is $p_* \circ q^!$, where

$$\mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{Dol}}(X) \times \mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{Dol}}(X) \xleftarrow{q} \mathbb{R}\underline{\mathrm{Coh}}^{\mathrm{ext}}_{\mathrm{Dol}}(X) \xrightarrow{p} \mathbb{R}\underline{\mathrm{Coh}}_{\mathrm{Dol}}(X)$$

stack of extensions

$$p : 0 \to E_1 \to E \to E_2 \to 0 \longmapsto E \quad \text{- proper}$$

$$q : \quad \text{———}//\text{———} \quad \longmapsto (E_2, E_1) \quad \text{-quasi-sm.}$$

Similarly, $\exists$ $\mathrm{COHA}^{ss,\nu}_{\mathrm{Dol}}(X)$ associated with $\mathbb{R}\underline{\mathrm{Coh}}^{ss,\nu}_{\mathrm{Dol}}(X)$

Using the theory of BPS Lie algebras for CY2 categories:

Theorem (Davison-Hennecart-Schegel-Mejía, 2023)

For any $\nu \in \mathbb{Q} \cup \{\infty\}$, $\exists$ a Lie algebra

$$\mathfrak{g}^{\nu}_X = \bigoplus_{\substack{(r,d) \\ d/r=\nu}} \mathfrak{g}^{\nu}_{r,d}$$

such that

$$\mathrm{COHA}^{ss,\nu}_{\mathrm{Dol}}(X) = U(\mathfrak{g}^{\nu}_X[u])$$

Moreover, for $g \geq 2$, we have

$$\mathfrak{g}^{\nu} \simeq \mathrm{FreeLie}\Big(\bigoplus_{d/r=\nu} IH^*(\mathcal{M}_{\mathrm{Dol}}(X; r, d))\Big)$$

Examples

- For $\nu=\infty$, $\mathfrak{g}^{\infty}_{0,d} \simeq H^*(X,\mathbb{Q})$ $\forall d$

- When genus $=0$, $\underline{\mathrm{Coh}}^{ss,0}_{\mathrm{Dol}}(\mathbb{P}^1; r,0) \simeq pt/GL(r,\mathbb{C})$

$$\Longrightarrow \mathrm{COHA}^{ss,0}_{\mathrm{Dol}}(\mathbb{P}^1) \simeq \bigoplus_{r\geq 1} \mathbb{Q}[t_1,\dots,t_r]^{\mathfrak{S}_r} + \text{shuffle product}$$

Attention (Diaconescu-Porta-S-Schiffmann-Vasserot):

$$\mathrm{COHA}_{\mathrm{Dol}}(\mathbb{P}^1) \simeq U(\mathfrak{n}_{ell})$$

where $\mathfrak{n}_{ell}$ = "non-standard" positive half of $\mathfrak{g}_{ell} := \text{u.c.e.}(\mathfrak{sl}(2)[z^{\pm 1}, u])$

- When genus $=1$, $\mathfrak{g}^{\nu}_{r,d} \simeq H^*(X;\mathbb{Q})$ $\forall r,d$

Attention: $\mathrm{COHA}_{\mathrm{Dol}}(X)$ should be $U(\widehat{\mathfrak{gl}}(1)^+)$

Attention ⚠:

Each $\mathfrak{g}^{\nu}_{r,d}$ is $\mathbb{Z}$-graded. coh. grading

For r,d coprime, we have

$$\boxed{\mathrm{gr.dim}\, \mathfrak{g}^{\nu}_{r,d} = q^{(g-1)r^2+1} A_{g,r,d}(q^{1/2}, \dots, q^{1/2})}$$

$\Longrightarrow$ $\mathfrak{g}^{\nu}_X$ could be the "right" candidate for solving Schiffmann's conjecture

The previous result has been refined:

Theorem (Davison-Kinjo-Hennecart-Schiffmann-Vasserot)

$\exists$ a Lie algebra

$$\mathfrak{g}_X[u] := \{x \in \mathrm{COHA}_{\mathrm{Dol}}(X) : \Delta(x) = x \otimes 1 + 1 \otimes x\} = \bigoplus_{(r,d)} \mathfrak{g}_{r,d}[u]$$

such that

- $U(\mathfrak{g}_X[u]) \subset \mathrm{COHA}_{\mathrm{Dol}}(X)$ is a subalgebra
- $\forall\, \nu \in \mathbb{Q} \cup \{\infty\}$, $\exists \bigoplus_{\substack{(r,d) \\ d/r = \nu}} \mathfrak{g}_{r,d}[u] \xrightarrow{\sim} \mathfrak{g}^{\nu}_X[u]$
- $\forall\, (r,d) \in \mathbb{N} \times \mathbb{Z}$, $\mathfrak{g}_{r,d}[u] \xrightarrow{\sim} \mathfrak{g}_{r,d+1}[u]$

The natural question at this point is:

Question: how do we endow the Lie algebra $\mathfrak{g}_X$ of the structure of a $GSp(2g, \mathbb{C})$-repr.?

Let $\mathcal{M}_g$ be the moduli stack of smooth projective curves$/_{\mathbb{C}}$. Then, recall:

$$0 \longrightarrow \text{Torelli subgroup} \longrightarrow \pi_1(\mathcal{M}_g) \longrightarrow Sp(2g, \mathbb{Z}) \longrightarrow 0$$

$\Longrightarrow$ As a first step, we could construct

$$\boxed{\pi_1(\mathcal{M}_g)\text{-repr.} \longleftrightarrow \text{local system over } \mathcal{M}_g}$$

Consider

$$\begin{array}{l} \mathcal{C}_g = \text{universal curve} \\ \downarrow \\ \mathcal{M}_g \end{array}$$

$\Longrightarrow$ $\underline{\mathbb{R}\mathrm{Coh}}_{\mathrm{Dol}/\mathcal{M}_g}(\mathcal{C}_g) \xrightarrow{\pi} \mathcal{M}_g$ and $\underline{\mathbb{R}\mathrm{Coh}}^{ss,\nu}_{\mathrm{Dol}/\mathcal{M}_g}(\mathcal{C}_g) \xrightarrow{\pi^\nu} \mathcal{M}_g$

$\Longrightarrow$ Define $\pi_* \pi^! \mathbb{Q}_{\mathcal{M}_g}$ and $\pi^\nu_* (\pi^\nu)^! \mathbb{Q}_{\mathcal{M}_g}$

Theorem (HPSS)

1. $\pi_* \pi^! \mathbb{Q}_{\mathcal{M}_g}$ and $\pi^\nu_* (\pi^\nu)^! \mathbb{Q}_{\mathcal{M}_g}$ are sheaves of associative algebras

2. $\forall\, X \overset{\iota_X}{\hookrightarrow} \mathcal{M}_g$, $\iota_X^* \pi_* \pi^! \mathbb{Q}_{\mathcal{M}_g} \simeq \mathrm{COHA}_{\mathrm{Dol}}(X)$

$$\iota_X^* \pi^\nu_* (\pi^\nu)^! \mathbb{Q}_{\mathcal{M}_g} \simeq \mathrm{COHA}^{ss,\nu}_{\mathrm{Dol}}(X)$$

3. $\exists$ a sheaf BPS_g of Lie algebras such that

$$\mathrm{Sym}\left(\mathrm{BPS}_g \otimes H^*_{\mathbb{C}^*}(pt)\right) \simeq \pi^\nu_* (\pi^\nu)^! \mathbb{Q}_{\mathcal{M}_g}$$

Attention:

(1) and (3) are obtained by generalizing to the "relative" setting what we saw before.

The proof of (2) is highly nontrivial since π and π^ν are not proper.

Theorem (HPSS)

1. $\pi^\nu_*(\pi^\nu)^! \mathbb{Q}_{\mathcal{M}_g}$ and BPS_g are local systems on $\mathcal{M}_g$

2. $\pi_* \pi^! \mathbb{Q}_{\mathcal{M}_g}$ is a local system on $\mathcal{M}_g$

Remark

Note that we can also define:

- de Rham and Betti COHAs - Porta-S.
- ——— " ——— BPS Lie algebras - Davison, Hennecart

$\Longrightarrow$ Davison, Hennecart proved that

$$\boxed{\begin{gathered}\text{Dolbeault COHA} \simeq \text{de Rham COHA} \simeq \text{Betti COHA} \\ \text{Dolbeault BPS L.} \simeq \text{de Rham BPS L.} \simeq \text{Betti BPS L.}\end{gathered}}$$

$\Longrightarrow$ This proves (1)

The proof of (2) follows from (1) using Harder-Narasimhan strata

Corollary $\forall$ smooth curve X, $\mathfrak{g}_X$ is a $\pi_1(\mathcal{M}_g)$-repr.

Theorem (HPSS)

1. The action of $\pi_1(\mathcal{M}_g)$ on $\mathfrak{g}_X$ descends to an action of $Sp(2g,\mathbb{Z})$.

2. The action of $Sp(2g,\mathbb{Z})$ on $\mathfrak{g}_X$ extends to an action of $GSp(2g,\mathbb{C})$

3. $ch_{GSp(2g,\mathbb{C})}(\mathfrak{g}_{r,d}) = q^{(g-1)r^2+1} A_{g,r,d}(q^{1/2},\dots,q^{1/2})$

$\Longrightarrow$ Schiffmann's conjecture is true.

Let me finish with some questions:

- Mellit computed the Betti numbers of the Dolbeault moduli spaces in the parabolic setting

  Q. What is the correct generalization of Schiffmann's conj.?

- Q: How do we extend this framework to stable curves?