Hahn-Banach theorems

Theorem 1 (Hahn-Banach I)**.** *Let $\mathcal{B}$ be a Banach space. Let V be a linear subspace of $\mathcal{B}$, C be a positive constant and $T : V \to \mathbb{R}$ be a linear application such that*

$$|T(v)| \leq C\|v\|_{\mathcal{B}} \quad \text{for every} \quad v \in V.$$

Then, there is a linear functional $S : \mathcal{B} \to \mathbb{R}$ such that:

(1) $S(v) = T(v)$ *per ogni* $v \in V$*;*

(2) $|S(b)| \leq C\|b\|_{\mathcal{B}}$ *for every* $b \in \mathcal{B}$.

Theorem 2 (Hahn-Banach II)**.** *Let $\mathcal{B}$ be a Banach space and let V be a closed linear subspace of $\mathcal{B}$. Then, there exists a bounded linear functional*

$$T : \mathcal{B} \to \mathbb{R}$$

such that $T \not\equiv 0$ and

$$T(x) = 0 \quad \text{for every} \quad x \in V.$$

The main lemma

The next lemma is the main ingredient in the proofs of Theorem 1 and Theorem 2.

Lemma 3. *Let $\mathcal{B}$ be a Banach space. Let V be a linear subspace of $\mathcal{B}$, C be a positive constant, and $T : V \to \mathbb{R}$ be a linear map such that*

$$|T(v)| \leq C\|v\|_{\mathcal{B}} \quad \text{for all} \quad v \in V.$$

Given a vector $z \in \mathcal{B}\backslash V$, let W be the linear subspace of $\mathcal{B}$ generated by V and z, that is:

$$W := \Big\{u + \alpha z : u \in V, \alpha \in \mathbb{R}\Big\}.$$

Then, there is a linear map

$$S : W \to \mathbb{R},$$

such that

(1) $S(v) = T(v)$ *for every* $v \in V$*;*

(2) $|S(w)| \leq C\|w\|_{\mathcal{B}}$ *for every* $w \in W$.

Proof. We notice that, since $z \notin V$, every element w of W can be written in a unique way in the form

$$w = u + \alpha z \quad \text{with} \quad v \in V \quad \text{and} \quad \alpha \in \mathbb{R}.$$

In particular, if we suppose that there is a linear map $S = W \to \mathbb{R}$ that extends $T : V \to \mathbb{R}$, then

$$S(w) = S(v + \alpha z) = S(v) + \alpha S(z) = T(v) + \alpha S(z),$$

which implies that:

$$\text{the map } S \text{ is uniquely determined by its value } S(z) \text{ in } z.$$

In what follows we will determine $S(z)$ in such a way that

$$|S(w)| \leq C\|w\|_{\mathcal{B}} \quad \text{for all} \quad w \in W.$$

In order to do this, for every $u, v \in V$ and every couple of positive real constants $\alpha, \beta > 0$, we write

$$\begin{aligned} T(\alpha u + \beta v) &\leq C\|\alpha u + \beta v\|_{\mathcal{B}} \\ &= C\|\alpha(u - \beta z) + \beta(v + \alpha z)\|_{\mathcal{B}} \\ &\leq C\|\alpha(u - \beta z)\|_{\mathcal{B}} + C\|\beta(v + \alpha z)\|_{\mathcal{B}} \\ &= C\alpha\|u - \beta z\|_{\mathcal{B}} + C\beta\|v + \alpha z\|_{\mathcal{B}}. \end{aligned}$$

We can rewrite the above inequality as:

$$\alpha\left(T(u) - C\|u - \beta z\|_{\mathcal{B}}\right) \leq -\beta\left(T(v) - C\|v + \alpha z\|_{\mathcal{B}}\right).$$

Since $\alpha > 0$ and $\beta > 0$ we can divide both sides by $\alpha\beta$, obtaining

$$\frac{1}{\beta}\left(T(u) - C\|u - \beta z\|_{\mathcal{B}}\right) \leq -\frac{1}{\alpha}\left(T(v) - C\|v + \alpha z\|_{\mathcal{B}}\right),$$

where we notice that the right-hand side depends only on u and β, while the right hand-side depends only on v and β. We next consider the sets of real numbers

$$\mathcal{I}_1 := \left\{ \frac{1}{\beta}\Big(T(u) - C\|u - \beta z\|_{\mathcal{B}}\Big) \ : \ u \in V,\ \beta > 0 \right\},$$

$$\mathcal{I}_2 := \left\{ -\frac{1}{\alpha}\Big(T(v) - C\|v + \alpha z\|_{\mathcal{B}}\Big) \ : \ v \in V,\ \alpha > 0 \right\}.$$

Since

$$a_1 \leq a_2 \quad \text{for all} \quad a_1 \in \mathcal{I}_1,\ a_2 \in \mathcal{I}_2\,,$$

we have that

$$\sup \mathcal{I}_1 \leq \inf \mathcal{I}_2.$$

Then, there is a real number $s \in \mathbb{R}$ such that

$$\sup \mathcal{I}_1 \leq s \leq \inf \mathcal{I}_2.$$

We define

$$S(z) := s.$$

By construction we have:

$$\begin{array}{lll} S(u - \beta z) & = T(u) - \beta s \leq C\|u - \beta z\|_{\mathcal{B}} & \text{for all} \quad u \in V, \beta > 0; \\ S(v + \alpha z) & = T(v) + \alpha s \leq C\|v + \alpha z\|_{\mathcal{B}} & \text{for all} \quad v \in V, \alpha > 0. \end{array}$$

This implies that

$$S(w) \leq C\|w\|_{\mathcal{B}} \quad \text{for all} \quad w \in W,$$

which, since S is linear, gives:

$$|S(w)| \leq C\|w\|_{\mathcal{B}} \quad \text{for all} \quad w \in W,$$

which concludes the proof of the lemma. □

Proof of Theorem 1 for separable spaces $\mathcal{B}$

We will prove Theorem 1 in full generality in the next section. Here, we only give an alternative proof of Theorem 1, which is based on a different construction, but works only for separable spaces.

Suppose that $\mathcal{B}$ is separable in the sense of the following definition.

Definition 4. *We say that a Banach space $\mathcal{B}$ is* **separable** *if there is a countable set $\mathcal{C} \subset \mathcal{B}$, dense in $\mathcal{B}$.*

Let $\mathcal{C} = \{\phi_n\}_{n\geq 1}$ be a countable dense subset of $\mathcal{B}$.
Consider the sequence of linear subspaces $V_n \subset \mathcal{B}$ defined as follows:

$$V_0 := V \quad \text{and} \quad T_0 = T : V_0 \to \mathbb{R},$$

and for every $n \geq 1$, we consider the space V_n defined as

$$V_n := \begin{cases} V_{n-1} & \text{if } \phi_n \in V_{n-1}; \\ \{u + \alpha\phi_n : u \in V_{n-1}, \alpha \in \mathbb{R}\} & \text{if } \phi_n \notin V_{n-1}. \end{cases}$$

and the map

$$T_n : V_n \to \mathbb{R}$$

constructed in Lemma 3 by extending T_{n-1} from V_{n-1} to V_n. We notice that, for every $n \geq k$ we have:

$$V_k \subset V_n \quad \text{and} \quad T_k = T_n \quad \text{on} \quad V_k.$$

On the vector space

$$V_\infty := \bigcup_{n\geq 1} V_n,$$

we consider the map

$$T_\infty : V_\infty \to \mathbb{R}$$

defined as

$$T_\infty(u) = T_k(u) \quad \text{per} \quad u \in V_k.$$

By construction, we have

$$|T_\infty(u)| \leq C\|u\|_{\mathcal{B}} \quad \text{for all} \quad u \in V_\infty.$$

Since V_∞ is a dense subspace of $\mathcal{B}$, we can extend the map T_∞ to a map $T : \mathcal{B} \to \mathbb{R}$ as follows: for every $u \in \mathcal{B}$ we define $T(u)$ as

$$T(u) = \lim_{n\to\infty} T_\infty(u_n),$$

where

$$u_n \in V_\infty \quad \text{for all} \quad n \geq 1 \quad \text{and where} \quad \lim_{n\to+\infty} \|u_n - u\|_{\mathcal{B}} = 0.$$

Since T_∞ is bounded we have that T is well defined (that is, the value $T(u)$ does not depend on the sequence u_n) and $T$is a bounded linear map on $\mathcal{B}$ satisfying

$$|T(u)| \leq C\|u\|_{\mathcal{B}} \quad \text{for all} \quad u \in \mathcal{B},$$

which concludes the proof of Theorem 1 for separable Banach space $\mathcal{B}$. □

Proof of Theorem 1 in the general case

Consider the family $\mathcal{X}$ of couples (W, S) such that:

- $V \subset W \subset \mathcal{B}$;
- $S : W \to \mathbb{R}$is a bounded linear functional on W satisfying

$$|S(w)| \leq C\|w\|_{\mathcal{B}} \quad \text{for all} \quad w \in W;$$

- $S \equiv T$ su V.

On the set $\mathcal{X}$ we consider the following order relation: we say that $(W_1, S_1) \leq (W_2, S_2)$, if

$$W_1 \subset W_2 \subset \mathcal{B} \quad \text{and} \quad S_2 \equiv S_1 \quad \text{on} \quad W_1.$$

Given a chain (a totally ordered subset) $\mathcal{C}$ in $\mathcal{X}$, consider the couple $(W_\infty, S_\infty) \in \mathcal{X}$ defined as

$$W_\infty := \bigcup_{(W,S)\in\mathcal{C}} W; \quad S_\infty(w) := S(w) \quad \text{for all} \quad w \in W \quad \text{and all} \quad (W, S) \in \mathcal{C}.$$

By construction, $(W_\infty, S_\infty) \in \mathcal{X}$ is an upper bound for the chain $\mathcal{C}$. By the Zorn's Lemma, there is a maximum

$$(W_{\max}, S_{\max}) \in \mathcal{X}.$$

Finally, thanks to Lemma 3, we have that necessarily $W_{\max} = \mathcal{B}$. □

Proof of Theorem 2

Let z be any element in $\mathcal{B}\backslash V$. Since the linear subspace V is closed, there exists a radius $\varepsilon > 0$ such that

$$B_\varepsilon(z) \subset \mathcal{B}\backslash V.$$

Let W be the linear subspace generated by V and z :

$$W = \{v + \alpha z : v \in V, \alpha \in \mathbb{R}\}.$$

We will define a bounded linear function $S : W \to \mathbb{R}$ such that

$$S(z) \neq 0 \quad \text{and} \quad S(v) = 0 \quad \text{for all} \quad v \in V$$

As in the proof of Lemma 3, we notice that every element w of W can be written in a unique way in the form

$$w = v + \alpha z \in W \quad \text{with} \quad v \in V \quad \text{and} \quad \alpha \in \mathbb{R}.$$

Thus, it is sufficient to choose the vale of S and z and then to define S as

$$S(w) = \alpha S(z).$$

In order to choose $S(z)$ we notice that since $B_\varepsilon(z) \cap V = \emptyset$, we have that

$$\inf_{v\in V} \|v - z\| \geq \varepsilon.$$

In particular, choosing

$$S(z) := \varepsilon > 0,$$

we obtain that for all $\alpha > 0$ we have the inequalities

$$0 \leq S(v + \alpha z) = \alpha S(z) = \alpha\varepsilon \leq \alpha \left\| \frac{1}{\alpha} v + z \right\|_{\mathcal{B}} = \|v + \alpha z\|_{\mathcal{B}},$$

$$0 \geq S(v - \alpha z) = -\alpha S(z) = -\alpha\varepsilon \geq -\alpha \left\| \frac{1}{\alpha} v - z \right\|_{\mathcal{B}} = -\|v - \alpha z\|_{\mathcal{B}}.$$

As a consequence, for every $\alpha \in \mathbb{R}$, we have:

$$|S(v + \alpha z)| \leq \|v + \alpha z\|_{\mathcal{B}} \quad \text{for all} \quad \alpha \in \mathbb{R},$$

which can be also written as

$$|S(w)| \leq \|w\|_{\mathcal{B}} \quad \text{for all} \quad w \in W.$$

By the first Hahn-Banach's theorem (Theorem 1), we can extend S to a bounded linear functional $T : \mathcal{B} \to \mathbb{R}$. □

A corollary of the first Hahn-Banach theorem

Corollary 5. *Let $\mathcal{B}$ be a Banach space. Then, for every $b \in \mathcal{B}$, there is a bounded linear functional*

$$T : \mathcal{B} \to \mathbb{R}$$

such that

$$T(b) = \|b\|_{\mathcal{B}} \qquad \text{and} \qquad \|T\|_{\mathcal{B}'} = 1,$$

where

$$\|T\|_{\mathcal{B}'} := \sup \Big\{ T(v) \ : \ v \in \mathcal{B}, \ \|v\|_{\mathcal{B}} \leq 1 \Big\}.$$

Proof. Consider the linear subspace V generated by b:

$$V = \big\{ \alpha b \ : \ \alpha \in \mathbb{R} \big\},$$

and the linear functional

$$S : V \to \mathbb{R}$$

defined as

$$S(\alpha b) = \alpha \|b\|_{\mathcal{B}}.$$

By construction we have that

$$S(b) = \|b\|_{\mathcal{B}} \quad \text{and} \quad |S(v)| \leq \|v\|_{\mathcal{B}} \quad \text{for all} \quad v \in V.$$

By the first theorem of Hahn-Banach we can find a functional

$$T : \mathcal{B} \to \mathbb{R}$$

such that

$$T = S \quad \text{on} \quad V,$$

and

$$|T(u)| \leq \|u\|_{\mathcal{B}} \quad \text{for all} \quad u \in \mathcal{B}.$$

This concludes the proof. □

In some cases the map T can be written explicitly.

Example 6. *Let $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set and let $p \in (1, +\infty)$. Let $\mathcal{B}$ be the Banach space $L^p(\Omega)$ and let $u \in L^p(\Omega)$ be a given non-zero function. Then, the function*

$$v(x) := \frac{1}{\|u\|_{L^p(\Omega)}}^{p-1} u(x)|u(x)|^{p-2},$$

is in $L^q(\Omega)$ and satisfies

$$\|v\|_{L^q(\Omega)} = 1 \quad \text{and} \quad \int_\Omega u(x)v(x)\,dx = \|u\|_{L^p(\Omega)}.$$

Thus, the functional

$$T_v : L^p(\Omega) \to \mathbb{R} \ , \qquad T_v(\phi) = \int_\Omega \phi(x)v(x)\,dx,$$

is a bounded linear functional on $L^p(\Omega)$ that satisfies

$$\|T_v\|_{(L^p(\Omega))'} = 1 \quad \text{and} \quad T_v(u) = \|u\|_{L^p(\Omega)}.$$

Example 7 (Example of a Banach space in which T is NOT unique)**.** *Let $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set. Let $\mathcal{B}$ be the Banach space $L^1(\Omega)$ and let $u \in L^1(\Omega)$ be a given non-zero function. Let $A_+ \subset \Omega$ and $A_- \subset \Omega$ be disjoint measurable sets such that*

$$\{g > 0\} \subset A_+ \quad \text{and} \quad \{g < 0\} \subset A_-, \tag{1}$$

and let

$$v(x) = \mathbb{1}_{A_+}(x) - \mathbb{1}_{A_-}(x).$$

Then, $v \in L^\infty(\Omega)$,

$$\|v\|_{L^\infty(\Omega)} = 1 \quad \text{and} \quad \int_\Omega u(x)v(x)\,dx = \|u\|_{L^1(\Omega)}.$$

Thus, the functional

$$T_v : L^1(\Omega) \to \mathbb{R}\ , \qquad T_v(\phi) = \int_\Omega \phi(x)v(x)\,dx,$$

is a bounded linear functional on $L^1(\Omega)$ that satisfies

$$\|T_v\|_{(L^1(\Omega))'} = 1 \quad \text{and} \quad T_v(u) = \|u\|_{L^1(\Omega)}. \tag{2}$$

In particular, this example shows that the map T might not be unique. Indeed, it is sufficient to consider a function $g \in L^1(\Omega)$ such that

$$|\Omega \setminus \{g \neq 0\}| > 0.$$

Then, there are multiple different choices of A_+ and A_- satisfying (1). Thus, there are multiple choices of functions $v(x) = \mathbb{1}_{A_+}(x) - \mathbb{1}_{A_-}(x)$ with associated operators T_v for which (2) holds.

Example 8 (Example of a Banach space in which T is unique)**.** *Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Then, for every non-zero vector $v \in \mathcal{H}$, there is a unique bounded linear functional $T : \mathcal{H} \to \mathbb{R}$ such that*

$$\|T\|_{\mathcal{H}'} = 1 \quad \text{and} \quad T(v) = \|v\|. \tag{3}$$

Let T be a functional satisfying (3). Since $\mathcal{H}$ is a Hilbert space, there is a vector $u \in \mathcal{H}$ such that

$$T(\phi) = \langle u, \phi \rangle \quad \text{for all} \quad \phi \in \mathcal{H}.$$

Moreover, we have that

$$\|T\|_{\mathcal{H}'} = \|u\| = 1.$$

Then,

$$\|v\| = T(v) = \langle u, v \rangle \leq \|u\|\|v\| = \|v\|.$$

Thus, we get that

$$\langle u, v \rangle = \|u\|\|v\| = \|v\|,$$

which implies

$$\|v - \|v\|u\|^2 = \|v\|^2 + \|v\|^2\|u\|^2 - 2\|v\|\langle u, v \rangle = \|v\|^2 + \|v\|^2 - 2\|v\|^2 = 0.$$

Thus, we obtain

$$u = \frac{v}{\|v\|},$$

which proves that T is unique.