

# Analyzing Asymptotics of Stochastic Processes through Optimal Transport

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<sup>1</sup>joint work with M. Mariani (arXiv:2307.10325)

# Plan

## 1 Introduction

- Occupation measure
- Optimal transport

## 2 Main results

## 3 Some ideas from the proof

## 4 Conclusion

# The empirical measure of a process

- Consider a (stochastic) process  $(X_t)_{t \geq 0}$  on  $E$  (metric Polish).
- Its occupation measure (at time  $T$ ) is the measure on  $E$

$$\mu_T^X = \int_0^T \delta_{X_s} ds$$

explicitly:

$$\mu_T^X(A) = \int_0^T I_{\{X_s \in A\}} ds \quad \forall A \in \mathcal{E}.$$

- Discrete time:  $(Y_n)_{n=0}^\infty \Rightarrow X_t = Y_{[t]}$ :

$$\mu_n^X = \sum_{i=0}^{n-1} \delta_{Y_i}.$$

- Applications:

• Ergodicity, mixing, etc.

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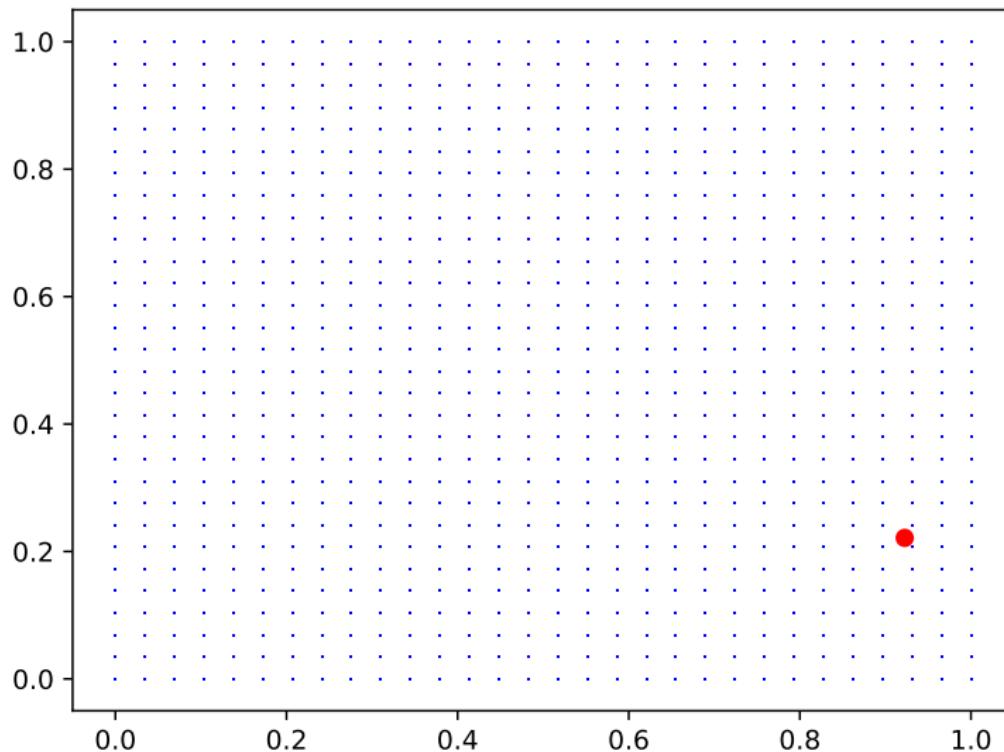
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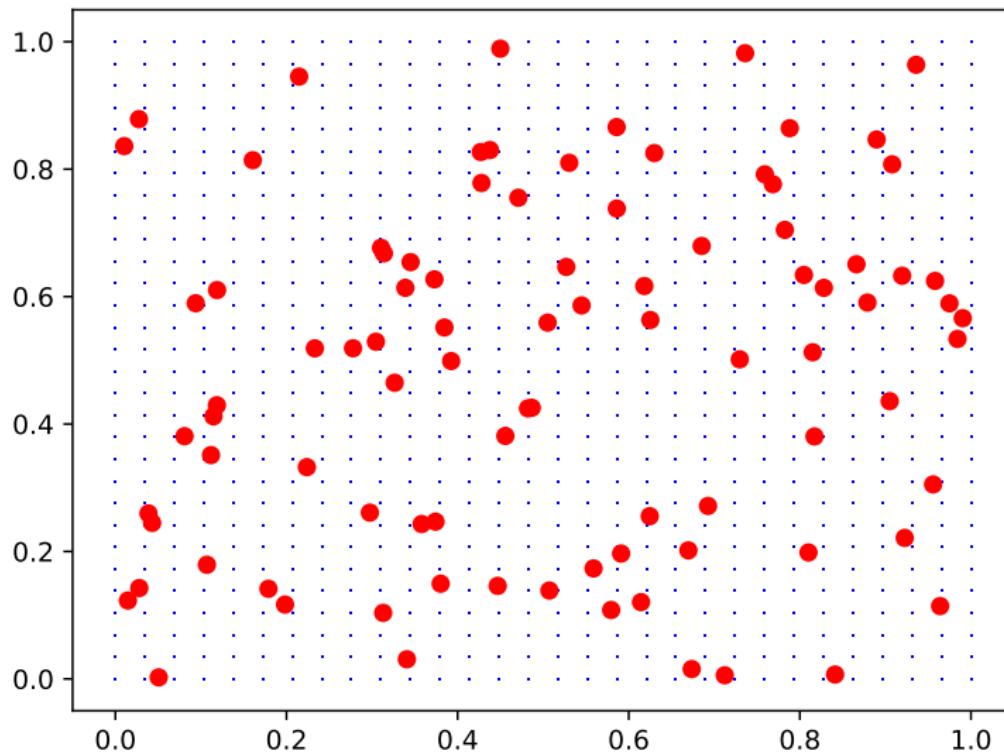
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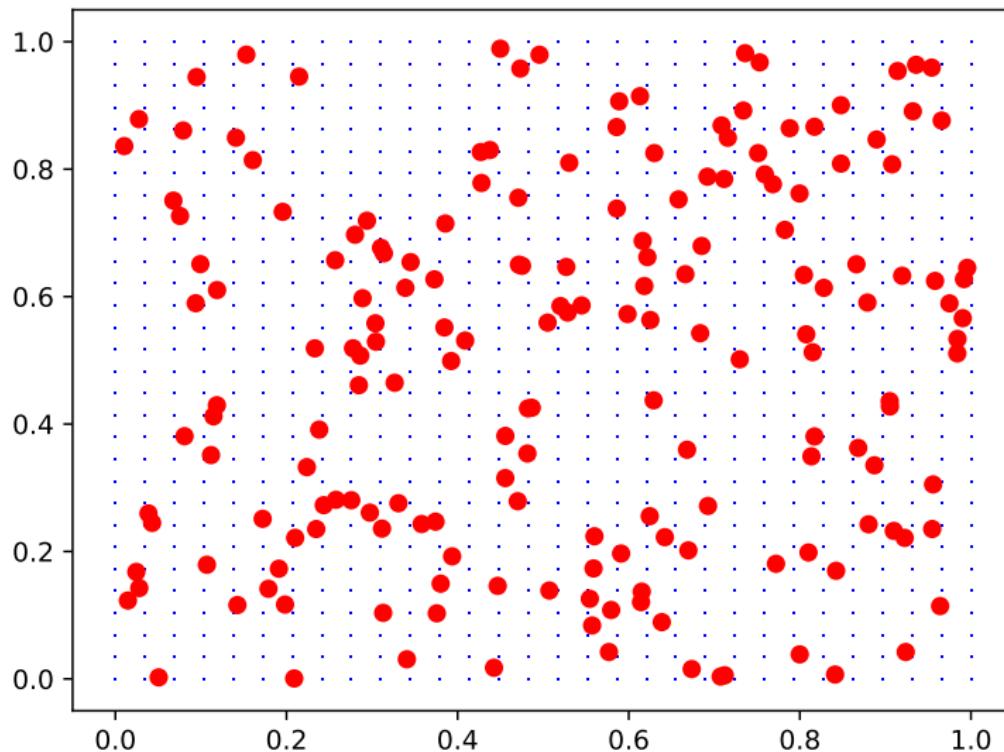
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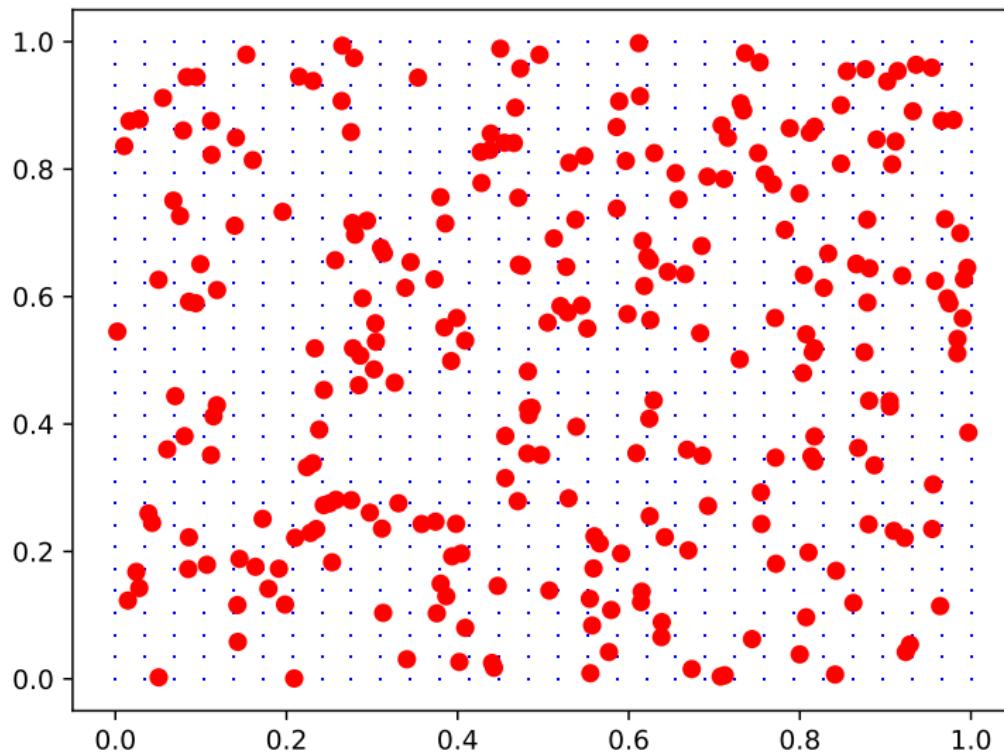
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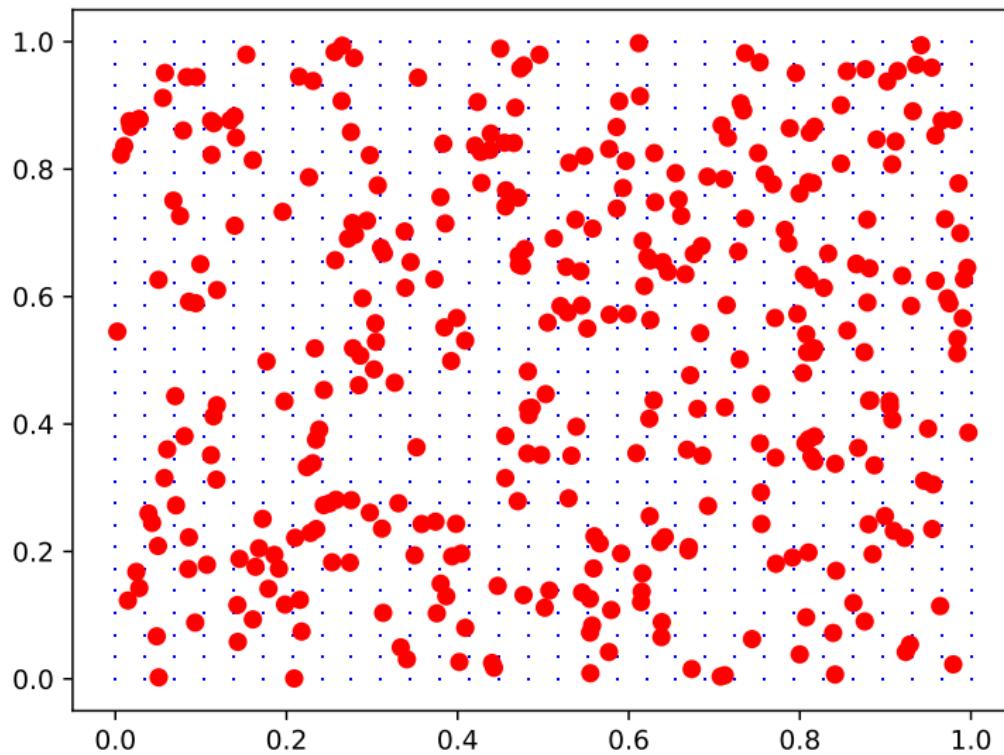
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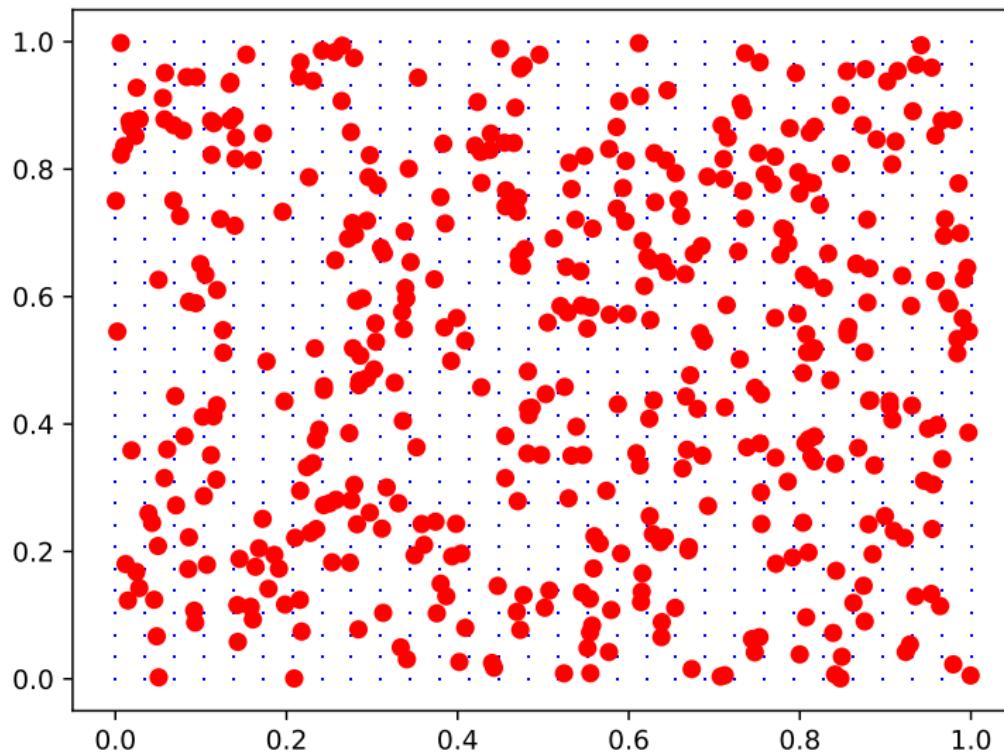
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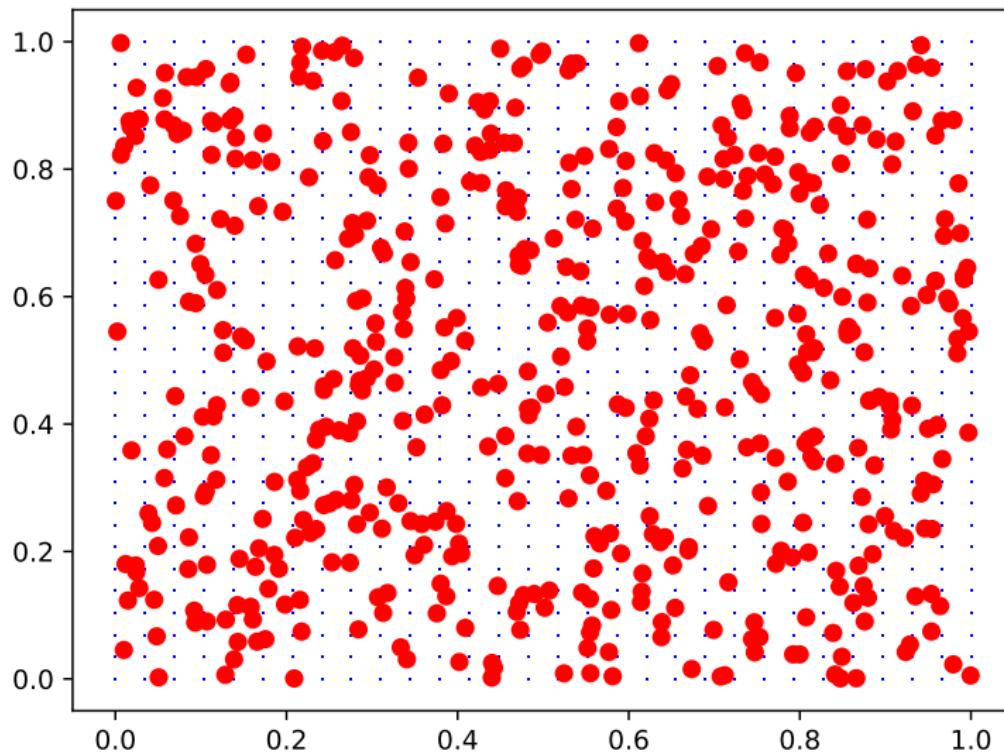
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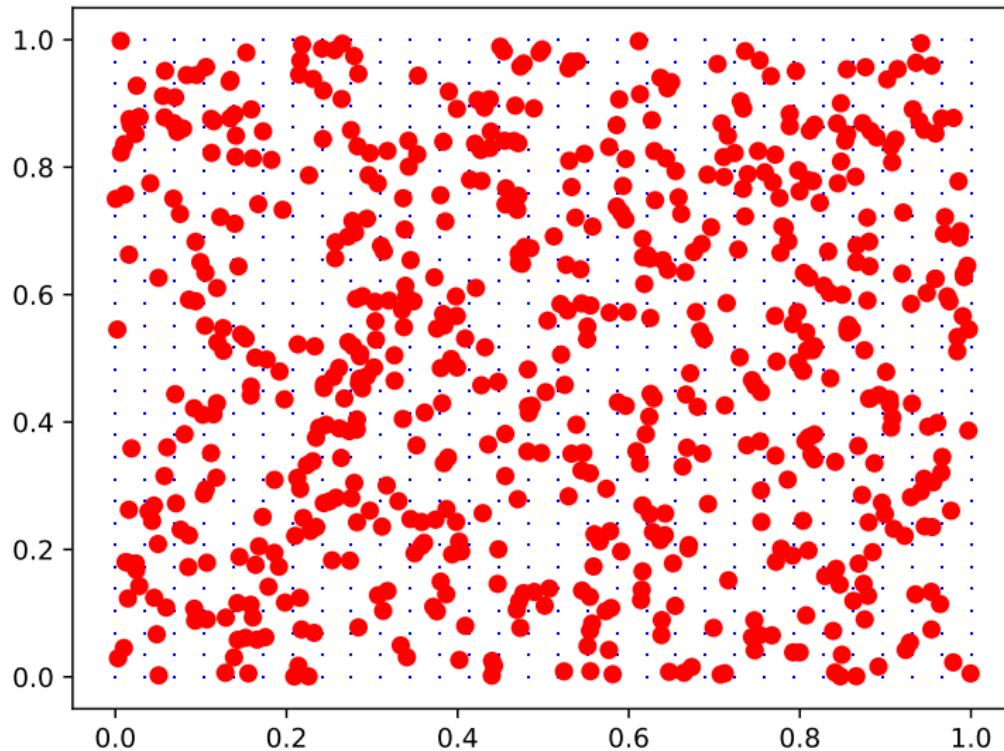
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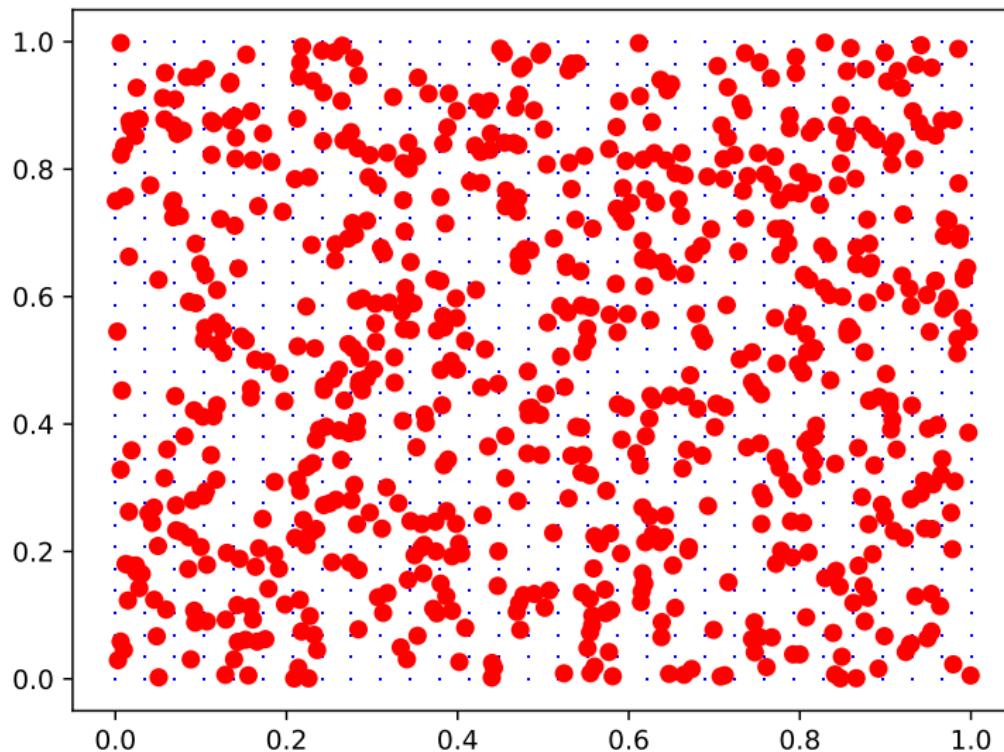
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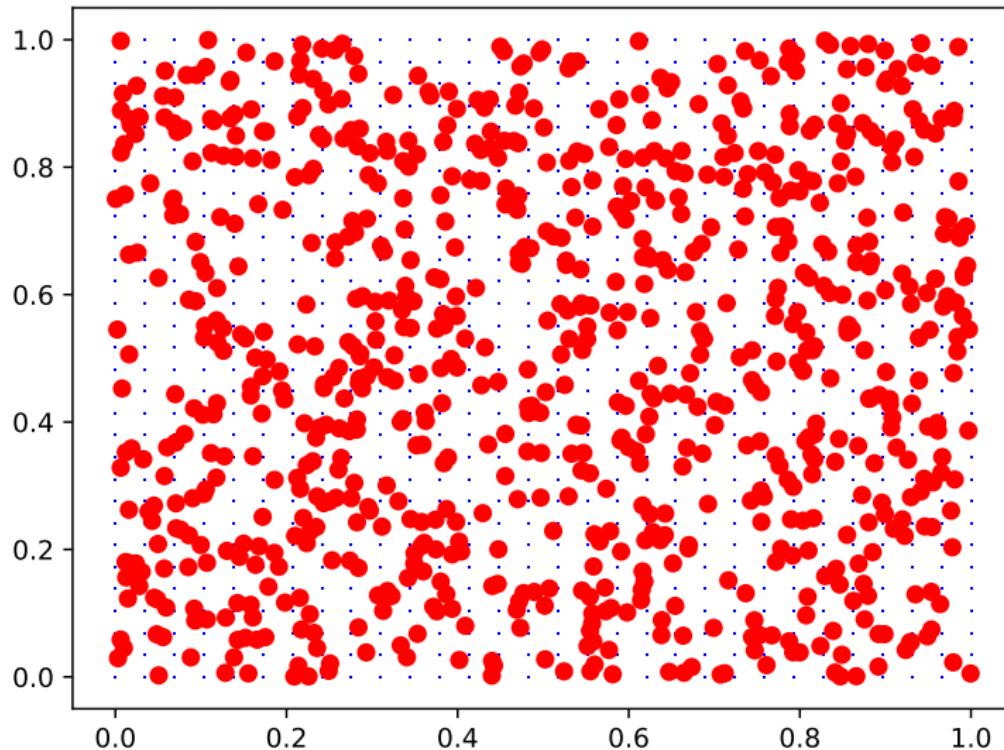
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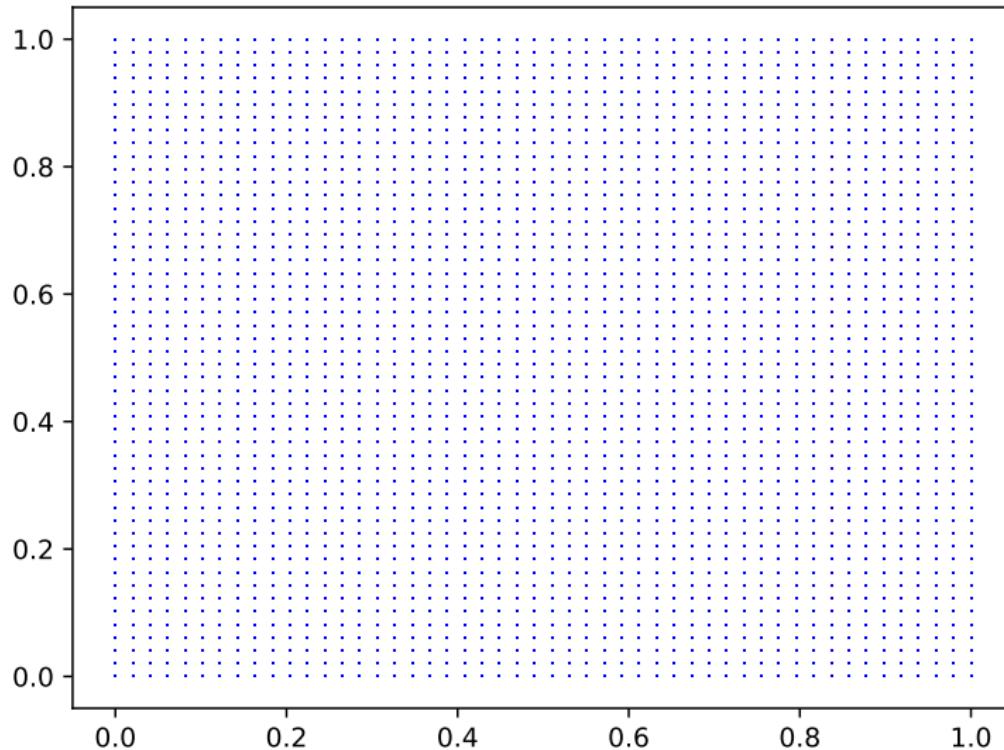
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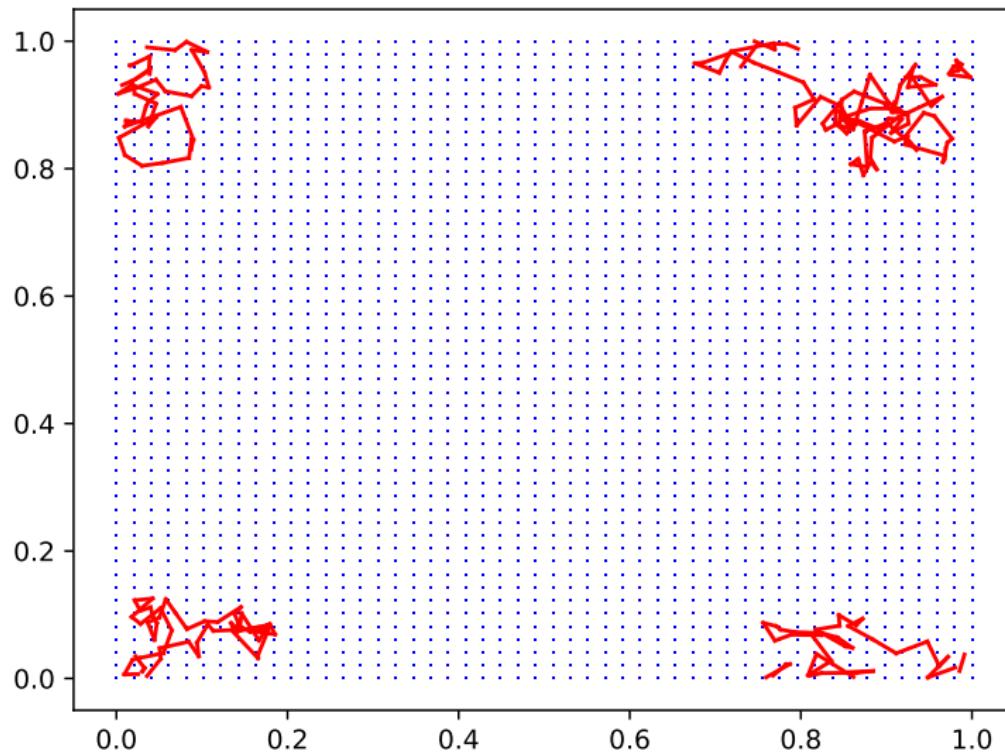
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$(X_t) = (B_t)_{t \geq 0}$  is Brownian motion on the torus  $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ .



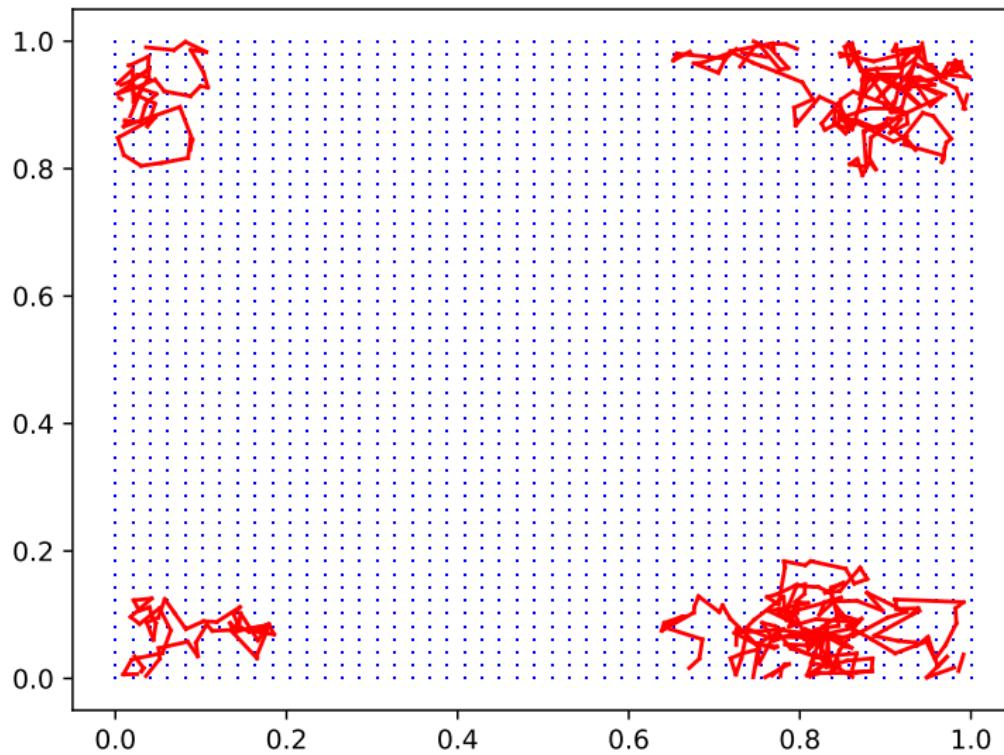
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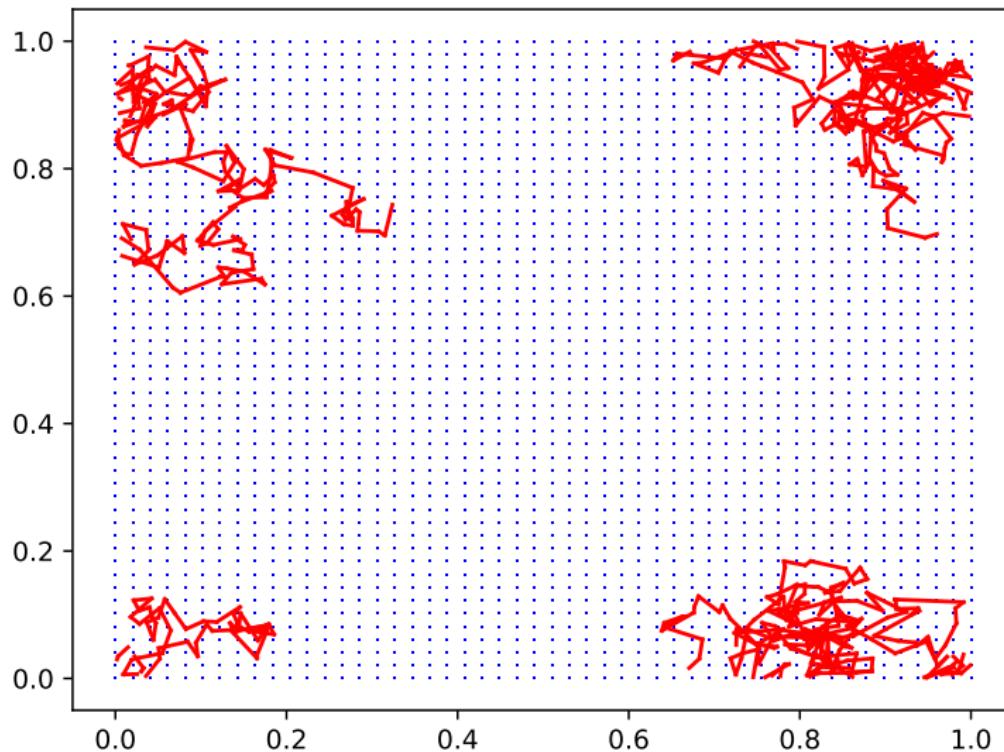
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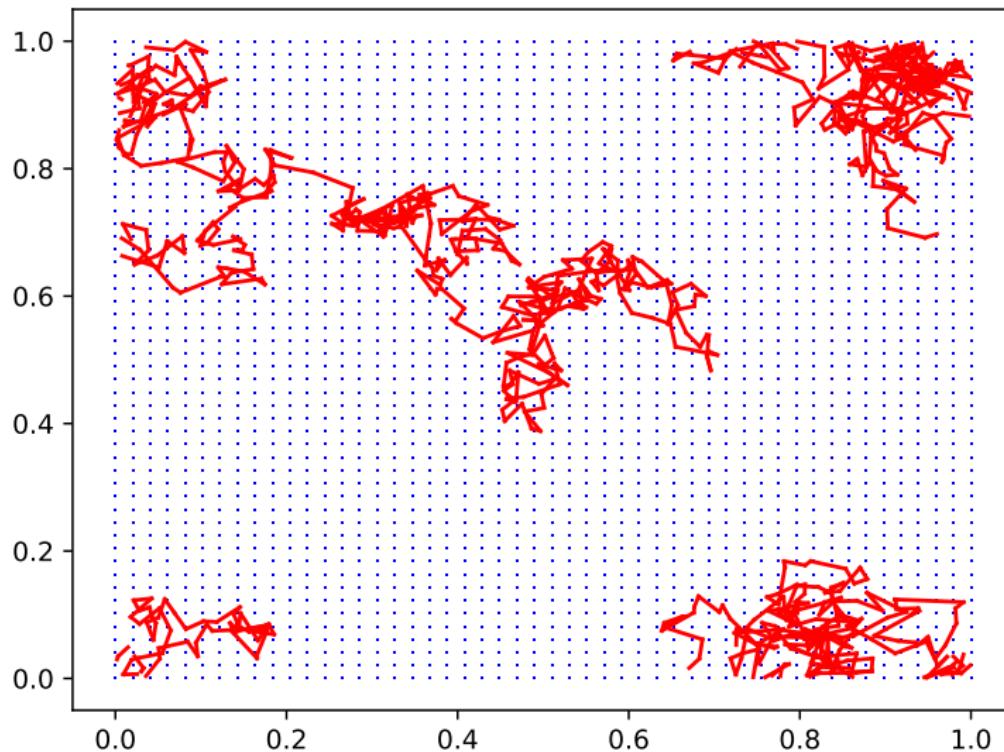
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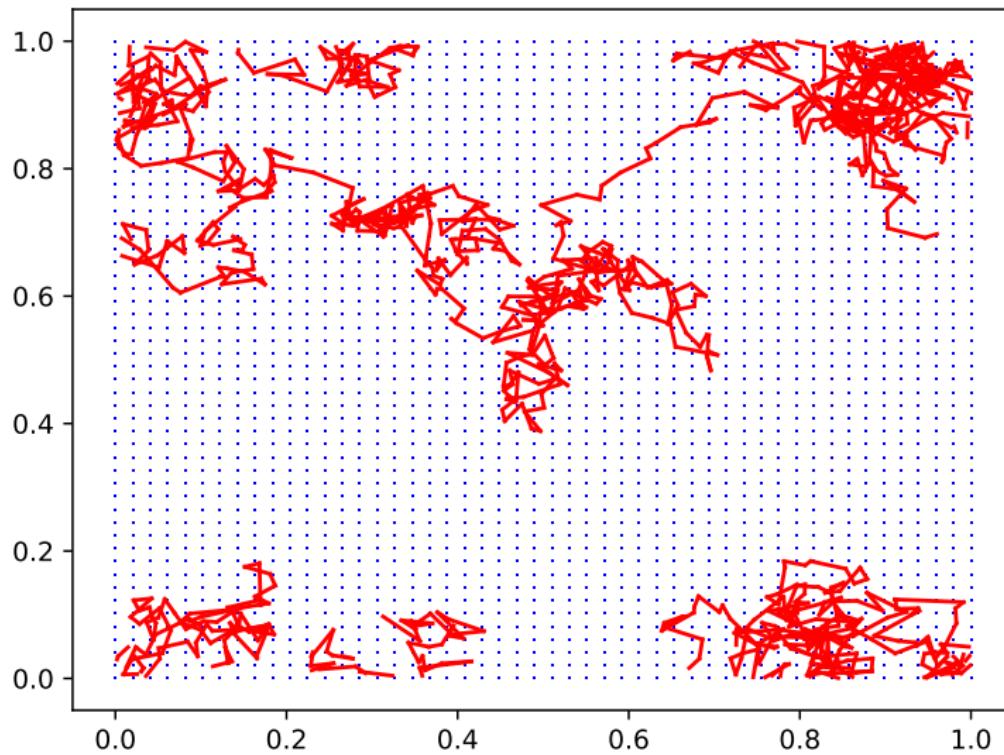
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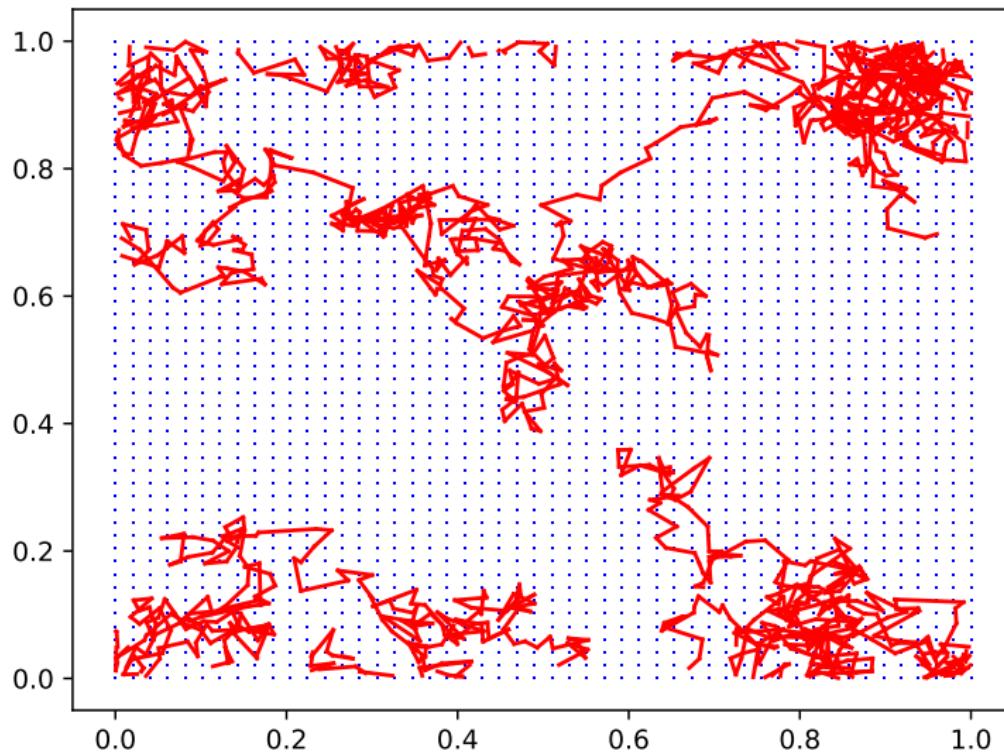
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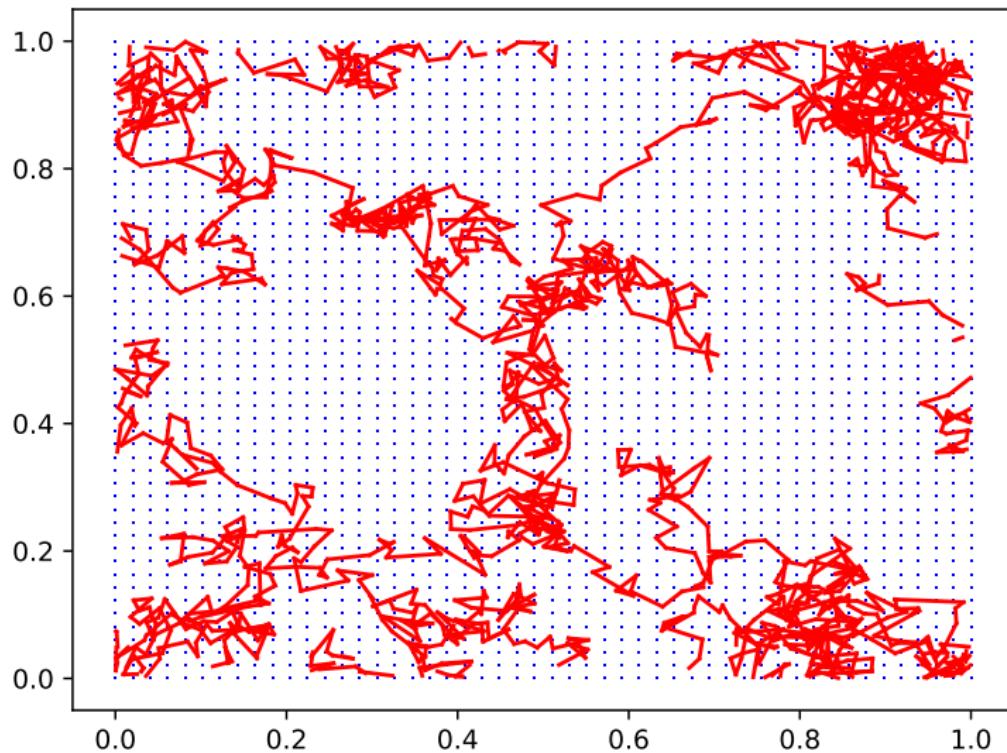
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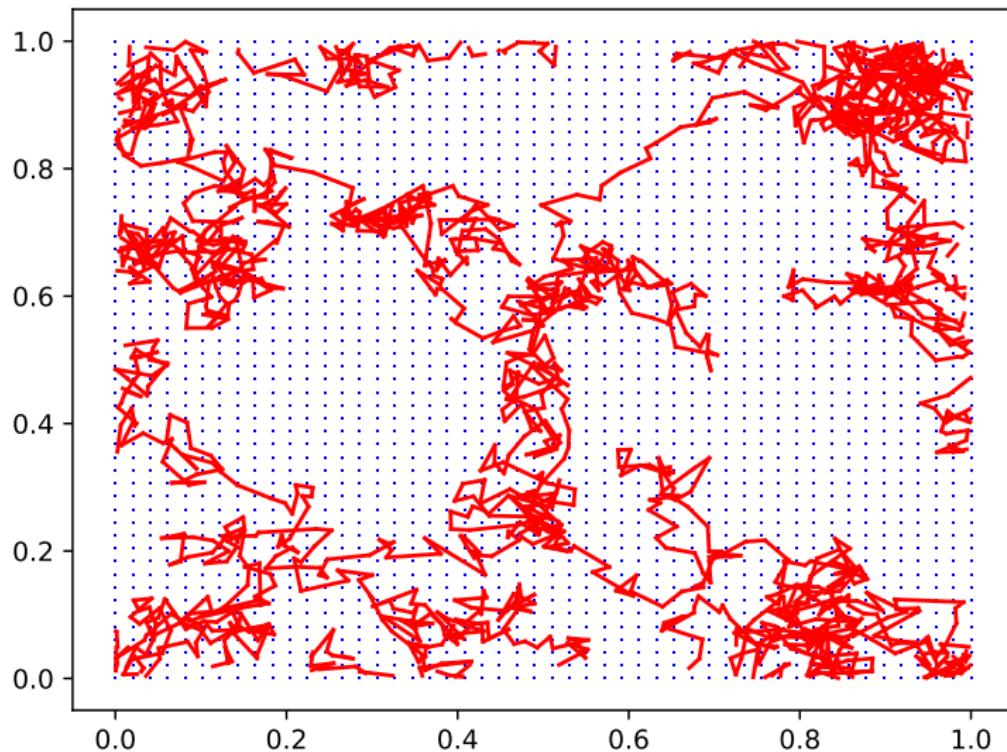
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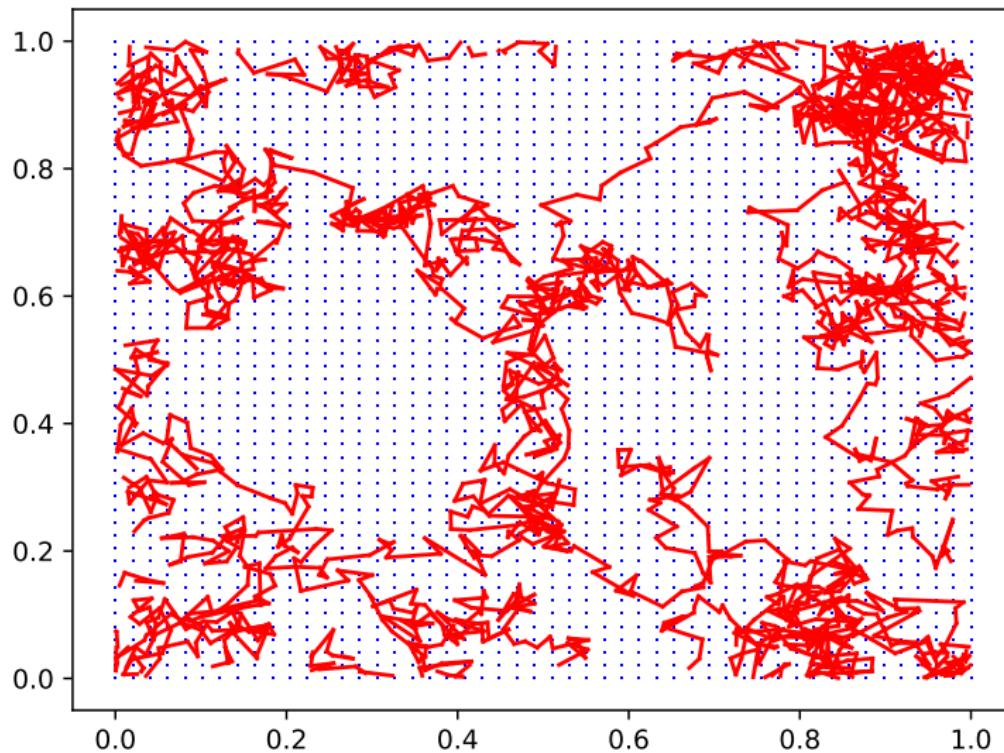
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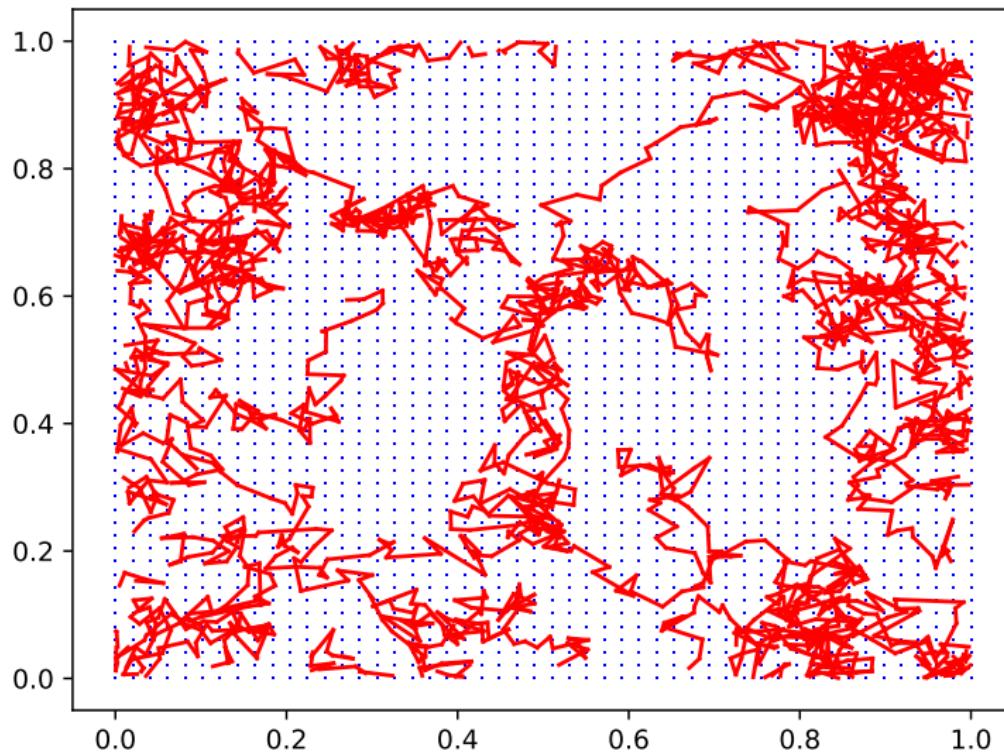
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# Convergence of Empirical Measure

- Assuming e.g. **stationarity** and **ergodicity** of  $X$  limit theorems are known:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mu_T^X = \mathfrak{m},$$

where  $\mathfrak{m}$  is (the) invariant measure of the process:

$$\forall f \in C_b(E) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(X_t) dt \rightarrow \int_E f d\mathfrak{m}.$$

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# Metric induced by a family of functions

Weak convergence  $\mu_n \rightarrow \nu$ :

$$\lim_{n \rightarrow \infty} \int_E f d\mu_n \rightarrow \int_E f d\nu \quad \forall f \in C_b(E).$$

Idea: fix a  $\mathcal{F} \subseteq C_b(E)$  and set

$$d_{\mathcal{F}}(\mu, \nu) = \sup_{f \in \mathcal{F}} \left| \int_E f d\mu - \int_E f d\nu \right|$$

•  $\mathcal{F} = \{ \|f\|_{\infty} \leq 1/2 \} \Rightarrow d_{\mathcal{F}}(\mu, \nu)$  total variation

•  $\mathcal{F} = \{ |f| \leq 1 \}$  bounded functions

•  $\mathcal{F} = \{ f \geq 0 \}$  non-negative functions

•  $\mathcal{F} = \{ f \text{ measurable} \}$  all measurable functions

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$\mathcal{F} = \{ \text{Lip}(f) \leq 1 \} \Rightarrow W^1(\mu, \nu)$  Wasserstein-Kantorovich

$\mathcal{F} = \{ f \geq 0, \int_E f d\mu = 1 \} \Rightarrow \text{KL}(\mu \| \nu)$  Kullback-Leibler

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# Optimal transport of mass

Idea: find the **cheapest** way to move the mass  $\mu$  into  $\nu$ .

- Fix a cost  $c(x, y) : E \times E \rightarrow [0, \infty)$  to transport a unit of mass from  $x$  to  $y$ :

$$d(x, y), \quad d(x, y)^p \quad (p > 0), \quad I_{\{x \neq y\}}$$

- A map  $T : E \rightarrow E$  pushes  $\mu$  into  $\nu$  if  $\mu(T^{-1}(A)) = \nu(A) \quad \forall A \subseteq E$  Borel.

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- A map  $T : E \rightarrow E$  pushes  $\mu$  into  $\nu$  if  $\mu(T^{-1}(A)) = \nu(A) \quad \forall A \subseteq E$  Borel.

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e.g.  $K(\cdot|x) = \delta_{T(x)}$ .

$$\min_K \int_{E \times} c(x, y) K(dy|x) \mu(dx) \quad (\text{Kantorovich})$$

# Optimal transport of mass

Idea: find the **cheapest** way to move the mass  $\mu$  into  $\nu$ .

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# Duality

Linear (convex) **duality** links the two viewpoints:

$$d_{TV}(\mu, \nu) = \min_K \int_{E \times E} I_{\{x \neq y\}} K(dy|x) \mu(dx)$$

$$W^1(\mu, \nu) = \min_K \int_{E \times E} d(x, y) K(dy|x) d\mu(dx)$$

$$d_{BL}(\mu, \nu) = \min_K \int_{E \times E} \min \{1, d(x, y)\} K(dy|x) d\mu(dx)$$

$$d_{Kol}(\mu, \nu) = (\text{exercise?})$$

# Plan

1 Introduction

2 Main results

3 Some ideas from the proof

4 Conclusion

# Notation

- Let

- ①  $E = \mathbb{R}^d$  with Euclidean distance  $d_{\mathbb{R}^d}(x, y) = |x - y|$
- ② or  $E = \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$  (with flat quotient distance)

$$d_{\mathbb{T}^d}(x, y) = \min_{k \in \mathbb{Z}^d} |x - y + k|$$

- Write  $|A|$  for Lebesgue measure  $\mathcal{L}^d$  for  $A \subseteq E$  (also on  $\mathbb{T}^d$ )

- For  $\Omega \subseteq E$ ,

$$W_\Omega^1(\mu, \nu) = W^1(\mu \llcorner \Omega, \nu \llcorner \Omega)$$

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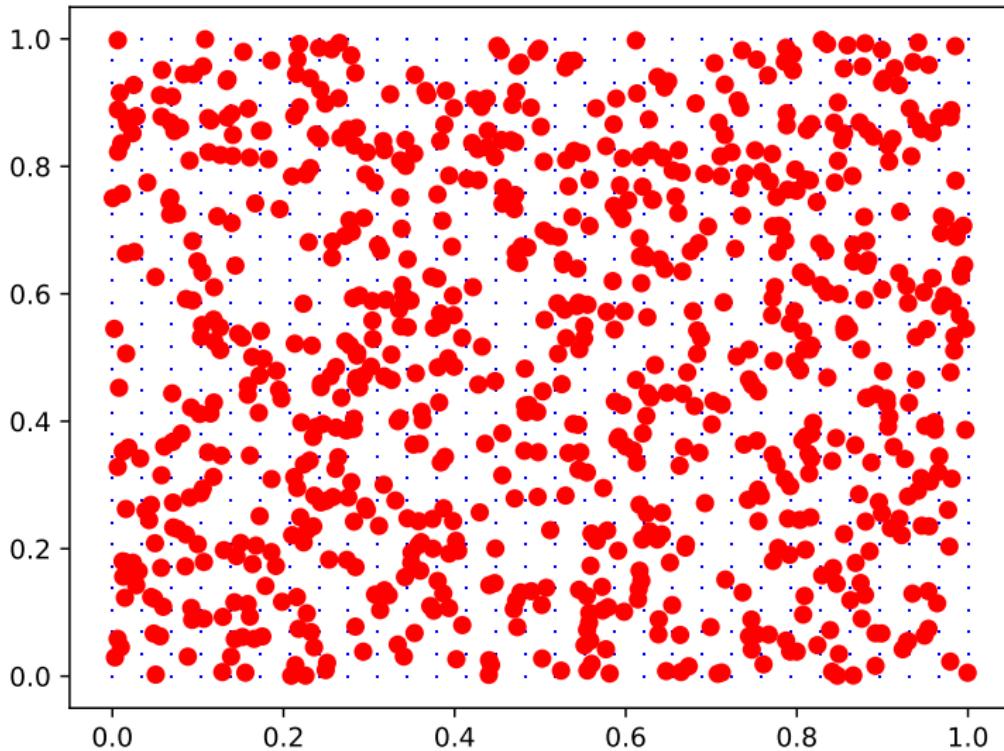
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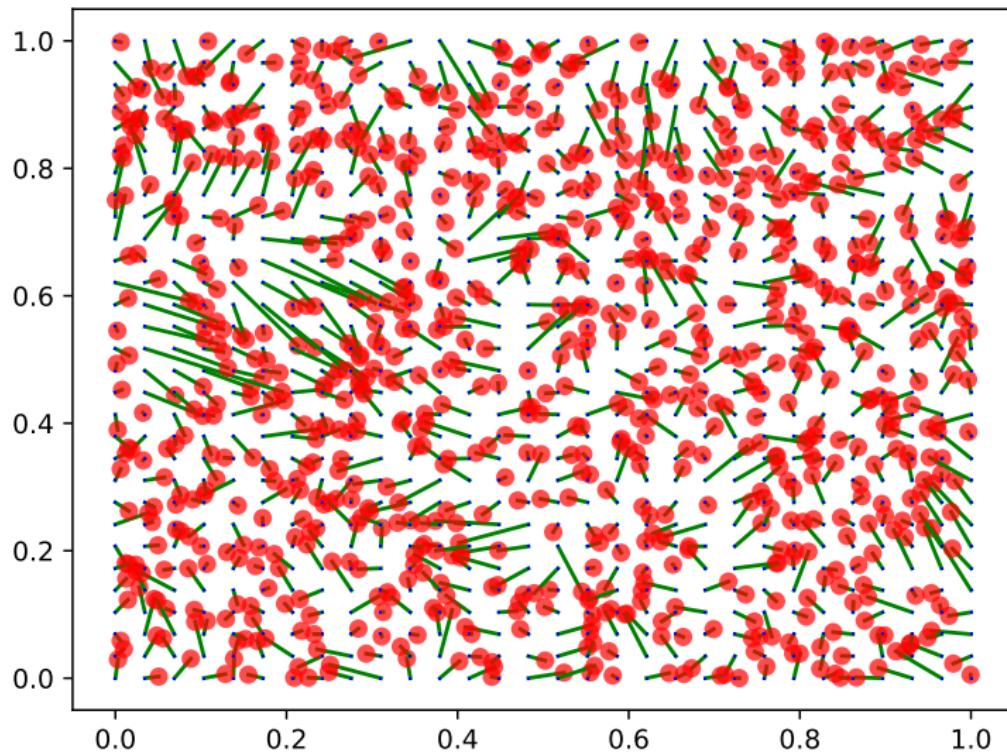
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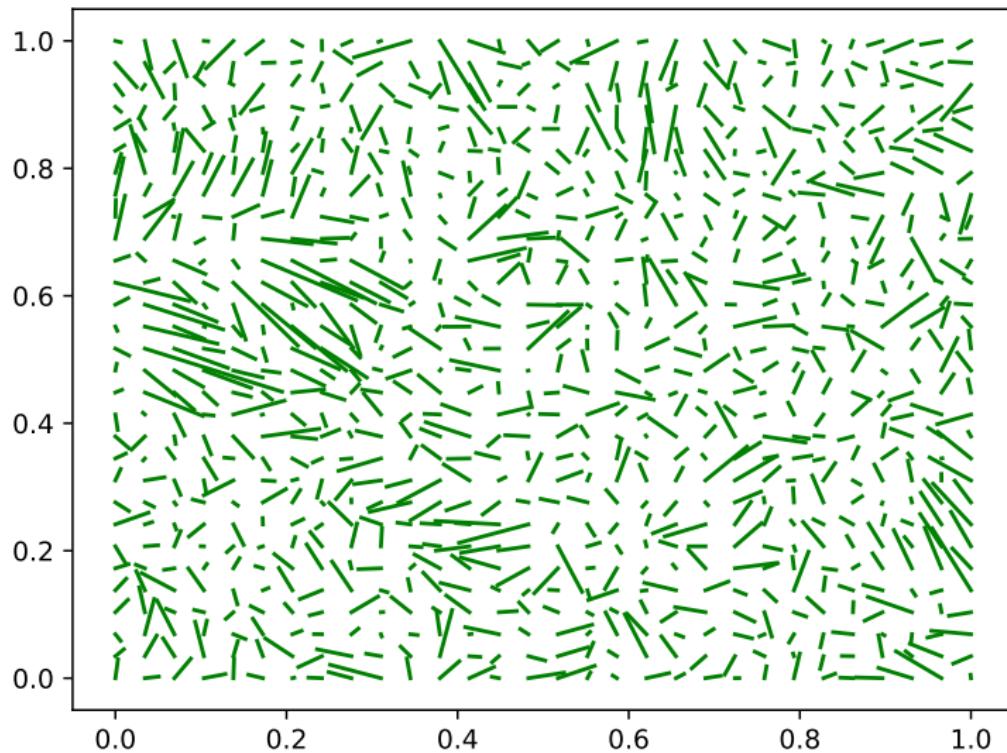
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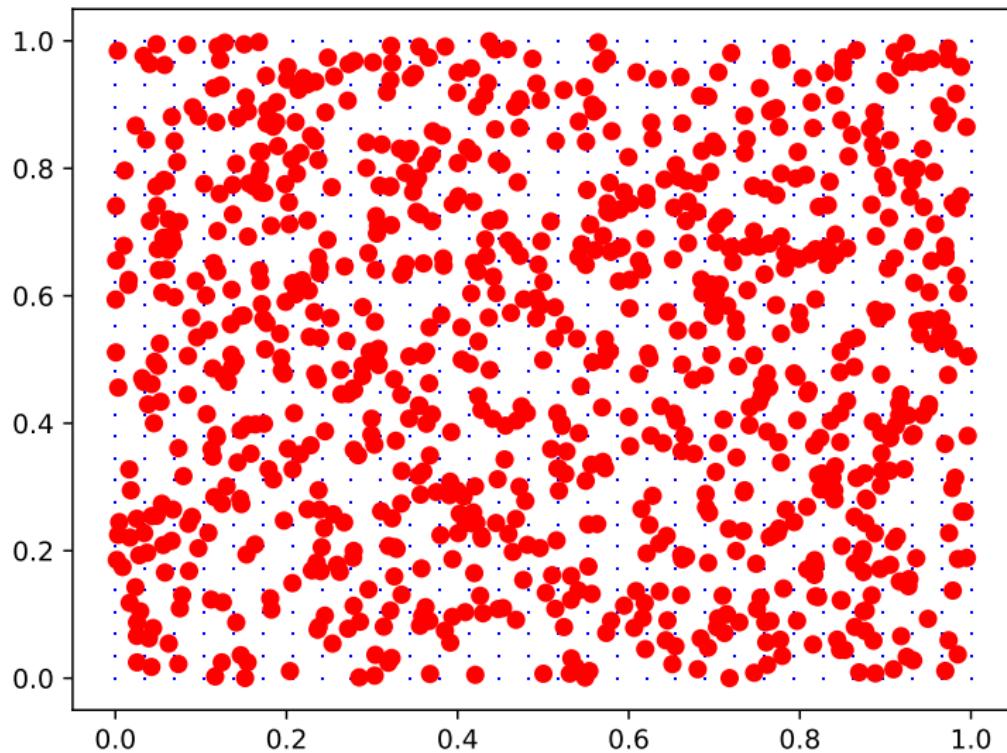
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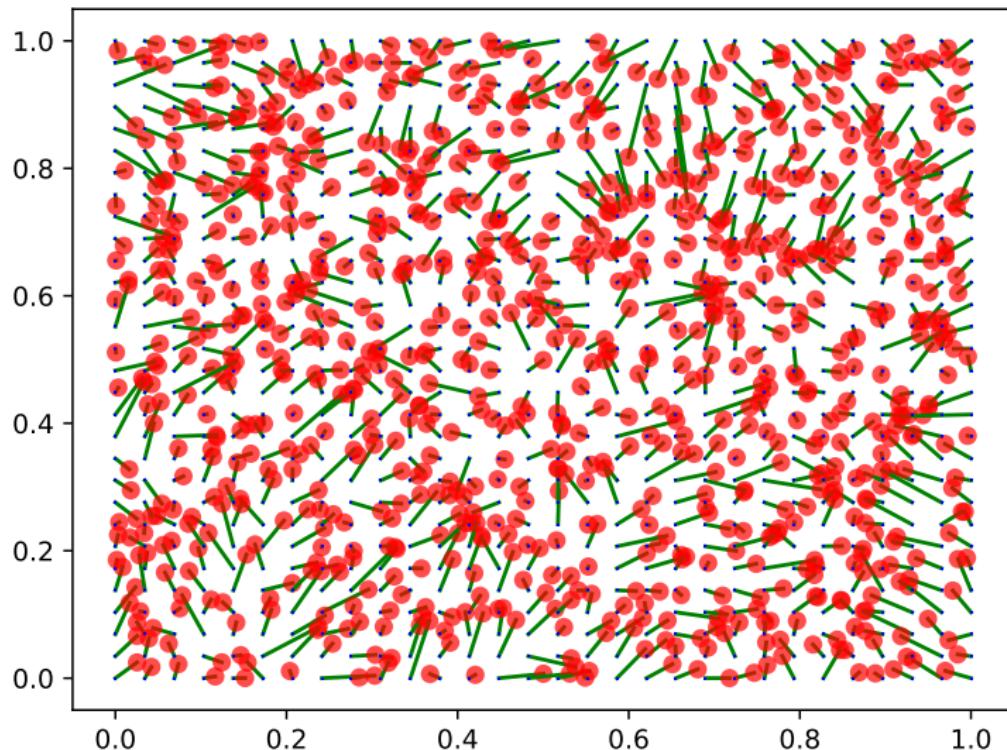
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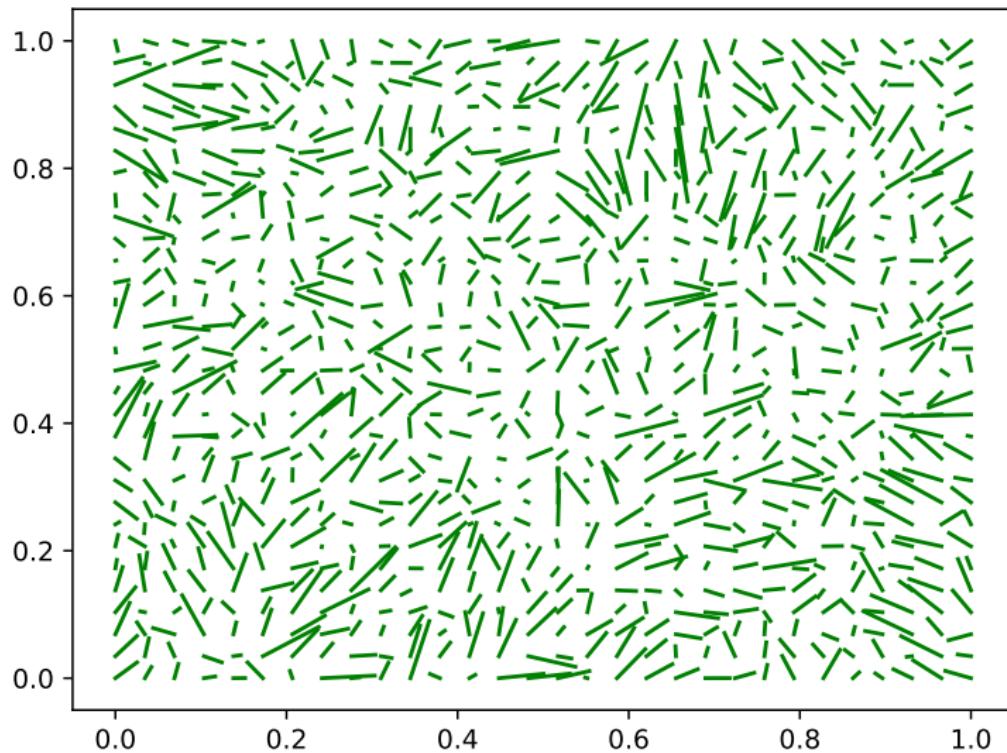
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- Consider i.i.d.  $(Y_n)_{n=1}^{\infty}$  uniform on  $[0, 1]^d$
- Problem: asymptotics of  $W_{[0,1]^d}^1\left(\sum_{i=1}^n \delta_{Y_i}\right)$  as  $n \rightarrow \infty$
- Heuristics: typical distances are  $n^{-1/d}$  (grid case)

$$\Rightarrow W_{[0,1]^d}^1\left(\sum_{i=1}^n \delta_{Y_i}\right) \sim n \cdot n^{-1/d}$$

What does it mean?  $\sum_{i=1}^n \delta_{Y_i}$  is a point cloud in  $[0, 1]^d$  with  $n$  points.

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Moreover, if  $d > 2$ , P.s.

$$\lim_{n \rightarrow \infty} \frac{W_{[0,1]^d}^1\left(\sum_{i=1}^n \delta_{Y_i}\right)}{\sqrt{n}} = \sigma(d) \in (0, \infty)$$

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Let's consider the case  $d = 1$ .  
 We have  $Y_1, Y_2, \dots, Y_n$  i.i.d. uniform on  $[0, 1]$ .  
 Then  $\sum_{i=1}^n \delta_{Y_i}$  is a discrete measure supported on  $\{Y_1, Y_2, \dots, Y_n\}$ .  
 The distance between the origin and the center of mass of this distribution is  

$$\sqrt{\frac{1}{n} \sum_{i=1}^n Y_i^2} = \sqrt{\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})^2 + \bar{Y}^2} = \sqrt{\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})^2} + \bar{Y}$$

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Theorem (Dudley, Billingsley-Komlos-Tusnady, Talagrand, Erdos-Gallai)

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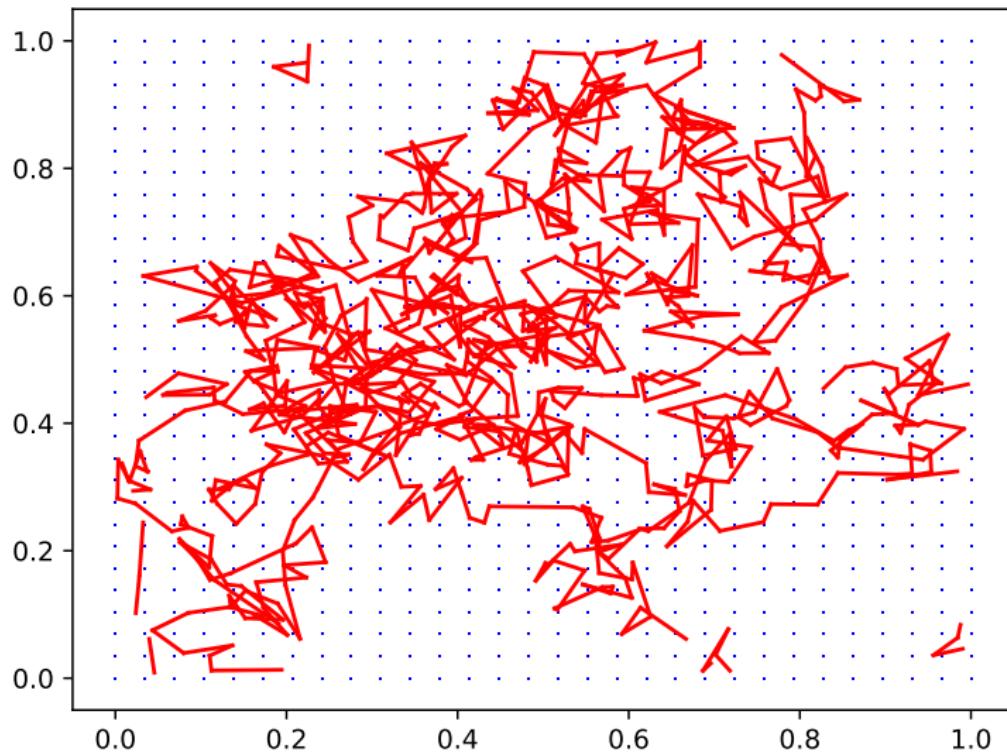
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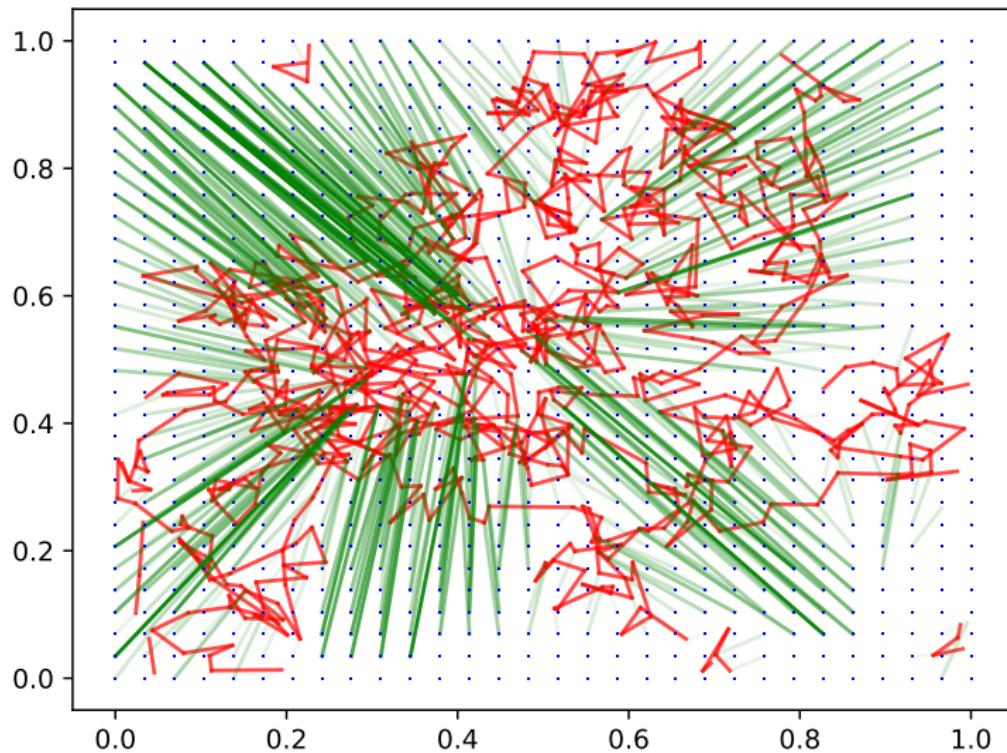
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Let us consider some simulations in  $d = 2$ :  $T = 1$ .



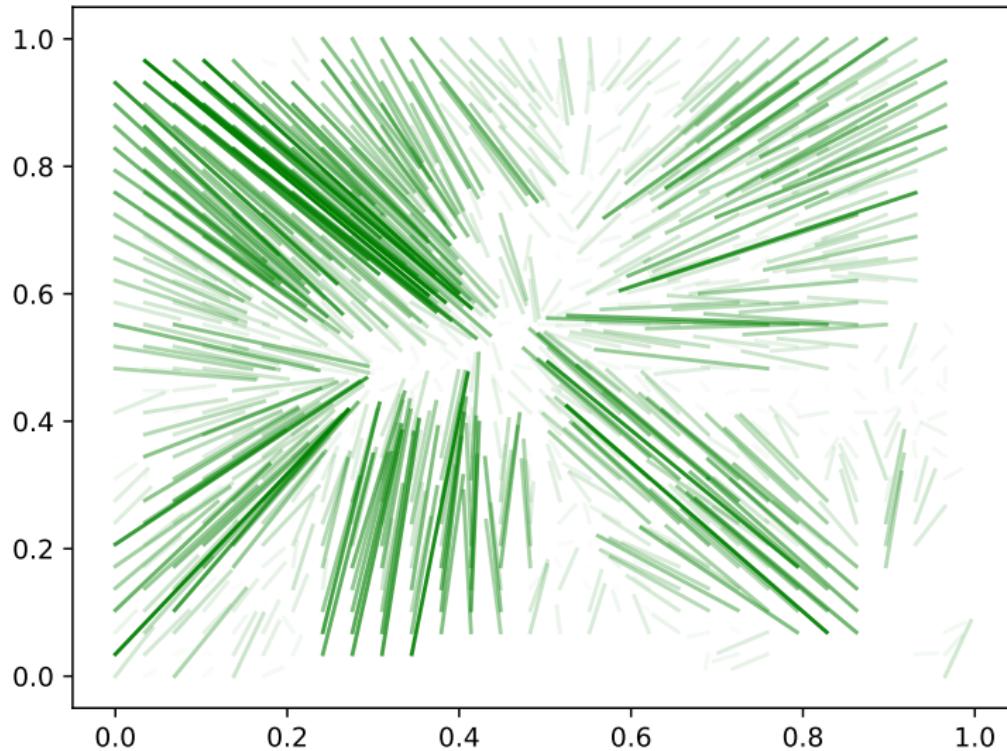
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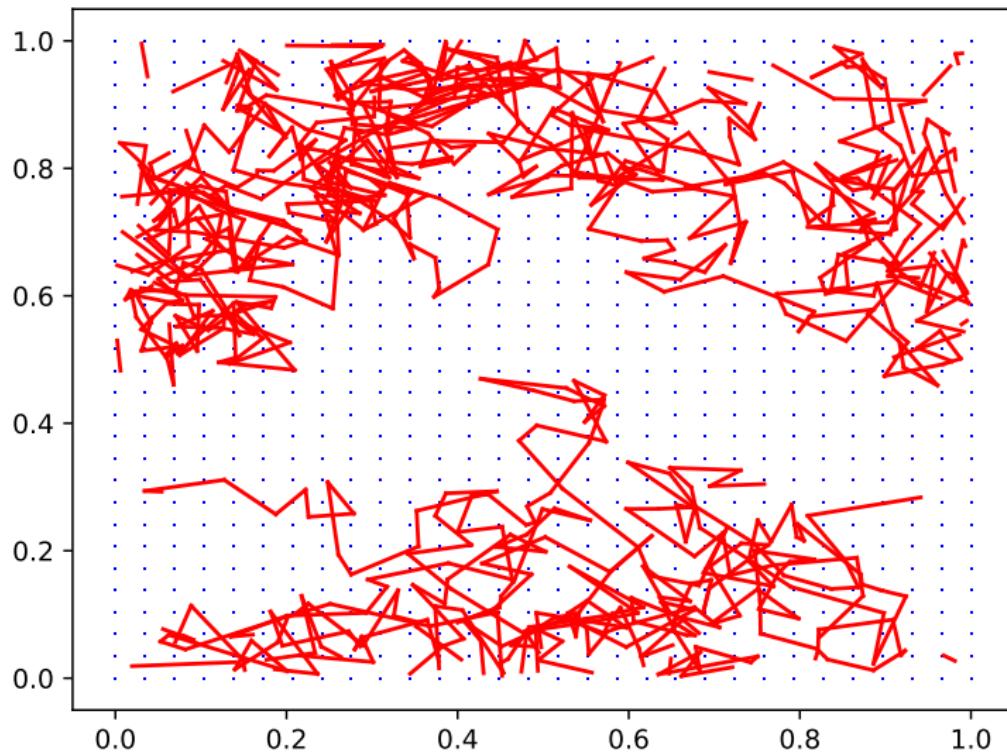
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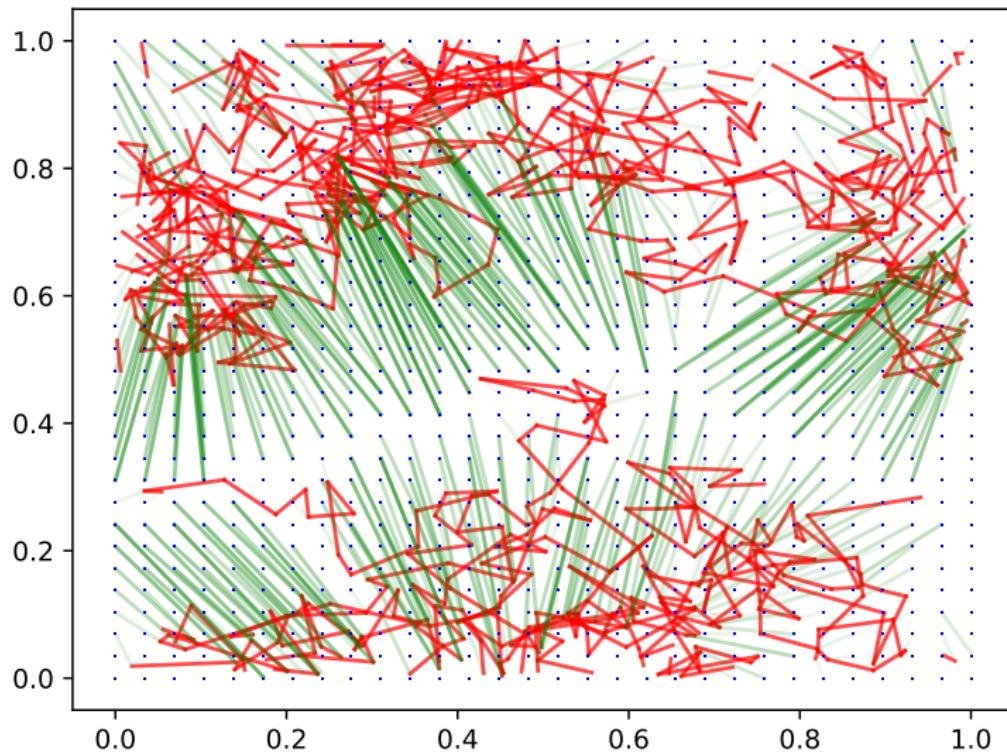
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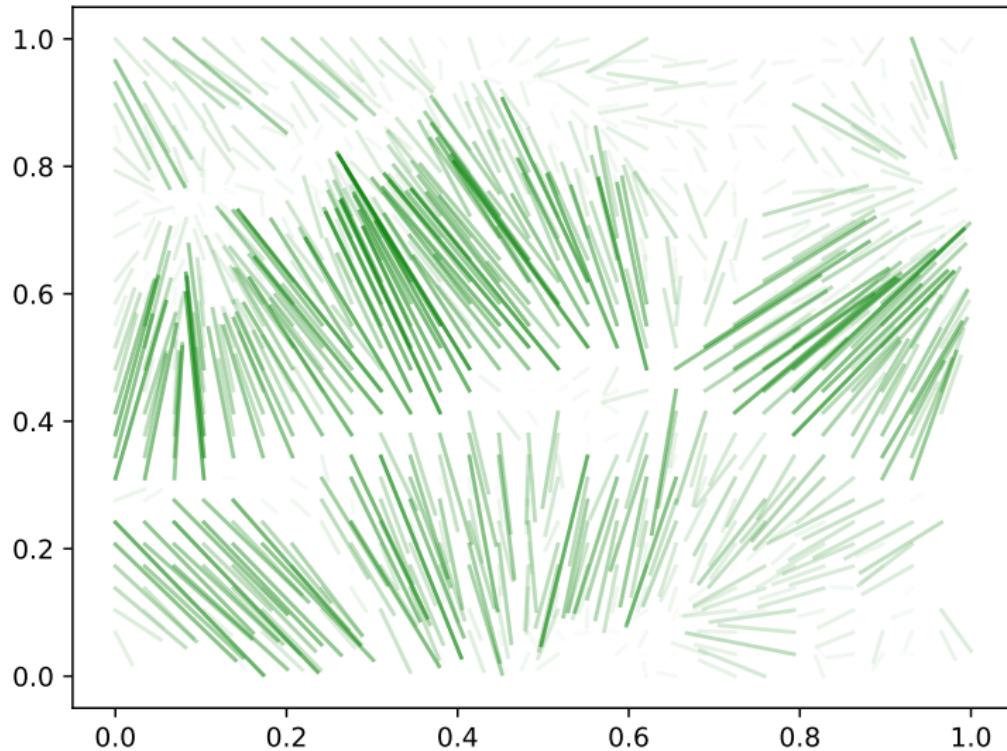
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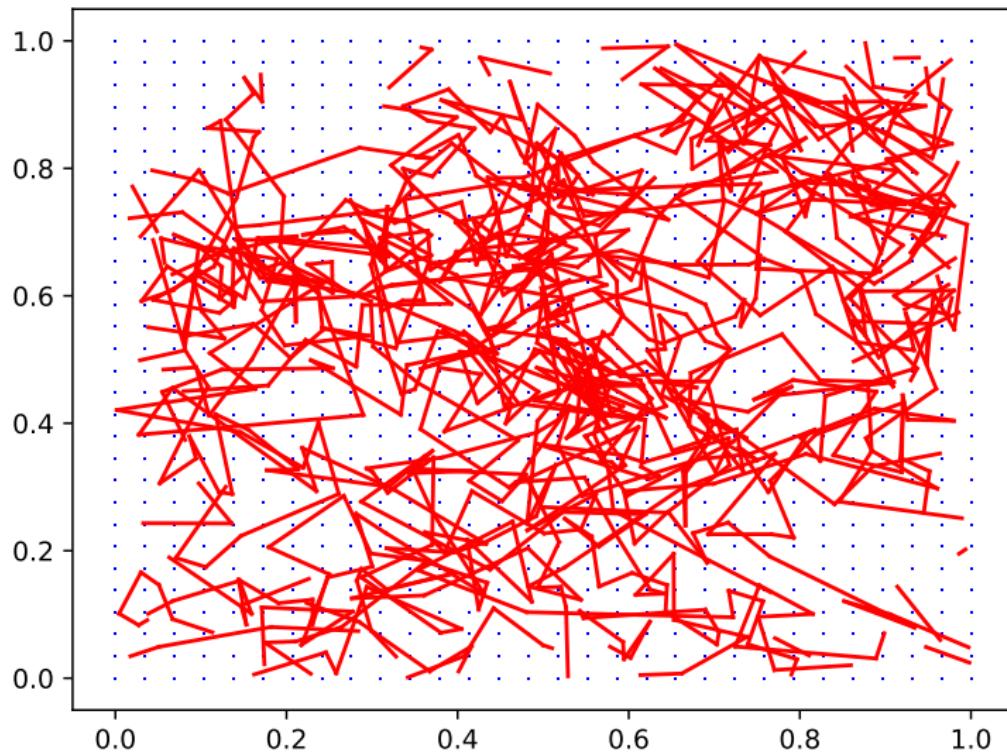
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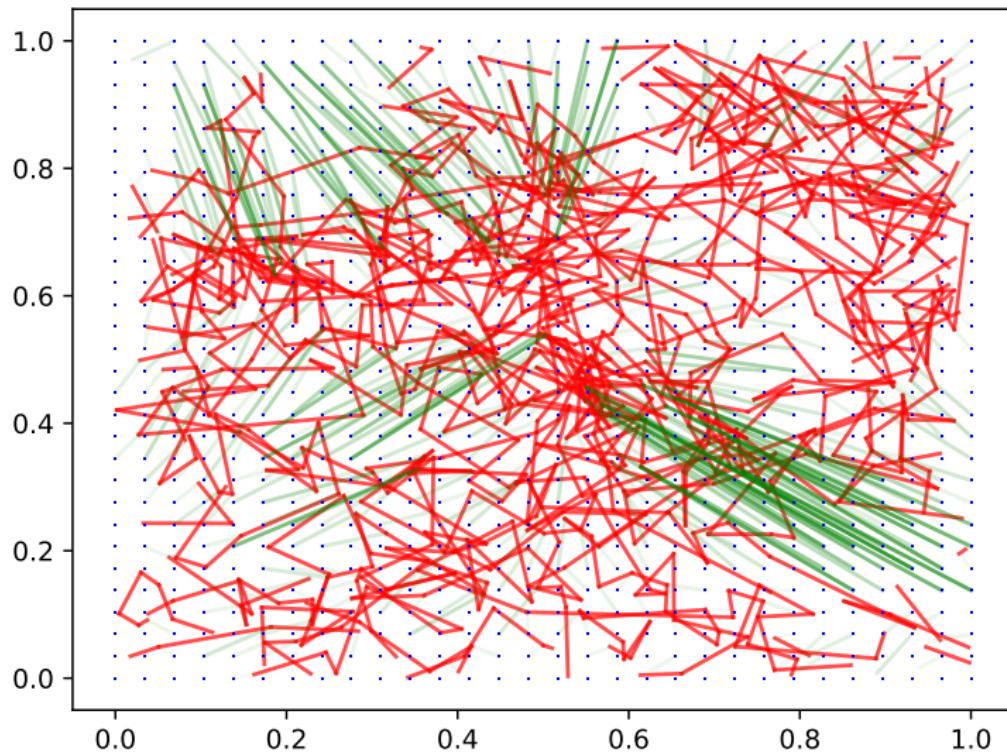
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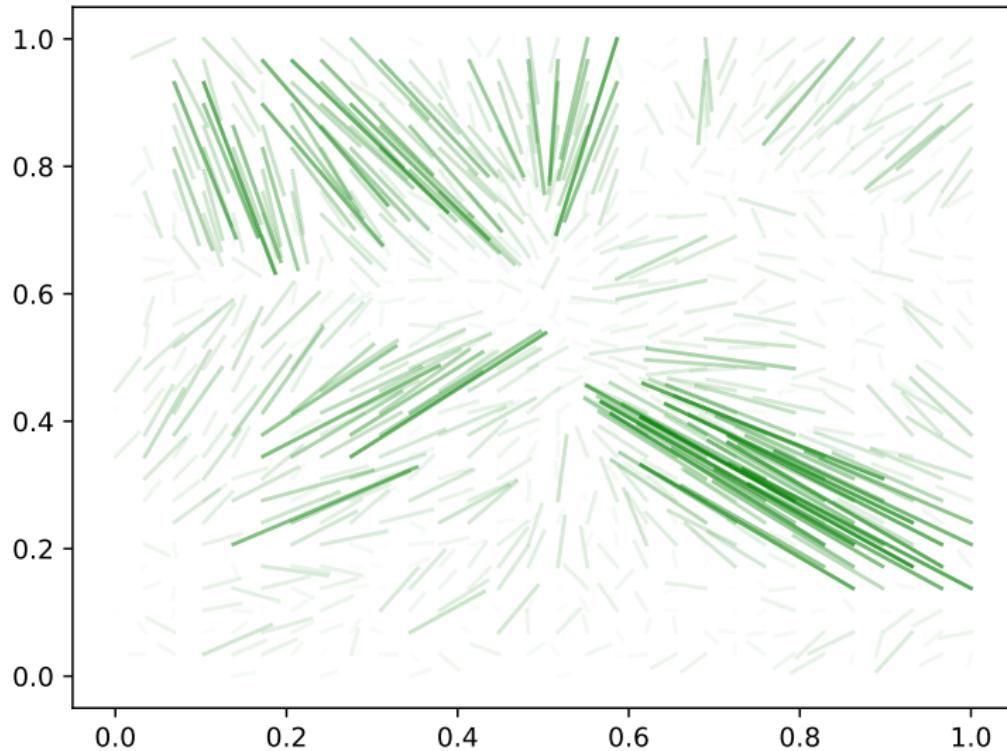
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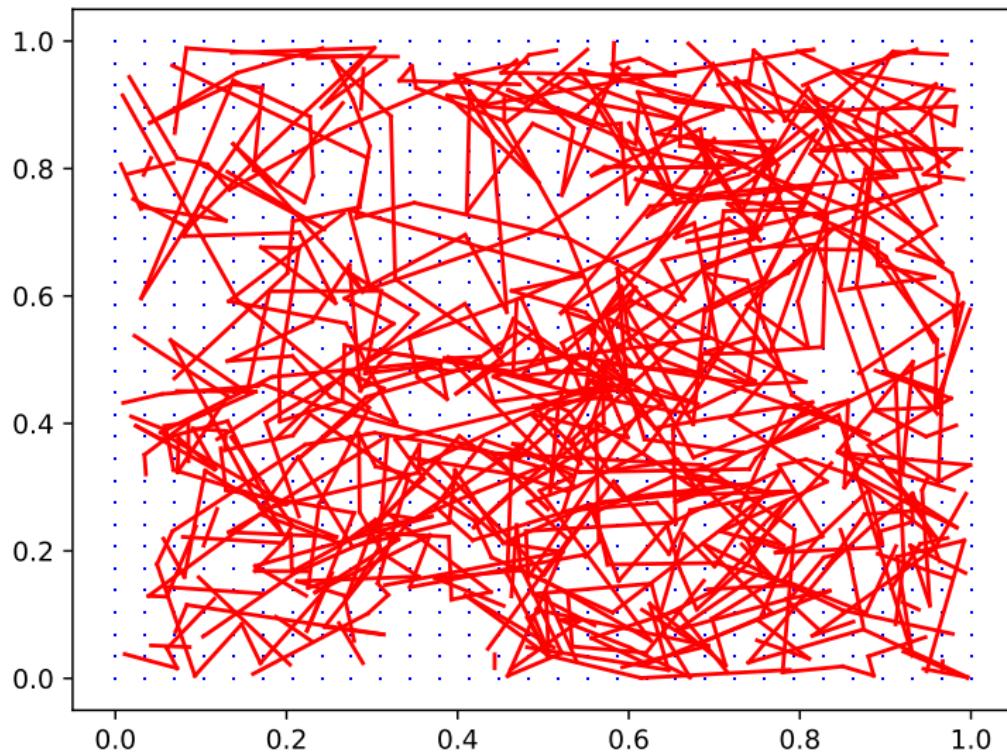
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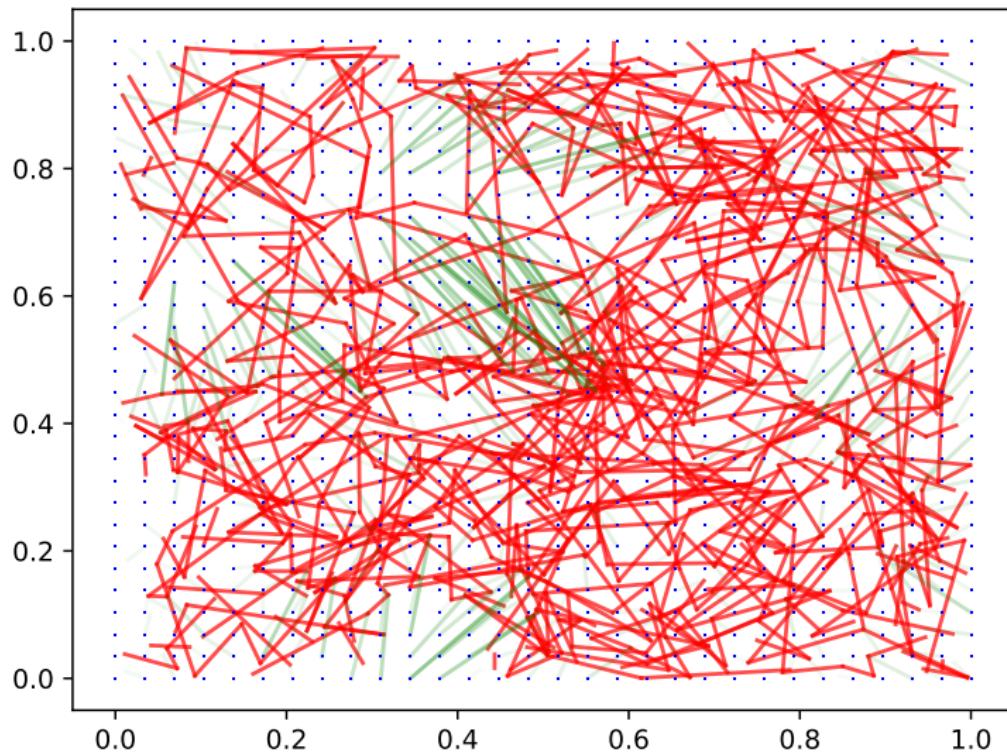
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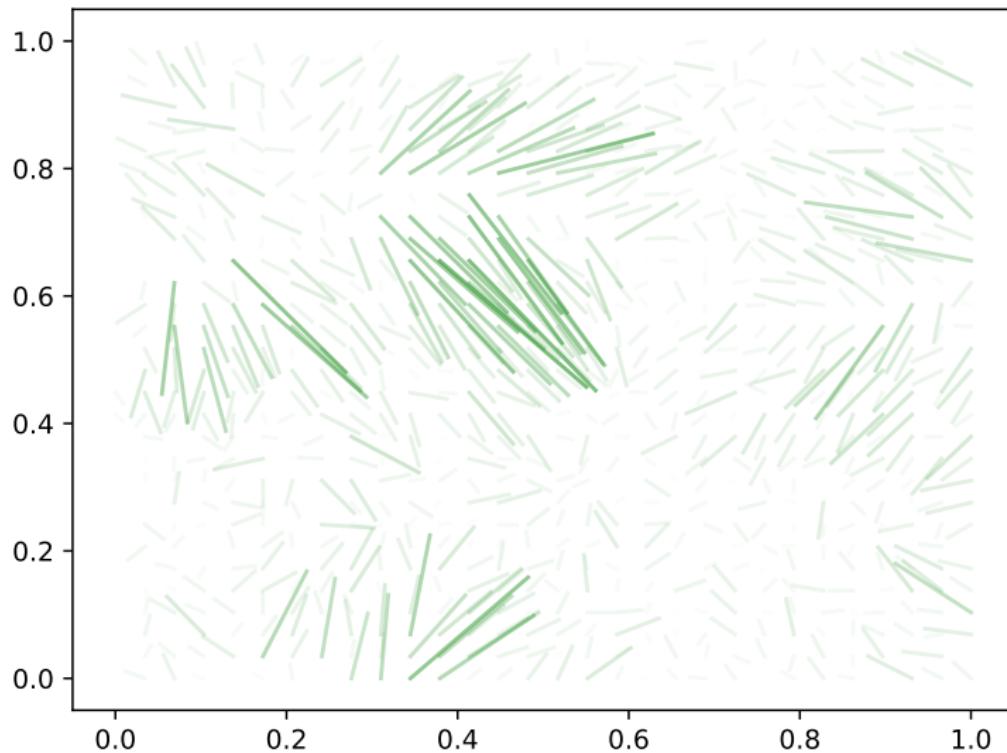
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Similar bounds are also known:

•  $L^p$  bounds

•  $L^p$  bounds for the derivative of the Brownian motion (with respect to the Lebesgue measure)

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Similar bounds are also known:

•  $L^p$  bounds

•  $L^p$  bounds for the difference between the empirical measure and the uniform measure

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# Main result

## Theorem (Mariani-T.)

If  $d > 4$ , then,  $\mathbb{P}$ -a.s.

$$\limsup_{T \rightarrow \infty} \frac{W_{\mathbb{T}^d}^1 \left( \int_0^T \delta_{B_s} ds \right)}{T \cdot T^{-1/(d-2)}} \leq c_{\mathcal{I}}(d) \in (0, \infty).$$

- $\mathcal{I}$  stands for “interlacement” (more on this later)
- **Conjecture:** the limit exists and  $=$  holds.

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If  $d > 4$  and  $\ell = T^{-\gamma}$  with  $\bar{\gamma}(d) < \gamma < 1/(d-2)$ , then,

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# Plan

- 1 Introduction
- 2 Main results
- 3 Some ideas from the proof
- 4 Conclusion

# Strategy in the i.i.d. case

## Theorem (BdMonvel-Martin)

Let  $d > 2$ ,  $(Y_n)_{n=1}^\infty$  be i.i.d. uniform on  $[0, 1]^d$ . Then,

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Proof can be split into two steps:

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→ existence follows by self-similarity and sub-additivity arguments

→ uniqueness

- ② Perform a rescaling and de-Poissonization argument to reduce to  $n$  (deterministic) i.i.d. points on  $[0, 1]^d$ .

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We need

- ➊ an analogue of the Poisson point process for Brownian motion on  $\mathbb{T}^d$ :  
⇒ Brownian interlacement occupation measure
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# Brownian interlacement

- First introduced by Sznitman, we study its **occupation measure**.

## Definition

Let  $d \geq 3$ , consider any  $L \geq 1$  and

1.i.i. Brownian motion on  $\mathbb{R}^d$  with initial law uniform on the sphere  $S^{d-1}$ ,

1.i.2. the set  $\Omega = \{B_t : t \geq 0\}$  of all paths  $B_t$  starting from a point in  $S^{d-1}$  and

1.i.3. the occupation measure  $\lambda_B$  defined by  $\lambda_B(A) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbf{1}_A(B_s) ds$  for every Borel set  $A \subset \mathbb{R}^d$ .

1.i.4. the set  $\mathcal{I} = \{\lambda_B : B \in \Omega\}$  of all occupation measures.

1.i.5. the set  $\mathcal{I}^*$  of all probability measures  $\mu$  on  $\mathcal{I}$  such that  $\mu(\{B : B(0) = x\}) = 1$  for every  $x \in S^{d-1}$ .

1.i.6. the set  $\mathcal{I}^{\text{unif}}$  of all probability measures  $\mu$  on  $\mathcal{I}$  such that  $\mu(\{B : B(0) = x\}) = 1$  for every  $x \in S^{d-1}$  and  $\mu(\{B : B(t) = y\}) = 1$  for every  $y \in \mathbb{R}^d$ .

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1.i.10. the set  $\mathcal{I}^{\text{unif}, \text{sym}, \text{erg}, \text{inv}, \text{rot}}$  of all probability measures  $\mu$  on  $\mathcal{I}$  such that  $\mu(\{B : B(0) = x\}) = 1$  for every  $x \in S^{d-1}$  and  $\mu(\{B : B(t) = y\}) = 1$  for every  $y \in \mathbb{R}^d$  and  $\mu(\{B : B(t) = -y\}) = 1$  for every  $y \in \mathbb{R}^d$  and  $\mu(\{B : B(t) = z\}) = 1$  for every  $z \in \mathbb{R}^d$  and  $\mu(\{B : B(t) = -z\}) = 1$  for every  $z \in \mathbb{R}^d$  and  $\mu(\{B : B(t) = \theta z\}) = 1$  for every  $z \in \mathbb{R}^d$  and  $\theta \in \mathbb{R}$ .

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# Brownian interlacement

- First introduced by Sznitman, we study its **occupation measure**.

## Definition

Let  $d \geq 3$ , consider any  $L \geq 1$  and

- ① i.i.d. Brownian motions on  $\mathbb{R}^d$  with initial law uniform on the sphere  $\partial D_L$
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and define

$$\mathcal{I} \llcorner D_L = \sum_{i=1}^N \mu_{\infty}^{B_i} \llcorner D_L,$$

Equality holds (in law)

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⇒ the **Brownian interlacement** occupation measure  $\mathcal{I}$  is well-defined.

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- $\mathcal{I}$  is **stationary** (translation invariant in law)  $\Rightarrow \mathcal{I}(A+x) = \mathcal{I}(A)$  (in law)
- $\mathbb{E}[\mathcal{I}(A)] = |A|$
- **Concentration:** if  $\text{diam}(A) \geq 1$ , then

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1. Concentration

2. Interlacing

3. De-interlacing

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# De-Poissonization

Direct applications of tools from the i.i.d. literature

- de-Poissonization

- geometric decomposition

⇒ result for a **deterministic** number of Brownian paths.

Then we can apply the same arguments as in the Poisson case, and get a similar result:

Given  $\mu$  a probability measure on  $\mathbb{R}^d$ , there exists a deterministic number  $n$  such that for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that if  $\lambda \in \mathbb{R}$  satisfies  $|\lambda| < \delta$  then  $\mu_\lambda$  is  $\epsilon$ -close to uniform on  $\partial D_1$ .

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Proposition

Let  $d \geq 5$  and  $(B^i)_{i \in \mathbb{Z}}^{\mathbb{Z}}$  be i.i.d. Brownian motions on  $\mathbb{R}^d$  with uniform initial law on the sphere  $\partial D_1$ . Then,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ w_n \left( \sum_{i=1}^n B^i \cdot R^n \delta_i ds \right) \right] = \text{vol}(\partial D_1)$$

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## Proposition

If  $d > 4$  and  $\ell = T^{-\gamma}$  with  $\bar{\gamma}(d) < \gamma < 1/(d-2)$ , then,

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Sketch of the argument:

- ① Split  $[0, T]$  into  $n \gg 1$  time intervals of length  $\rho \sim T/n$  and replace a single Brownian path with  $n$  i.i.d. stationary  $(B^i)_{i=1}^n$  on  $\mathbb{T}^d$ .
- ② Consider the rare events  $A_i$  that  $B^i$  hits  $D_\ell$  before  $\rho$ :

$$P(A_i) \sim \rho \ell^{d-2} \text{Cap}(D_1) \ll 1 \quad \text{if } \rho \ll \ell^{d-2}.$$

- ③ By law of large numbers, the contribution of  $\sum_{i=1}^n \mu_\rho^{B^i} \llcorner D_\ell$  comes from

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Step 2 uses a result on hitting probabilities. **Notation:**

$\tau_{D_\ell}$  = first hitting time of  $D_\ell$  for process  $B$

$$\nu_\rho(A) := \mathbb{P}(B_{\tau_{D_\ell}} \in A | 0 < \tau_{D_\ell} \leq \rho).$$

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Let  $d \geq 3$ ,  $0 < \gamma < d - 2$ , and

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Moreover,

$$W_{D_\ell}^1(\nu_\rho, \operatorname{Unif}_{\partial D_\ell}) \lesssim \ell \cdot (\rho \ell^{d-2} |\log \ell| + \ell^2 / \rho),$$

- The first bound is well-known in the literature (also used by Sznitman)
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# Plan

- 1 Introduction
- 2 Main results
- 3 Some ideas from the proof
- 4 Conclusion

# Concluding remarks

In brief: tools and ideas from

- random matching of i.i.d. points
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are useful to analyze asymptotics of stochastic processes through optimal transport.

Open questions:

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$$\limsup_{n \rightarrow \infty} \frac{W_{\alpha,d}(\sum_{j=1}^n \delta_{X_j}, \mu)}{\sqrt{n - n^{1/d}}} \leq c(d).$$

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