The dual space of $L^\infty(\Omega)$ contains $L^1(\Omega)$, but is not $L^1(\Omega)$

The dual space of $L^\infty(\Omega)$ contains $L^1(\Omega)$

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set. For every function $g \in L^1(\Omega)$, we define*

$$T_g : L^\infty(\Omega) \to \mathbb{R}$$

to be the functional acting on $L^\infty(\Omega)$ in the following way:

$$T_g(f) = \int_\Omega f(x)g(x)dx \quad \text{for all} \quad f \in L^\infty(\Omega).$$

Then, T_g is a bounded linear functional on $L^\infty(\Omega)$ and its norm

$$\|T_g\|_{(L^\infty(\Omega))'} := \sup\left\{T_g(f) : f \in L^\infty(\Omega), \|f\|_{L^\infty(\Omega)} \le 1\right\}$$

is given by

$$\|T_g\|_{(L^\infty(\Omega))'} = \|g\|_{L^1(\Omega)}.$$

Proof. Since the Lebesgue integral is linear, also the functional T_g is linear.
The boundedness of T_g follows from the inequality

$$|T_g(f)|_{(L^\infty(\Omega))'} = \left|\int_\Omega f(x)g(x)dx\right| \le \|f\|_{L^\infty(\Omega)}\|g\|_{L^1(\Omega)} \quad \text{for all} \quad f \in L^\infty(\Omega),$$

which also gives the inequality

$$\|T_g\|_{(L^\infty(\Omega))'} \le \|g\|_{L^1(\Omega)}.$$

In order to prove that we have an equality, we consider the function

$$f(x) := \mathbb{1}_{\{g>0\}}(x) - \mathbb{1}_{\{g<0\}}(x).$$

Then, $\|f\|_{L^\infty(\Omega)} = 1$ and

$$T_g(f) = \int_\Omega f(x)g(x)dx = \int_\Omega \Big(\mathbb{1}_{\{g>0\}}(x) - \mathbb{1}_{\{g<0\}}(x)\Big)g(x)dx = \int_\Omega |g(x)|dx = \|g\|_{L^1(\Omega)}.$$

which implies that

$$\|T_g\|_{(L^\infty(\Omega))'} = \sup\left\{T_g(f) : f \in L^\infty(\Omega), \|f\|_{L^\infty(\Omega)} \le 1\right\} \ge T_g(f) = \|g\|_{L^1(\Omega)},$$

and provides the equality

$$\|T_g\|_{(L^\infty(\Omega))'} = \|g\|_{L^1(\Omega)},$$

which concludes the proof. □

The dual space of $L^\infty(\Omega)$ is not $L^1(\Omega)$

Theorem 2. *Let Ω be a Lebesgue measurable set in $\mathbb{R}^d$. Then, there is a bounded linear functional*

$$T : L^\infty(\Omega) \to \mathbb{R},$$

for which there is NO function $g \in L^1(\Omega)$ such that

$$T(f) = \int_\Omega f(x)g(x)dx \quad \text{for all} \quad f \in L^\infty(\Omega).$$

We will give two different proofs of this fact, based on the two Hahn-Banach theorems from the previous section.

First proof of Theorem 2. Consider the space of functions

$$V = \Big\{ u : \Omega \to \mathbb{R} \ : \ u \in L^\infty(\Omega) \text{ and there is a function } \phi \in C_c(\mathbb{R}^d) \text{ such that } u = \phi \text{ a.e. on } \Omega \Big\},$$

where $C_c(\mathbb{R}^d)$ is the space of continuous functions on $\mathbb{R}^d$ with compact support. Clearly, V is a linear subspace of $L^\infty(\Omega)$. Pick a point $x_0 \in \mathbb{R}^d$ such that

$$\lim_{r \to 0} \frac{|\Omega \cap B_r(x_0)|}{|B_r|} = 1.$$

We notice that the property above implies that

$$|\Omega \cap B_r(x_0)| > 0 \quad \text{for all} \quad r > 0.$$

(Remark: If Ω has non-empty interior, then we can take as x_0 any point in the interior of Ω.)

Consider the functional

$$\delta_{x_0} : V \to \mathbb{R},$$

defined as

$$\delta_{x_0}(u) := \lim_{r \to 0} \frac{1}{|\Omega \cap B_r(x_0)|} \int_{\Omega \cap B_r(x_0)} u(x)\, dx,$$

which is well defined thanks to the fact that u coincides a.e. on Ω with a function ϕ, which is continuous in x_0. In fact, we have

$$\delta_{x_0}(u) := \lim_{r \to 0} \frac{1}{|\Omega \cap B_r(x_0)|} \int_{\Omega \cap B_r(x_0)} u(x)\, dx = \lim_{r \to 0} \frac{1}{|\Omega \cap B_r(x_0)|} \int_{\Omega \cap B_r(x_0)} \phi(x)\, dx = \phi(x_0).$$

Clearly δ_{x_0} is a linear functional on V. Moreover, δ_0 is bounded on V; for any $u \in V$ and $r > 0$, we have

$$\frac{1}{|\Omega \cap B_r(x_0)|} \int_{\Omega \cap B_r(x_0)} u(x)\, dx \le \frac{1}{|\Omega \cap B_r(x_0)|} \int_{\Omega \cap B_r(x_0)} |u(x)|\, dx \le \|u\|_{L^\infty(\Omega)}.$$

By the first theorem of Hahn-Banach, we can extend δ_{x_0} to a bounded linear functional

$$T : L^\infty(\Omega) \to \mathbb{R},$$

with norm

$$\|T\|_{(L^\infty(\Omega))'} = 1.$$

We will show that T cannot be written in the form $T = T_g$ for some function $g \in L^1(\Omega)$, where T_g is the functional associated to g from Theorem 1. Suppose by contradiction that there is $g \in L^1(\Omega)$ such that $T = T_g$. Then, g is non-zero. Indeed, by Theprem 1,

$$1 = \|T\|_{(L^\infty(\Omega))'} = \|T_g\|_{(L^\infty(\Omega))'} = \|g\|_{L^1(\Omega)}.$$

Next, consider the sequence of continuous functions

$$\phi_n \mathbb{R}^d \to \mathbb{R} \ , \qquad \phi_n(x) := (1 - n|x - x_0|)_+ \, .$$

By construction, we have:

$$\phi_n \in C(\mathbb{R}^d) \ , \quad \|\phi_n\|_{L^\infty(\mathbb{R}^d)} = 1 \ , \quad \phi_n(x_0) = 1 \ , \quad \phi_n \equiv 0 \quad \text{in} \quad \mathbb{R}^d \setminus B_{1/n}(x_0).$$

We now apply T to the (restriction on Ω of) ϕ_n. From one side, we have

$$T(\phi_n) = \delta_{x_0}(\phi_n) = \phi_n(x_0) = 1 \quad \text{for each} \quad n \ge 1.$$

On the other hand,

$$T(\phi_n) = T_g(\phi_n) = \int_\Omega \phi_n(x) g(x)\, dx \le \int_{\Omega \cap B_{1/n}(x_0)} |g(x)|\, dx,$$

so passing to the limit as $n \to +\infty$, we get

$$\lim_{n \to +\infty} T(\phi_n) \le \lim_{n \to +\infty} \int_{\Omega \cap B_{1/n}(x_0)} |g(x)|\, dx = 0,$$

which is a contradiction. □

Second proof of Theorem 2. We consider again the space of functions

$$V = \Big\{ u : \Omega \to \mathbb{R} \ : \ u \in L^\infty(\Omega) \text{ and there is a function } \phi \in C_c(\mathbb{R}^d) \text{ such that } u = \phi \text{ a.e. on } \Omega \Big\},$$

where as above $C_c(\mathbb{R}^d)$ is the space of continuous functions on $\mathbb{R}^d$ with compact support. Consider the closure W of V in $L^\infty(\Omega)$ (with respect to the norm $\|\cdot\|_{L^\infty(\Omega)}$). Then,

$$W \text{ is a closed linear subspace of } L^\infty(\Omega) \text{ and } W \neq L^\infty(\Omega).$$

By the second theorem of Hahn-Banach, there is a bounded linear functional

$$T : L^\infty(\Omega) \to \mathbb{R},$$

such that

$$\|T\|_{(L^\infty(\Omega))'} = 1 \qquad \text{and} \qquad T \equiv 0 \quad \text{on} \quad W.$$

Suppose, by contradiction, that there is $g \in L^1(\Omega)$ such that $T = T_g$. Then, by Theprem 1, we have

$$1 = \|T\|_{(L^\infty(\Omega))'} = \|T_g\|_{(L^\infty(\Omega))'} = \|g\|_{L^1(\Omega)},$$

which in particular gives that g is non-zero. Consider the extension

$$\widetilde{g} \in L^1(\mathbb{R}^d) \ , \qquad \widetilde{g}(x) = \begin{cases} g(x) & \text{if} \quad x \in \Omega, \\ 0 & \text{if} \quad x \in \mathbb{R}^d \setminus \Omega. \end{cases}$$

By construction, for all functions $\psi \in C_c(\mathbb{R}^d)$, we have that

$$\int_{\mathbb{R}^d} \widetilde{g}(x)\psi(x)\,dx = \int_\Omega g(x)\psi(x)\,dx = T_g(\psi) = T(\psi) = 0.$$

At this point, we can conclude the proof in two different ways.

First way: by approximation argument using the convolution. Fix a function $\phi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \phi(x)\,dx = 1 \ , \quad \phi \geq 0 \quad \text{on} \quad \mathbb{R}^d \ , \quad \phi(x) = \phi(-x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^d,$$

and its rescalings

$$\phi_\varepsilon(x) := \varepsilon^d \phi(x/\varepsilon).$$

Then, we know that the convolution

$$(\widetilde{g} * \phi_\varepsilon)(x) = \int_{\mathbb{R}^d} \phi_\varepsilon(x-y)\widetilde{(y)}\,dy,$$

is a continuous function on $\mathbb{R}^d$ and

$$\lim_{\varepsilon \to 0} \|\widetilde{g} * \phi_\varepsilon - \widetilde{g}\|_{L^1(\mathbb{R}^d)} = 0.$$

Now, for all $\psi \in C_c(\mathbb{R}^d)$ we have that $\phi_\varepsilon * \psi \in C_c^\infty(\mathbb{R}^d)$, so we have

$$T(\psi_\varepsilon * \psi) = 0.$$

Thus,

$$\int_{\mathbb{R}^d} (\widetilde{g} * \phi_\varepsilon)(x)\psi(x)\,dx = \int_{\mathbb{R}^d} \widetilde{g}(y)(\psi * \phi_\varepsilon)(y)\,dy = T(\psi_\varepsilon * \psi) = 0,$$

for all $\psi \in C_c(\mathbb{R}^d)$. Since $\widetilde{g} * \phi_\varepsilon$ is continuous this implies that $\widetilde{g} * \phi_\varepsilon \equiv 0$ on $\mathbb{R}^d$. But then, passing to the limit as $\varepsilon \to 0$, we get that $\widetilde{g} \equiv 0$ on $\mathbb{R}^d$ which is a contradiction.

Second way: by the regularity of the Lebesgue measure. Suppose that $\widetilde{g} \neq 0$. Then, there are $R > 0$ and $t > 0$ such that

$$|B_R \cap \{|\widetilde{g}| > t\}| = |B_R \cap \{\widetilde{g} > t\}| + |B_R \cap \{\widetilde{g} < -t\}| > 0.$$

Without loss of generality, we can suppose that

$$|B_R \cap \{\widetilde{g} > t\}| > 0.$$

We recall the following properties of the Lebesgue measure:

- By the regularity of the Lebesgue measure, for every $\delta > 0$, we can find a compact set $K \subset \mathbb{R}^d$ and a (bounded) open set $A \subset \mathbb{R}^d$ such that:

$$\Big| K \setminus \Big(B_R \cap \{\widetilde{g} > t\} \Big) \Big| = 0, \quad \Big| \Big(B_R \cap \{\widetilde{g} > t\} \Big) \setminus A \Big| = 0, \quad A \subset K, \quad |A \setminus K| \leq \delta.$$

- For every $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\int_\omega |\widetilde{g}| \leq \varepsilon \quad \text{for all Lebesgue measurable } \omega \text{ with } |\omega| < \delta.$$

Fix now $\varepsilon > 0$ and choose $\delta \leq \varepsilon$ such that both points above hold. Consider the continuous function

$$\phi(x) = \operatorname{dist}(x, \mathbb{R}^d \setminus A).$$

Since $K \subset A$ is compact and ϕ is continuous, we have that

$$m := \min_{x \in K} \phi(x) > 0.$$

Consider the function

$$\psi(x) = 1 \wedge \frac{\phi(x)}{m}.$$

Then, ψ is continuous on $\mathbb{R}^d$, $0 \leq \psi \leq 1$ on $\mathbb{R}^d$, $\{\psi > 0\} = A$ and $\psi \equiv 1$ on K. Then, we have

$$\begin{aligned}
0 = \int_{\mathbb{R}^d} \widetilde{g}(x)\psi(x)\,dx &= \int_A \widetilde{g}(x)\psi(x)\,dx \\
&= \int_{B_R \cap \{\widetilde{g} > t\}} \widetilde{g}(x)\psi(x)\,dx + \int_{A \setminus (B_R \cap \{\widetilde{g} > t\})} \widetilde{g}(x)\psi(x)\,dx \\
&\geq t \int_{B_R \cap \{\widetilde{g} > t\}} \psi(x)\,dx - \int_{A \setminus (B_R \cap \{\widetilde{g} > t\})} |\widetilde{g}(x)|\,dx \\
&\geq t \int_K \psi(x)\,dx - \int_{A \setminus K} |\widetilde{g}(x)|\,dx \\
&= t|K| - \int_{A \setminus K} |\widetilde{g}(x)|\,dx \\
&= t|B_R \cap \{\widetilde{g} > t\}| - t|(B_R \cap \{\widetilde{g} > t\}) \setminus K| - \int_{A \setminus K} |\widetilde{g}(x)|\,dx \\
&\geq t|B_R \cap \{\widetilde{g} > t\}| - t|A \setminus K| - \int_{A \setminus K} |\widetilde{g}(x)|\,dx \\
&\geq t|B_R \cap \{\widetilde{g} > t\}| - (1+t)\varepsilon.
\end{aligned}$$

Since we can choose ε arbitrarily small, we get that

$$t|B_R \cap \{\widetilde{g} > t\}| = 0,$$

which is a contradiction. □