The dual space of $L^1(\Omega)$

A special class of functionals on $L^1(\Omega)$

Proposition 1. *Let $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set. Fixed a function $g \in L^\infty(\Omega)$, consider the linear functional*

$$T_g : L^1(\Omega) \to \mathbb{R}$$

defined as

$$T_g(f) = \int_\Omega f(x)g(x)\,dx \quad \text{for all} \quad f \in L^1(\Omega).$$

Then, T_g is a bounded operator and its norm

$$\|T_g\|_{(L^1(\Omega))'} := \sup\left\{T_g(f) : f \in L^1(\Omega), \|f\|_{L^1(\Omega)} \le 1\right\},$$

is given by

$$\|T_g\|_{(L^1(\Omega))'} = \|g\|_{L^\infty(\Omega)}.$$

Proof. The fact that T_g is bounded simply follows from the inequality

$$|T_g(f)| = \left|\int_\Omega f(x)g(x)dx\right| \le \|f\|_{L^1(\Omega)}\|g\|_{L^\infty(\Omega)} \quad \text{for all} \quad f \in L^1(\Omega),$$

which also proves that

$$\|T_g\|_{(L^1(\Omega))'} \le \|g\|_{L^\infty(\Omega)}.$$

In order to prove that an equality holds, we consider a sequence

$$t_n \uparrow \|g\|_{L^\infty(\Omega)}.$$

Since for each n we have $t_n < \|g\|_{L^\infty(\Omega)}$, we have that

$$|\{|g| > t_n\}| \neq 0.$$

Now, for any fixed $n \ge 1$, we pick a set of finite measure

$$\omega_n \subset \{|g| > t_n\},$$

and we consider the test function

$$f_n(x) := \frac{1}{|\omega_n|}\Big(\mathbb{1}_{\omega_n\cap\{g>t_n\}}(x) - \mathbb{1}_{\omega_n\cap\{g<-t_n\}}(x)\Big).$$

By construction we have that $f_n \in L^1(\Omega)$ and

$$\begin{aligned}\int_\Omega |f_n(x)|\,dx &= \int_\Omega \frac{1}{|\omega_n|}\Big(\mathbb{1}_{\omega_n\cap\{g>t_n\}}(x) + \mathbb{1}_{\omega_n\cap\{g<-t_n\}}(x)\Big)\,dx\\ &= \frac{1}{|\omega_n|}\Big(|\omega_n\cap\{g>t_n\}| + |\omega_n\cap\{g<-t_n\}|\Big) = 1.\end{aligned}$$

Then,

$$\begin{aligned}T_g(f_n) &= \int_\Omega g(x)f_n(x)\,dx\\ &= \int_\Omega g(x)\frac{1}{|\omega_n|}\Big(\mathbb{1}_{\omega_n\cap\{g>t_n\}}(x) - \mathbb{1}_{\omega_n\cap\{g<-t_n\}}(x)\Big)\,dx\\ &= \frac{1}{\omega_n}\left(\int_{\omega_n\cap\{g>t_n\}} g(x)dx - \int_{\omega_n\cap\{g<-t_n\}} g(x)dx\right)\\ &\ge \frac{1}{\omega_n}\left(\int_{\omega_n\cap\{g>t_n\}} t_n dx + \int_{\omega_n\cap\{g<-t_n\}} t_n dx\right)\\ &= \frac{t_n}{|\omega_n|}\Big(|\omega_n\cap\{g>t_n\}| + |\omega_n\cap\{g<-t_n\}|\Big) = t_n.\end{aligned}$$

This implies that

$$\begin{aligned}\|T_g\|_{(L^1(\Omega))'} &= \sup\left\{T_g(f) : f \in L^1(\Omega), \|f\|_{L^1(\Omega)} \le 1\right\}\\ &\ge \sup_{n\ge 1} T_g(f_n)\\ &\ge \sup_{n\ge 1} t_n\\ &= \|g\|_{L^\infty(\Omega)},\end{aligned}$$

so we have obtained the opposite inequality

$$\|T_g\|_{(L^1(\Omega))'} \ge \|g\|_{L^\infty(\Omega)},$$

which concludes the proof. □

A remark on the quantity $\|T_g\|_{(L^1(\Omega))'}$

We notice that, given a Lebesgue measurable set $\Omega \subset \mathbb{R}^d$ and a function $g \in L^\infty(\Omega)$, the norm

$$\|T_g\|_{(L^1(\Omega))'} := \sup\left\{T_g(f) : f \in L^1(\Omega), \|f\|_{L^1(\Omega)} \le 1\right\},$$

of the functional

$$T_g : L^1(\Omega) \to \mathbb{R}\ , \qquad T_g(f) = \int_\Omega f(x)g(x)\,dx,$$

might not be achieved. Here are few examples.

Proposition 2. *Consider $\Omega = \mathbb{R}$ and the function $g \in L^\infty(\mathbb{R})$ defined as*

$$g(x) = \begin{cases} 1 - \dfrac{1}{x} & \text{for} \quad x \ge 1;\\ 0 & \text{for} \quad x \le 1.\end{cases}$$

Then, the supremum

$$\|T_g\|_{(L^1(\mathbb{R}))'} := \sup\left\{T_g(f) = \int_{-\infty}^{+\infty} f(x)g(x)\,dx \ :\ f \in L^1(\mathbb{R}), \|f\|_{L^1(\mathbb{R})} \le 1\right\},$$

is not achieved.

Proof. We clearly have

$$\|g\|_{L^\infty(\mathbb{R})} = 1,$$

and so, thanks to Proposition 1,

$$\|T_g\|_{(L^1(\mathbb{R}))'} = \|g\|_{L^\infty(\mathbb{R})} = 1.$$

Suppose by contradiction that there is a function $f \in L^1(\mathbb{R})$ such that

$$\|f\|_{L^1(\mathbb{R})} = \int_{-\infty}^{+\infty} |f(x)|\,dx \le 1 \qquad \text{and} \qquad T_g(f) = \int_{-\infty}^{+\infty} f(x)g(x)\,dx = 1.$$

Then, for every $n \in \mathbb{N}$, $n \ge 1$, we have

$$\begin{aligned}1 = \int_{-\infty}^{+\infty} f(x)g(x)\,dx &= \int_{-\infty}^{n} f(x)g(x)\,dx + \int_{n}^{+\infty} f(x)g(x)\,dx\\ &\le \int_{-\infty}^{n} |f(x)|g(x)\,dx + \int_{n}^{+\infty} |f(x)|g(x)\,dx\\ &\le \left(1 - \frac{1}{n}\right)\int_{-\infty}^{n} |f(x)|\,dx + \int_{n}^{+\infty} |f(x)|\,dx\\ &= \int_{-\infty}^{+\infty} |f(x)|\,dx - \frac{1}{n}\int_{-\infty}^{n} |f(x)|\,dx \le 1 - \frac{1}{n}\int_{-\infty}^{n} |f(x)|\,dx.\end{aligned}$$

In particular, this implies that necessarily

$$\int_{-\infty}^{n} |f(x)|\,dx = 0.$$

Since n is arbitrary, we get that $f \equiv 0$ on $\mathbb{R}$, which is a contradiction since $|f|$ should integrate to 1 on $\mathbb{R}$. □

Proposition 3. *Consider $\Omega = B_1$, where B_1 is the unit ball in $\mathbb{R}^d$, with $d \geq 1$. Let $g \in L^\infty(B_1)$ be the function*

$$g(x) = 1 - |x|^2.$$

Then, the supremum

$$\|T_g\|_{(L^1(B_1))'} := \sup\left\{T_g(f) = \int_{B_1} f(x)g(x)\,dx \ : \ f \in L^1(B_1), \|f\|_{L^1(B_1)} \leq 1\right\},$$

is not achieved.

Proof. We clearly have that

$$\|g\|_{L^\infty(B_1)} = 1,$$

and so, thanks to Proposition 1,

$$\|T_g\|_{(L^1(B_1))'} = \|g\|_{L^\infty(B_1)} = 1.$$

Suppose by contradiction that there is a function $f \in L^1(B_1)$ such that

$$\|f\|_{L^1(B_1)} = \int_{B_1} |f(x)|\,dx \leq 1 \qquad \text{and} \qquad T_g(f) = \int_{B_1} f(x)g(x)\,dx = 1.$$

Then, for every $\varepsilon > 0$, we have

$$\begin{aligned}
1 = \int_{B_1} f(x)g(x)\,dx &= \int_{B_\varepsilon} f(x)g(x)\,dx + \int_{B_1\setminus B_\varepsilon} f(x)g(x)\,dx \\
&\leq \int_{B_\varepsilon} |f(x)|g(x)\,dx + \int_{B_1\setminus B_\varepsilon} |f(x)|g(x)\,dx \\
&\leq \int_{B_\varepsilon} |f(x)|\,dx + \left(1-\varepsilon^2\right)\int_{B_1\setminus B_\varepsilon} |f(x)|\,dx \\
&= \int_{B_1} |f(x)|\,dx - \varepsilon^2 \int_{B_1\setminus B_\varepsilon} |f(x)|\,dx \\
&\leq 1 - \varepsilon^2 \int_{B_1\setminus B_\varepsilon} |f(x)|\,dx.
\end{aligned}$$

In particular, this implies that necessarily

$$\int_{B_1\setminus B_\varepsilon} |f(x)|\,dx = 0.$$

Since $\varepsilon > 0$ is arbitrary, we get that $|f| \equiv 0$ on B_1, which is a contradiction. □

The examples from Propositions 2 and 3 can be deduced also from the following more general fact.

Proposition 4. *Let Ω be a Lebesgue measurable set in $\mathbb{R}^d$. Let $g \in L^\infty(\Omega)$ be a given function. Then, the following are equivalent:*

1. *the set $\{x \in \Omega \ : \ g(x) = L^\infty(\Omega)\}$ has non-zero Lebesgue measure;*
2. *the supremum*

$$\|T_g\|_{(L^1(\Omega))'} := \sup\left\{T_g(\phi) = \int_{\Omega} \phi(x)g(x)\,dx \ : \ \phi \in L^1(\Omega), \|\phi\|_{L^1(\Omega)} \leq 1\right\},$$

is achieved.

Proof. For simplicity we set

$$M := \|g\|_{L^\infty(\Omega)}.$$

Suppose first that (1) holds. Then, there is a set of measure set $\omega \subset \{g = M\}$ such that

$$0 < |\omega| < +\infty.$$

Then, the function

$$f(x) = \frac{1}{|\omega|}\mathbb{1}_\omega(x),$$

is such that

$$f \in L^1(\Omega)\ , \quad \|f\|_{L^1(\Omega)} = \int_\Omega f(x)\,dx = 1\ , \quad \text{and} \quad \int_\Omega f(x)g(x)\,dx = \frac{1}{\omega}\int_\omega g(x)\,dx = M.$$

Thus, we get the inequality

$$M = T_g(f) \leq \sup\left\{T_g(\phi) = \int_\Omega \phi(x)g(x)\,dx \ : \ \phi \in L^1(\Omega), \|f\|_{L^1(\Omega)} \leq 1\right\}.$$

On the other hand, Proposition 1 implies that

$$\|T_g\|_{(L^1(\Omega))'} \leq \|g\|_{L^\infty(\Omega)} = M,$$

which proves that

$$T_g(f) = \sup\left\{T_g(\phi) = \int_\Omega \phi(x)g(x)\,dx \ : \ \phi \in L^1(\Omega), \|\phi\|_{L^1(\Omega)} \leq 1\right\}.$$

This proves (2).

Suppose that (1) does not hold and suppose by contradiction that (2) holds. Then, there is a function $f \in L^1(\Omega)$ such that

$$\|f\|_{L^1(\Omega)} = \int_\Omega |f(x)|\,dx = 1 \qquad \text{and} \qquad T_g(f) = \int_\Omega f(x)g(x)\,dx = 1.$$

Fix a ball B_R in $\mathbb{R}^d$ and $\varepsilon > 0$. Then

$$\begin{aligned}
M = \int_\Omega f(x)g(x)\,dx &= \int_{\Omega\cap(B_R\cap\{|g|<M-\varepsilon\})} f(x)g(x)\,dx + \int_{\Omega\setminus(B_R\cap\{|g|<M-\varepsilon\})} f(x)g(x)\,dx \\
&\leq \int_{\Omega\cap(B_R\cap\{|g|<M-\varepsilon\})} |f|(x)|g|(x)\,dx + \int_{\Omega\setminus(B_R\cap\{|g|<M-\varepsilon\})} |f|(x)|g|(x)\,dx \\
&\leq (M-\varepsilon)\int_{\Omega\cap(B_R\cap\{|g|<M-\varepsilon\})} |f|(x)\,dx + M\int_{\Omega\setminus(B_R\cap\{|g|<M-\varepsilon\})} |f|(x)\,dx \\
&= M\int_\Omega |f(x)|\,dx - \varepsilon\int_{\Omega\cap(B_R\cap\{|g|<M-\varepsilon\})} |f|(x)\,dx \\
&\leq M - \varepsilon\int_{\Omega\cap(B_R\cap\{|g|<M-\varepsilon\})} |f|(x)\,dx.
\end{aligned}$$

In particular, this implies that

$$\int_{\Omega\cap(B_R\cap\{|g|<M-\varepsilon\})} |f|(x)\,dx = 0.$$

and in particular that $|f| = 0$ almost everywhere on the set $\Omega \cap (B_R \cap \{|g| < M - \varepsilon\})$. Since ε is arbitrary, we get that $|f| = 0$ a.e. on $\Omega \cap (B_R \cap \{|g| < M\})$. Since $|\{|g| = M\}| = 0$ we get that $|f| = 0$ a.e. on $\Omega \cap B_R$. Now, since R is arbitrary, we get that $|f| = 0$ a.e. on Ω. □

Representation theorem for the bounded linear functionals on $L^1(\Omega)$

Theorem 5. *Let $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set and let*

$$T : L^1(\Omega) \to \mathbb{R},$$

be a bounded linear functional on $L^1(\Omega)$. Then, there is a function $g \in L^\infty(\Omega)$ such that

$$T(f) = \int_\Omega f(x)g(x)dx \quad \text{for all} \quad f \in L^1(\Omega).$$

Moreover, the function g is unique.

We start with the following lemma, which guarantees the uniqueness of the function g representing T.

Lemma 6. *Let $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set. Suppose that $g_1 \in L^\infty(\Omega)$ and $g_2 \in L^\infty(\Omega)$ are two functions such that*

$$\int_\Omega f(x)g_2(x)dx = \int_\Omega f(x)g_1(x)dx \quad \text{for all} \quad f \in L^1(\Omega).$$

Then $g_1 = g_2$.

Proof. Suppose that g_1 and g_2 are two different elements of $L^\infty(\Omega)$. Then, the set $\{g_1 \neq g_2\}$ has a non-zero measure. In particular, there is $\varepsilon > 0$ for which the set $\{|g_2 - g_1| > \varepsilon\}$ has a non-zero Lebesgue measure. Then, there exists a measurable set $\omega_\varepsilon \subset \{|g_2 - g_1| > \varepsilon\}$ such that

$$0 < |\omega_\varepsilon| < +\infty.$$

Consider the function

$$f_\varepsilon(x) = \mathbb{1}_{\omega_\varepsilon \cap \{g_2 - g_1 > \varepsilon\}} - \mathbb{1}_{\omega_\varepsilon \cap \{g_2 - g_1 < -\varepsilon\}}.$$

Then

$$\begin{aligned} 0 &= \int_\Omega f_\varepsilon(x)\Big(g_2(x) - g_1(x)\Big)\,dx \\ &= \int_\Omega \Big(\mathbb{1}_{\omega_\varepsilon \cap \{g_2 - g_1 > \varepsilon\}} - \mathbb{1}_{\omega_\varepsilon \cap \{g_2 - g_1 < -\varepsilon\}}\Big)\Big(g_2(x) - g_1(x)\Big)\,dx \\ &= \int_\Omega \mathbb{1}_{\omega_\varepsilon \cap \{|g_2 - g_1| > \varepsilon\}} |g_2(x) - g_1(x)|\,dx \geq \varepsilon |\omega_\varepsilon|, \end{aligned}$$

which is a contradiction. □

Proof of Theorem 5. In what follows we will use the notation

$$\|T_g\|_{(L^1(\Omega))'} := \sup\left\{ T_g(\phi) = \int_\Omega \phi(x) g(x)\,dx \;:\; \phi \in L^1(\Omega), \|\phi\|_{L^1(\Omega)} \leq 1 \right\}.$$

In particular, this implies that

$$|T(f)| \leq \|T_g\|_{(L^1(\Omega))'} \|f\|_{L^1(\Omega)} \quad \text{for all} \quad f \in L^1(\Omega).$$

We proceed in two steps.

Step 1. We first consider the case $|\Omega| < +\infty$.
Construction of g. Since we have the inclusion

$$L^2(\Omega) \subset L^1(\Omega),$$

and the inequality

$$\|f\|_{L^1(\Omega)} \leq |\Omega|^{1/2} \|f\|_{L^2(\Omega)} \quad \text{for all} \quad f \in L^2(\Omega),$$

we have that the functional

$$T : L^2(\Omega) \to \mathbb{R},$$

is a bounded linear functional on $L^2(\Omega)$ and it holds

$$|T(f)| \leq \|T_g\|_{(L^1(\Omega))'} |\Omega|^{1/2} \|f\|_{L^2(\Omega)} \quad \text{for all} \quad f \in L^2(\Omega).$$

Thus, we can find a function $g \in L^2(\Omega)$ such that

$$T(f) = \int_\Omega f(x) g(x) dx \quad \text{for all} \quad f \in L^2(\Omega).$$

Boundedness of g. We will show that $|g| \leq \|T_g\|_{(L^1(\Omega))'}$ almost-everywhere on Ω.
Suppose that there exists a level $t > 0$ such that

$$t > \|T_g\|_{(L^1(\Omega))'} \quad \text{and} \quad |\{|g| > t\} \cap \Omega| > 0.$$

Consider the function

$$f(x) := \mathbb{1}_{\{g > t\} \cap \Omega}(x) - \mathbb{1}_{\{g < -t\} \cap \Omega}(x).$$

Observe that $f \in L^1(\Omega)$ and

$$\|f\|_{L^1(\Omega)} = \int_\Omega |f(x)| dx = \int_\Omega \Big(\mathbb{1}_{\{g > t\}}(x) - \mathbb{1}_{\{g < -t\}}(x)\Big) dx = |\{|g| > t\} \cap \Omega|.$$

We also have that $f \in L^2(\Omega)$, so by the definition of g, we get

$$\begin{aligned} T(f) = \int_\Omega f(x) g(x) dx &= \int_\Omega \Big(\mathbb{1}_{\{g > t\}}(x) - \mathbb{1}_{\{g < -t\}}(x)\Big) g(x) dx \\ &= \int_{\{|g| > t\} \cap \Omega} |g(x)| dx \geq t |\{|g| > t\} \cap \Omega|. \end{aligned}$$

On the other hand, since T is bounded on L^1, we have

$$T(f) \leq \|T_g\|_{(L^1(\Omega))'} \|f\|_{L^1(\Omega)} = \|T_g\|_{(L^1(\Omega))'} |\{|g| > t\} \cap \Omega|.$$

Thus, we have obtained

$$t|\{|g| > t\} \cap \Omega| \leq T(f) \leq \|T_g\|_{(L^1(\Omega))'} |\{g > t\} \cap \Omega|,$$

which is a contradiction. This implies that

$$g(x) \leq \|T_g\|_{(L^1(\Omega))'} \quad \text{for Lebesgue almost-every} \quad x \in \Omega.$$

In other words

$$g \in L^\infty(\Omega) \quad \text{and} \quad \|g\|_{L^\infty(\Omega)} \leq \|T_g\|_{(L^1(\Omega))'}.$$

Step 2. Suppose now that $|\Omega| = +\infty$.

For each $n \in \mathbb{N}$, $n \geq 1$, consider the set

$$\Omega_n := B_n \cap \Omega.$$

To each set Ω_n we associate the extension map

$$e_n : L^1(\Omega_n) \to L^1(\Omega), \quad e_n(f)(x) = \begin{cases} f(x) & \text{if} \quad x \in \Omega_n \\ 0 & \text{otherwise.} \end{cases}$$

For every $n \geq 1$, we define the operator

$$T_n : L^1(\Omega_n) \to \mathbb{R}, \quad T_n(f) := T(e_n(f)).$$

Then, T_n is linear and bounded: for each $f \in L^1(\Omega_n)$, we have

$$|T_n(f)| = |T(e_n(f))| \leq \|T\|_{(L^1(\Omega))'} \|e_n(f)\|_{L^1(\Omega)} \leq \|T\|_{(L^1(\Omega))'} \|f\|_{L^1(\Omega)}.$$

In other words, we have

$$\|T_n\|_{(L^1(\Omega_n))'} \leq \|T\|_{(L^1(\Omega))'}.$$

By Step 1, we can find $g_n \in L^\infty(\Omega_n)$ such that

$$\|g_n\|_{L^\infty(\Omega_n)} = \|T_n\|_{(L^1(\Omega_n))'} \leq \|T\|_{(L^1(\Omega))'} \quad \text{and} \quad T_n(f) = \int_{\Omega_n} f(x) g_n(x) dx \quad \text{for all} \quad f \in L^1(\Omega_n).$$

Consider now $m \geq n \geq 1$. Let $f \in L^1(\Omega)$. be a function such that

$$f \equiv 0 \quad \text{on} \quad \Omega \setminus B_n.$$

Since $B_n \subset B_m$, we also have that

$$f \equiv 0 \quad \text{on} \quad \Omega \setminus B_m.$$

In terms of the extensions operators $e_n : L^1(\Omega_n) \to L^1(\Omega)$ and $e_m : L^1(\Omega_m) \to L^1(\Omega)$, we have that

$$e_n(f|_{\Omega_n}) = e_m(f|_{\Omega_m}) = f.$$

In particular

$$T(f) = T(e_n(f|_{\Omega_n})) = T_n(f|_{\Omega_n}) = \int_{\Omega_n} f(x) g_n(x)\, dx,$$

and similarly

$$T(f) = T(e_m(f|_{\Omega_m})) = T_m(f|_{\Omega_m}) = \int_{\Omega_m} f(x) g_m(x)\, dx = \int_{\Omega_n} f(x) g_m(x)\, dx.$$

This implies that

$$\int_{\Omega_n} f(x) \Big(g_m - g_n \Big)\, dx = 0 \quad \text{for all} \quad f \in L^1(\Omega_n),$$

which in turn leads to

$$g_m = g_n \quad \text{on} \quad \Omega_n,$$

quad text for all $m \geq n$.

We define the function $g : \Omega \to \mathbb{R}$ as follows

$$g(x) = g_n(x) \quad \text{for all} \quad x \in \Omega_n \quad \text{and all} \quad n \geq 1.$$

By construction we have that

$$g \in L^\infty(\Omega) \qquad \text{and} \qquad \|g\|_{L^\infty(\Omega)} \leq \|T\|_{(L^1(\Omega))'}.$$

Consider now a function $f \in L^1(\Omega)$ and consider the sequence

$$f_n(x) = f(x)\mathbb{1}_{B_n}(x).$$

Then f_n converges in $L^1(\Omega)$ to f and in particular, by the continuity of T, we get

$$T(f) = \lim_{n\to+\infty} T(f_n).$$

On the other hand, since $f_n \equiv 0$ outside $\Omega_n = \Omega \cap B_n$, we have

$$T(f_n) = T(e_n(f_n)) = \int_{\Omega_n} f_n(x)g_n(x)\,dx = \int_{\Omega} f_n(x)g(x)\,dx.$$

Passing to the limit as $n \to +\infty$, we get

$$\lim_{n\to+\infty} T(f_n) = \lim_{n\to+\infty} \int_{\Omega} f_n(x)g(x)\,dx = \int_{\Omega} f(x)g(x)\,dx,$$

which concludes the proof. □