The dual space of $L^p(\Omega)$

A class of bounded linear functionals on $L^p(\Omega)$

Proposition 1. *Let Ω be a Lebesgue measurable set in $\mathbb{R}^d$. Let $p \in (1,+\infty)$ and $q := \frac{p}{p-1} \in (1,+\infty)$. Given a function $g \in L^q(\Omega)$, we define the functional*

$$T_g : L^p(\Omega) \to \mathbb{R}\ , \qquad T_g(f) = \int_\Omega f(x)g(x)dx.$$

Then, T_g is a bounded linear functional on $L^p(\Omega)$ and

$$\|T_g\|_{(L^p(\Omega))'} = \|g\|_{L^q(\Omega)},$$

where $\|T_g\|_{(L^p(\Omega))'}$ is the the operator norm defined as follows:

$$\|T_g\|_{(L^p(\Omega))'} := \sup\Big\{T_g(f)\ :\ f \in L^p(\Omega),\ \|f\|_{L^p(\Omega)} \le 1\Big\}.$$

Proof. For every $f \in L^p(\Omega)$ with $\|f\|_{L^p(\Omega)} \le 1$, we have

$$|T_g(f)| = \left|\int_\Omega f(x)g(x)dx\right| \le \|g\|_{L^q(\Omega)}\|f\|_{L^p(\Omega)} \le \|g\|_{L^q(\Omega)}.$$

Taking the supremum over all $f \in L^p(\Omega)$ with $\|f\|_{L^p(\Omega)} \le 1$, we get the inequality

$$\|T_g\|_{(L^p(\Omega))'} \le \|g\|_{L^q(\Omega)}.$$

On the other hand, choosing the function

$$f(x) := \frac{1}{\|g\|^{q-1}_{L^q(\Omega)}}|g(x)|^{q-2}g(x),$$

we get that

$$\|f\|^p_{L^p(\Omega)} = \frac{1}{\|g\|^{(q-1)p}_{L^q(\Omega)}}\int_\Omega |g|^{p(q-1)}dx = \frac{1}{\|g\|^q_{L^q(\Omega)}}\int_\Omega |g|^q dx = 1,$$

$$T_g(f) = \int_\Omega f(x)g(x)dx = \frac{1}{\|g\|^{q-1}_{L^q(\Omega)}}\int_\Omega |g|^q dx = \|g\|_{L^q(\Omega)}.$$

Thus we get the opposite inequality

$$\|T_g\|_{(L^p(\Omega))'} \ge T_g(f) = \|g\|_{L^q(\Omega)},$$

which concludes the proof. □

Teorema di rappresentazione di Riesz per gli spazi $L^p(\Omega)$

Theorem 2. *Let Ω be a Lebesgue measurable set in $\mathbb{R}^d$ and let $p \in (1,+\infty)$. Let*

$$T : L^p(\Omega) \to \mathbb{R}$$

be a continuous linear functional on $L^p(\Omega)$. Then, there exists a unique function

$$g \in L^q(\Omega) \quad \text{with} \quad q := \frac{p}{p-1}$$

such that

$$T(f) = \int_\Omega f(x)g(x)dx \quad \text{for all} \quad f \in L^p(\Omega).$$

In particular,

$$\|T\|_{(L^p(\Omega))'} := \sup\big\{T(f) : f \in L^p(\Omega), \|f\|_{L^p(\Omega)} \le 1\big\} = \|g\|_{L^q(\Omega)}.$$

We will consider two different cases: $1 < p \le 2$ and $2 \le p < +\infty$. For each of these cases we will give two different arguments. Both arguments are based on the Clarkson's inequality. We notice that the each of these arguments can be adapted and extended to treat both cases.

Proof of Theorem 2 in the case $1 < p \leq 2$

We proceed in three steps.

Step 1. Proof for $1 < p \leq 2$ and $\Omega := [-R, R]^d$. For every $n \geq 1$ we consider the partition

$$\mathcal{P}_n := \left\{Q_j : j = 1, \dots, 2^{nd}\right\}$$

of Ω into 2^{nd} disjoint cubes $Q_j, j = 1, \dots, n^d$ of side $R/2^n$. Let

$$V_n \subset L^p(\Omega),$$

be the space of all functions $u \in L^p(\Omega)$ of the form.

$$u(x) = \sum_{j=1}^{2^{nd}} a_j \mathbb{1}_{Q_j}(x),$$

for some real constants $a_1, \dots, a_{2^{nd}}$. We notice that, by construction, each space V_n is a closed finite dimensional subspace of $L^p(\Omega)$. We also notice that the sequence of spaces $(V_n)_{n\geq 1}$ is increasing, that is,

$$V_n \subset V_m \quad \text{for all} \quad m \geq n \geq 1.$$

Moreover, given $u \in V_n$ of the form

$$u(x) = \sum_{j=1}^{2^{nd}} a_j \mathbb{1}_{Q_j}(x),$$

we can compute

$$T(u) = \sum_{j=1}^{2^{nd}} a_j T\left(\mathbb{1}_{Q_j}\right).$$

Define the function

$$g_n(x) := \sum_{j=1}^{2^{nd}} \frac{T\left(\mathbb{1}_{Q_j}\right)}{|Q_j|} \mathbb{1}_{Q_j}(x).$$

We notice that, by construction, we have

$$T(u) = \int_\Omega g_n(x)u(x)dx \quad \text{for all} \quad u = \sum_{j=1}^{2^{nd}} a_j \mathbb{1}_{Q_j} \in V_n.$$

Indeed, since

$$\mathbb{1}_{Q_i}\mathbb{1}_{Q_j} = \begin{cases} \mathbb{1}_{Q_i} & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases}$$

we have that

$$\begin{aligned}
\int_\Omega g_n(x)u(x)dx &= \int_\Omega \left(\sum_{i=1}^{2^{nd}} \frac{T\left(\mathbb{1}_{Q_i}\right)}{|Q_i|} \mathbb{1}_{Q_i}(x)\right)\left(\sum_{j=1}^{2^{nd}} a_j \mathbb{1}_{Q_j}(x)\right) dx \\
&= \int_\Omega \left(\sum_{j=1}^{2^{nd}} \frac{a_j T\left(\mathbb{1}_{Q_j}\right)}{|Q_j|} \mathbb{1}_{Q_j}(x)\right) dx \\
&= \sum_{j=1}^{2^{nd}} a_j T\left(\mathbb{1}_{Q_j}\right) \int_\Omega \frac{1}{|Q_j|} \mathbb{1}_{Q_j}(x)\, dx = \sum_{j=1}^{2^{nd}} a_j T\left(\mathbb{1}_{Q_j}\right) = T(u).
\end{aligned}$$

We will next prove that

$$\|g_n\|_{L^q(\Omega)} \leq \|T\|_{(L^p(\Omega))'}. \tag{1}$$

Indeed, thanks to the Hölder inequality, we have

$$T(u) = \left|\int_\Omega g_n(x)u(x)dx\right| \leq \|u\|_{L^p(\Omega)}\|g_n\|_{L^q(\Omega)} \quad \text{for all} \quad u \in V_n,$$

and so

$$\|g_n\|_{L^q(\Omega)} = \sup\left\{\int_\Omega g_n(x)u(x)dx : u \in V_n, \|u\|_{L^p(\Omega)} \le 1\right\}$$
$$= \sup\left\{\int_\Omega g_n(x)u(x)dx : u \in V_n, \|u\|_{L^p(\Omega)} \le 1\right\} \le \|T\|_{(L^p(\Omega))'}.$$

We next claim that there is a function $f_n : \Omega \to \mathbb{R}$ such that

$$f_n \in V_n\ , \qquad \|f_n\|_{L^p(\Omega)} = 1 \qquad \text{and} \qquad T(f_n) = \int_\Omega f_n(x)g_n(x)\,dx = \|g_n\|_{L^q(\Omega)}. \tag{2}$$

Indeed, it is sufficient to choose f_n as follows

$$f_n(x) := \frac{1}{\|g_n\|^{q-1}_{L^q(\Omega)}} \sum_{j=1}^{2^{nd}} \left(\frac{T\left(\mathbb{1}_{Q_j}\right)}{|Q_j|}\right)^{q-1} \mathbb{1}_{Q_j}(x).$$

In order to prove that $\|f_n\|_{L^p} = 1$, using the identity $q = p(q-1)$, we compute

$$\|f_n\|_{L^p(\Omega)} = \frac{1}{\|g_n\|^{q-1}_{L^q(\Omega)}} \left(\sum_{j=1}^{2^{nd}} \left(\frac{T\left(\mathbb{1}_{Q_j}\right)}{|Q_j|}\right)^{p(q-1)} |Q_j|\right)^{1/p}$$
$$= \frac{1}{\|g_n\|^{q-1}_{L^q(\Omega)}} \left(\sum_{j=1}^{2^{nd}} \left(\frac{T\left(\mathbb{1}_{Q_j}\right)}{|Q_j|}\right)^{q} |Q_j|\right)^{1/p} = \frac{1}{\|g_n\|^{q-1}_{L^q(\Omega)}} \left(\|g_n\|^q_{L^q(\Omega)}\right)^{1/p} = 1.$$

In order to show that $\int_\Omega f_n g_n\,dx = \|g_n\|_{L^q}$, we write

$$\|g_n\|_{L^q(\Omega)} = \left(\sum_{j=1}^{2^{nd}} \left(\frac{T\left(\mathbb{1}_{Q_j}\right)}{|Q_j|}\right)^{q} |Q_j|\right)^{1/q}$$
$$= \left(\sum_{j=1}^{2^{nd}} \left(\frac{T\left(\mathbb{1}_{Q_j}\right)}{|Q_j|}\right)^{q} |Q_j|\right)^{\frac{1}{q}-1} \sum_{j=1}^{2^{nd}} \left(\frac{T\left(\mathbb{1}_{Q_j}\right)}{|Q_j|}\right)^{q} |Q_j|$$
$$= \frac{1}{\|g_n\|^{q-1}_{L^q(\Omega)}} \sum_{j=1}^{2^{nd}} T\left(\mathbb{1}_{Q_j}\right) \left(\frac{T\left(\mathbb{1}_{Q_j}\right)}{|Q_j|}\right)^{q-1} = \int_\Omega g_n(n)f_n(x)dx$$

Our next move is to show that the sequence of norms $\|g_n\|_{L^q(\Omega)}$ is increasing. In order to do this, we take two integers

$$m \ge n \ge 1,$$

and we consider the function f_n defined in (2). We notice that, since $V_n \subset V_m$, we have

$$f_n \in V_m,$$

so we can write $T(f_n)$ in two different ways, as

$$T(f_n) = \int_\Omega f_n(x)g_n(x)\,dx \quad \text{and} \quad T(f_n) = \int_\Omega f_n(x)g_m(x)\,dx.$$

Thus, thanks to the properties of f_n, we have

$$\|g_n\|_{L^q(\Omega)} = \int_\Omega f_n(x)g_n(x)dx = T\left(f_n\right) = \int_\Omega f_n(x)g_m(x)dx \le \|g_m\|_{L^q(\Omega)},$$

which proves that

$$\|g_n\|_{L^q(\Omega)} \le \|g_m\|_{L^q(\Omega)} \quad \text{for all} \quad m \ge n \ge 1. \tag{3}$$

By (1) and (3), we know that the sequence $\|g_n\|_{L^q(\Omega)}$ is increasing and bounded. Thus, it converges to some

$$L := \lim_{n\to\infty} \|g_n\|_{L^q(\Omega)} \le \|T\|_{(L^p(\Omega))'}.$$

(We notice that, from the fact that $\cup_{n\geq 1}V_n$ is dense in $L^p(\Omega)$ one can deduce that $L=\|T\|$.)
Now, since $q\geq 2$, the Clarkson's inequality implies

$$\begin{aligned}\left\|\frac{g_m-g_n}{2}\right\|^q_{L^q(\Omega)} &\leq \frac{1}{2}\|g_m\|^q_{L^q(\Omega)}+\frac{1}{2}\|g_n\|^q_{L^q(\Omega)}-\left\|\frac{g_m+g_n}{2}\right\|^q_{L^q(\Omega)}\\ &\leq \frac{1}{2}\|g_m\|^q_{L^q(\Omega)}+\frac{1}{2}\|g_n\|^q_{L^q(\Omega)}-\left|\int_\Omega f_n(x)\frac{g_m(x)+g_n(x)}{2}dx\right|^q\\ &= \frac{1}{2}\|g_m\|^q_{L^q(\Omega)}+\frac{1}{2}\|g_n\|^q_{L^q(\Omega)}-\left|\int_\Omega f_n(x)g_n(x)dx\right|^q\\ &= \frac{1}{2}\|g_m\|^q_{L^q(\Omega)}+\frac{1}{2}\|g_n\|^q_{L^q(\Omega)}-\|g_n\|^q_{L^q(\Omega)}\\ &= \frac{1}{2}\|g_m\|^q_{L^q(\Omega)}-\frac{1}{2}\|g_n\|^q_{L^q(\Omega)}\end{aligned}$$

Thus, the $(g_n)_{n\geq 1}$ is a Cauchy sequence in $L^q(\Omega)$ and so it converges in $L^q(\Omega)$ to some function

$$g\in L^q(\Omega),\quad g=\lim_{n\to\infty}g_n.$$

We next prove that g is the function that represents the functional T. Consider a function

$$u\in L^p(\Omega)$$

We notice that there is a sequence $u_n\in L^p(\Omega)$ such that

$$u=\lim_{n\to\infty}u_n\quad\text{ and }\quad u_n\in V_n\quad\text{ for all }\quad n\geq 1.$$

Now, using the continuity of T, we get

$$T(u)=\lim_{n\to\infty}T(u_n)=\lim_{n\to\infty}\int_\Omega g_n(x)u_n(x)dx=\int_\Omega u(x)g(x)dx,$$

which concludes the proof in the case $1<p\leq 2$ and $\Omega:=[-R,R]^d$.

Step 2. Proof for $1<p\leq 2$ and for bounded Ω. Let $\Omega\subset\mathbb{R}^d$ be a bounded Lebesgue measurable set. For every $R>0$ we define

$$\Omega_R:=[-R,R]^d.$$

We choose R large enough in order to have

$$\Omega\subset\Omega_R:=[-R,R]^d.$$

For Ω_R chosen above, we consider the restriction operator

$$\pi:L^p(\Omega_R)\to L^p(\Omega),$$

defined, for all $u\in L^p(\Omega_R)$ as

$$\pi(u)(x):=u(x)\quad\text{ for all }\quad x\in\Omega.$$

Let

$$S:L^p(\Omega_R)\to\mathbb{R}$$

be the functional defined as

$$S(u)=T(\pi(u))\quad\text{for all}\quad u\in L^p(\Omega_R).$$

Then, S is linear and bounded on $L^p(\Omega_R)$. Indeed, for every $u\in L^p(\Omega_R)$, we have

$$\begin{aligned}|S(u)|=|T(\pi(u))| &\leq \|T\|_{(L^p(\Omega))'}\|\pi(u)\|_{L^p(\Omega)}\\ &= \|T\|_{(L^p(\Omega))'}\left(\int_\Omega|\pi(u)|^p(x)\,dx\right)^{1/p}\\ &= \|T\|_{(L^p(\Omega))'}\left(\int_\Omega|u|^p(x)\,dx\right)^{1/p}\\ &\leq \|T\|_{(L^p(\Omega))'}\left(\int_{\Omega_R}|u|^p(x)\,dx\right)^{1/p}=\|T\|_{(L^p(\Omega))'}\|u\|_{L^p(\Omega_R)}.\end{aligned}$$

(Notice that the above inequality can be written in the simpler form $\|S\|_{(L^p(\Omega_R))'}\leq\|T\|_{(L^p(\Omega))'}$).

By Step 1, there is a function

$$g \in L^p\left(\Omega_R\right),$$

such that

$$S(v) = \int_{\Omega_R} v(x)g(x)dx \quad \text{for all} \quad v \in L^p\left(\mathbb{R}^d\right).$$

For any function $f \in L^p(\Omega)$, let

$$e(f) \in L^p\left(\Omega_R\right), \quad e(f)(x) := \begin{cases} f(x) & \text{se} \quad x \in \Omega, \\ 0 & \text{se} \quad x \in \Omega_R \backslash \Omega. \end{cases}$$

Since

$$\pi(e(f))(x) = f(x) \quad \text{for all} \quad x \in \Omega,$$

we have that

$$T(f) = T(\pi(e(f))) = S(e(f)) = \int_{\Omega_R} e(f)(x)g(x)dx = \int_{\Omega} f(x)g(x)dx.$$

This concludes the proof for $1 < p \leq 2$ for bounded sets Ω.

Step 3. Proof for $1 < p \leq 2$ and for any Lebesgue measurable $\Omega \subset \mathbb{R}^d$.
For every natural $n \geq 1$, consider the set $\Omega_n := \Omega \cap B_n$ and define the extension operator

$$e_n : L^p(\Omega \cap B_n) \to L^p(\Omega)\ , \qquad e_n(f)(x) := \begin{cases} f(x) & \text{if} \quad x \in \Omega \cap B_n, \\ 0 & \text{if} \quad x \in \Omega \backslash B_n. \end{cases}$$

Consider the functional

$$T_n : L^p(\Omega \cap B_n) \to \mathbb{R}\ , \qquad T_n(f) = T(e_n(f)).$$

Then, T_n is a bounded linear functional on $L^p(\Omega \cap B_n)$. Indeed, for any $f \in L^p(\Omega \cap B_n)$, we have

$$|T_n(f)| = |T(e_n(f))| \leq \|T\|_{(L^p(\Omega))'} \|e_n(f)\|_{L^p(\Omega)} = \|T\|_{(L^p(\Omega))'} \|f\|_{L^p(\Omega \cap B_n)}.$$

(Notice that the above inequality can be written in the form $\|T_n\|_{(L^p(\Omega \cap B_n))'} \leq \|T\|_{(L^p(\Omega))'}$).
By the previous Step, we can find a function $g_n \in L^q(\Omega \cap B_n)$ such that

$$T_n(f) = \int_{\Omega \cap B_n} f(x)g_n(x)\,dx \quad \text{for all} \quad f \in L^p(\Omega \cap B_n).$$

We next consider the sequence of spaces

$$W_n := \Big\{ f \in L^p(\Omega)\ :\ f = 0 \ \text{ Lebesgue a.e. on } \ \Omega \setminus B_n \Big\},$$

and the functions

$$\widetilde{g}_n \in L^q(\Omega)\ , \qquad \widetilde{g}_n(x) = \begin{cases} g_n(x) & \text{if} \quad x \in \Omega \cap B_n, \\ 0 & \text{if} \quad x \in \Omega \backslash B_n. \end{cases}$$

We claim that

$$T(f) = \int_{\Omega} \widetilde{g}_n(x) f(x)\,dx \quad \text{for all} \quad f \in W_n.$$

Indeed, for every $f \in W_n$ we have that $f = e_n(f)$ and so

$$T(f) = T(e_n(f)) = \int_{\Omega \cap B_n} g_n(x) f(x)\,dx = \int_{\Omega} \widetilde{g}_n(x) f(x)\,dx.$$

The rest of the proof follows closely the argument from Step 1. Since the argument is very general, we give directly a more general statement in Proposition 3. This concludes the proof of Theorem 2 in the case $1 < p \leq 2$ for general Lebesgue measurable $\Omega \subset \mathbb{R}^d$. □

Proposition 3. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set in $\mathbb{R}^d$ and let $p \in (1, 2]$.*
Let T be a bounded linear functional on $L^p(\Omega)$ with

$$\|T\|_{(L^p(\Omega))'} = \sup \Big\{ T(f)\ :\ f \in L^p(\Omega),\ \|f\|_{L^p(\Omega)} \leq 1 \Big\}.$$

Suppose there is a sequence of linear subspaces $V_n \subset L^p(\Omega)$, $n \geq 1$, such that:

(a) **monotonicity:** *$V_n \subset V_m$ for every $m \geq n \geq 1$;*
(b) **density:** *for every $f \in L^p(\Omega)$ there is a sequence $f_n \in V_n$ converging to f strongly in $L^p(\Omega)$;*
(c) **representation of T on V_n:** *for every $n \geq 1$, there is a function $g_n \in L^q(\Omega)$ such that*

$$T(f) = \int_{\Omega} g_n(x) f(x)\,dx \quad \textit{for every} \quad f \in V_n;$$

(d) **norm of g_n:** *for every $n \geq 1$ we have*

$$\|g_n\|_{L^q(\Omega)} = \sup\Big\{T(f) \ : \ f \in V_n, \ \|f\|_{L^p(\Omega)} \leq 1\Big\}.$$

Then, the following holds:

(i) **boundedness:** $\|g_n\|_{L^q(\Omega)} \leq \|T\|_{(L^p(\Omega))'}$ *for every* $n \geq 1$;

(ii) **monotonicity:** $\|g_n\|_{L^q(\Omega)} \leq \|g_m\|_{L^q(\Omega)}$ *for every* $m \geq n \geq 1$;

(iii) **convergence:** *the sequence g_n converges strongly in $L^q(\Omega)$ to some $g \in L^q(\Omega)$;*

(iv) **representation of T on $L^p(\Omega)$:** *the limit function g satisfies*

$$T(f) = \int_\Omega g(x)f(x)\,dx \quad \textit{for every} \quad f \in L^p(\Omega).$$

Proof. In order to prove (i), we simply notice that

$$\begin{aligned}\|\widetilde{g}_n\|_{L^q(\Omega)} &= \sup\left\{\int_\Omega g_n(x)u(x)dx : u \in V_n, \|u\|_{L^p(\Omega)} \leq 1\right\}\\ &= \sup\left\{\int_\Omega g_n(x)u(x)dx : u \in V_n, \|u\|_{L^p(\Omega)} \leq 1\right\} \leq \|T\|_{(L^p(\Omega))'}.\end{aligned}$$

Now, in order to prove (ii), we fix $f \in V_n \subset V_m$ with $\|f\|_{L^p(\Omega)} \leq 1$ and we compute

$$\int_\Omega g_n f\,dx = T(f) = \int_\Omega g_m f\,dx \leq \|g_m\|_{L^q(\Omega)}\|f\|_{L_p(\Omega)} \leq \|g_m\|_{L^q(\Omega)}.$$

Taking the supremum over all $f \in V_n$ with $\|f\|_{L^p(\Omega)} \leq 1$ and using the hypothesis (c), we get (ii).

In order to prove (iii), we use the Clarkson's inequality with q:

$$\left\|\frac{g_m - g_n}{2}\right\|^q_{L^q(\Omega)} \leq \frac{1}{2}\|g_m\|^q_{L^q(\Omega)} + \frac{1}{2}\|g_n\|^q_{L^q(\Omega)} - \left\|\frac{g_m + g_n}{2}\right\|^q_{L^q(\Omega)}.$$

Next notice that for every fixed $f \in V_n \subset V_m$ with $\|f\|_{L^p} = 1$, we have

$$\left|\int_\Omega f_n(x)\frac{g_m(x) + g_n(x)}{2}dx\right| \leq \left\|\frac{g_m + g_n}{2}\right\|_{L^q(\Omega)}.$$

Thus,

$$\begin{aligned}\left\|\frac{g_m - g_n}{2}\right\|^q_{L^q(\Omega)} &\leq \frac{1}{2}\|g_m\|^q_{L^q(\Omega)} + \frac{1}{2}\|g_n\|^q_{L^q(\Omega)} - \left\|\frac{g_m + g_n}{2}\right\|^q_{L^q(\Omega)}\\ &\leq \frac{1}{2}\|g_m\|^q_{L^q(\Omega)} + \frac{1}{2}\|g_n\|^q_{L^q(\Omega)} - \left|\int_\Omega f(x)\frac{g_m(x) + g_n(x)}{2}dx\right|^q\\ &= \frac{1}{2}\|g_m\|^q_{L^q(\Omega)} + \frac{1}{2}\|g_n\|^q_{L^q(\Omega)} - \left|\int_\Omega f(x)g_n(x)dx\right|^q,\end{aligned}$$

where the last equality follows since

$$\int_\Omega f(x)\frac{g_m(x) + g_n(x)}{2}dx = \frac{1}{2}T(f) + \frac{1}{2}\int_\Omega f(x)g_n(x)\,dx = \int_\Omega f(x)g_n(x)\,dx.$$

Thus, we get

$$\left\|\frac{g_m - g_n}{2}\right\|^q_{L^q(\Omega)} \leq \frac{1}{2}\|g_m\|^q_{L^q(\Omega)} + \frac{1}{2}\|g_n\|^q_{L^q(\Omega)} - \left|\int_\Omega f(x)g_n(x)dx\right|^q.$$

Now, taking the supremum over $f \in V_n$ with $\|f\|_{L^p} \leq 1$ and using the hypothesis (d), we have

$$\begin{aligned}\left\|\frac{g_m - g_n}{2}\right\|^q_{L^q(\Omega)} &= \frac{1}{2}\|g_m\|^q_{L^q(\Omega)} + \frac{1}{2}\|g_n\|^q_{L^q(\Omega)} - \|g_n\|^q_{L^q(\Omega)}\\ &= \frac{1}{2}\|g_m\|^q_{L^q(\Omega)} - \frac{1}{2}\|g_n\|^q_{L^q(\Omega)}.\end{aligned}$$

Now, since the sequence of norms $\|g_n\|_{L^q(\Omega)}$ is bounded and monotone, it converges to some limit $L \in \mathbb{R}$. In particular, this implies that g_n is a Cauchy sequence in $L^q(\Omega)$ and so, it converges to some $g \in L^q(\Omega)$. This concludes the proof of (iii).

Finally, in order to prove (iv), for every $f \in L^p(\Omega)$ we consider a sequence $f_n \in V_n \subset L^p(\Omega)$ such that $f_n \to f$ in $L^p(\Omega)$. By the continuity of T we have $T(f_n) \to T(f)$. Thus,

$$T(u) = \lim_{n\to\infty} T(u_n) = \lim_{n\to\infty} \int_\Omega g_n(x)u_n(x)dx = \int_\Omega u(x)g(x)dx,$$

which concludes the proof of (iv). □

Proof of Theorem 2 in the case $2 \le p < +\infty$

In this case we don't have to distinguish between the case $|\Omega| < +\infty$ and $|\Omega| = +\infty$ as the argument allows to treat both cases at once.

Step 1. *Consider the kernel K of the map T:*

$$K := \{f \in L^p(\Omega) : T(f) = 0\}.$$

We claim that for every $u \in L^p(\Omega)$ there is a unique function $v \in K$ such that

$$\|u - v\|_{L^p(\Omega)} = \inf\Big\{\|u - \varphi\|_{L^p(\Omega)} \ : \ \varphi \in K\Big\}.$$

We will call v the projection of u on K.

If $u \in K$, we trivially have that $v = u$. Thus, we only need to consider the case $u \notin K$.
Let $\phi_n \in K$ be a minimizing sequence, that is

$$\lim_{n\to+\infty} \|u - \phi_n\|^p_{L^p(\Omega)} = I,$$

where we set

$$I := \inf\Big\{\|u - \varphi\|^p_{L^p(\Omega)} \ : \ \varphi \in K\Big\}.$$

Without loss of generality, we can assume that

$$\|u - \phi_m\|_{L^p(\Omega)} \le \|u - \phi_n\|_{L^p(\Omega)} \quad \text{for all} \quad m \ge n \ge 1.$$

We notice that, by the Clarkson's inequality in the case $p \ge 2$, we have

$$\begin{aligned}
\left\|\frac{\phi_m - \phi_n}{2}\right\|^p_{L^p(\Omega)} &= \left\|\frac{(u-\phi_m) - (u-\phi_n)}{2}\right\|^p_{L^p(\Omega)} \\
&\le \frac{1}{2}\|u - \phi_m\|^p_{L^p(\Omega)} + \frac{1}{2}\|u - \phi_n\|^p_{L^p(\Omega)} - \left\|u - \frac{\phi_m + \phi_n}{2}\right\|^p_{L^p(\Omega)} \\
&\le \frac{1}{2}\|u - \phi_m\|^p_{L^p(\Omega)} + \frac{1}{2}\|u - \phi_n\|^p_{L^p(\Omega)} - I,
\end{aligned}$$

where in the last inequality we used that $\frac{1}{2}(\phi_n + \phi_m) \in K$. This implies that $(\phi_n)_{n\ge1}$ is a Cauchy sequence in $L^p(\Omega)$. Thus, the limit

$$v := \lim_{n\to+\infty} \phi_n,$$

exists in $L^p(\Omega)$ and so

$$\|u - v\|_{L^p(\Omega)} = \lim_{n\to+\infty} \|u - \phi_n\|_{L^p(\Omega)} = I.$$

On the other hand, the continuity of T implies that

$$T(v) = \lim_{n\to+\infty} T(\phi_n) = 0,$$

which proves that $v \in K$. This proves the existence of an element $v \in K$ such that $\|u - v\|^p_{L^p} = I$. In order to prove the uniqueness of the projection v, suppose that there are two functions $v_1, v_2 \in K$ such that

$$\|u - v_1\|^p_{L^p(\Omega)} = \|u - v_2\|^p_{L^p(\Omega)} = I.$$

Using again the Clarkson's inequality and the fact that $\frac{1}{2}(v_1 + v_2) \in K$, we get that

$$\begin{aligned}
\left\|\frac{v_1 - v_2}{2}\right\|^p_{L^p(\Omega)} &= \left\|\frac{(u-v_1) - (u-v_2)}{2}\right\|^p_{L^p(\Omega)} \\
&\le \frac{1}{2}\|u - v_2\|^p_{L^p(\Omega)} + \frac{1}{2}\|u - v_2\|^p_{L^p(\Omega)} - \left\|u - \frac{v_1 + v_2}{2}\right\|^p_{L^p(\Omega)} \le \frac{I}{2} + \frac{I}{2} - I = 0,
\end{aligned}$$

which proves that $v_1 = v_2$. (We notice that, for the uniqueness of the minimizer v, it is sufficient to observe that we have already have that *every minimizing sequence converges*. This can only be true if the the minimizer is unique.)

Step 2. Let $u \in L^p(\Omega)$ such that $u \notin K$. Then, the unique projection v of u on K satisfies

$$\int_\Omega \varphi(x)(u-v)|u-v|^{p-2}dx = 0 \quad \text{for every} \quad \varphi \in K.$$

We will prove this fact in a separate lemma.

Step 3. Suppose that the map T is not identically zero. Then, we can find a function $u \in L^p(\Omega)$ such that

$$T(u) = 1.$$

Let v be the projection of u on K. Setting $w = u - v$, we have that

$$T(w) = 1$$

and

$$\int_\Omega \varphi w|w|^{p-2}dx = 0 \quad \text{for every} \quad \varphi \in K. \tag{4}$$

We next define the function

$$g := \frac{1}{\|w\|_{L^p}^p} w|w|^{p-2},$$

and we notice that $w \in L^q(\Omega)$. Now, for any $f \in L^p(\Omega)$, the linearity of T gives that

$$f - T(f)w \in K.$$

Using $\phi = f - T(f)w$ as test function in the integral identity (4), we get that

$$\int_\Omega g(f - T(f)w)dx = 0,$$

which can also be written as ,

$$\int_\Omega g(x)f(x)\,dx = T(f)\int_\Omega g(x)w(x)dx.$$

Finally, by the choice of g we have

$$\int_\Omega g(x)w(x)dx = \frac{1}{\|w\|_{L^p}^p}\int_\Omega |w(x)|^p dx = 1,$$

which implies that

$$\int_\Omega g(x)f(x)\,dx = T(f),$$

and concludes the proof. □