

On the separability of $\mathcal{B}$ and $\mathcal{B}'$

Theorem 1. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{B}'$ be its dual. If $\mathcal{B}'$ is separable, then also $\mathcal{B}$ is separable.*

Proof. Let $(T_n)_{n \geq 1}$ be a dense subset of $\mathcal{B}'$ with respect to the topology induced by the norm $\|\cdot\|_{\mathcal{B}'}$. By the definition of $\|T_n\|_{\mathcal{B}'}$, for every T_n , we can find $x_n \in \mathcal{B}$ such that:

$$\|x_n\|_{\mathcal{B}} = 1 \quad \text{and} \quad \frac{1}{2} \|T_n\|_{\mathcal{B}'} \leq T_n(x_n) \leq \|T_n\|_{\mathcal{B}'}.$$

Consider the countable set

$$V_{\mathbb{Q}} = \left\{ \sum_{i=1}^N x_i q_i : q_i \in \mathbb{Q}, N \geq 1 \right\},$$

and its closure V in $\mathcal{B}$ with respect to the strong topology of $\mathcal{B}$. Clearly, V is a closed linear subspace of $\mathcal{B}$.

Suppose that $V \neq \mathcal{B}$. Then, by the Hahn-Banach's theorem, there is a functional $T \in \mathcal{B}'$ such that

$$\|T\|_{\mathcal{B}'} = 1 \quad \text{and} \quad T \equiv 0 \quad \text{on} \quad V.$$

Since $(T_n)_{n \geq 1}$ is dense we can find a sequence $(T_{n_k})_{k \geq 1}$ such that

$$\lim_{k \rightarrow +\infty} \|T_{n_k} - T\|_{\mathcal{B}'} = 0.$$

We then have

$$\begin{aligned} \frac{1}{2} \|T_{n_k}\|_{\mathcal{B}'} &\leq |T_{n_k}(x_{n_k})| = |T_{n_k}(x_{n_k}) - T(x_{n_k})| \\ &\leq \|T_{n_k} - T\|_{\mathcal{B}'} \|x_{n_k}\|_{\mathcal{B}} \\ &\leq \|T_{n_k} - T\|_{\mathcal{B}'} . \end{aligned}$$

On the other hand

$$\lim_{k \rightarrow +\infty} \|T_{n_k}\|_{\mathcal{B}'} = 1 \quad \text{and} \quad \lim_{k \rightarrow +\infty} \|T_{n_k} - T\|_{\mathcal{B}'} = 0,$$

which is a contradiction. Thus, $\mathcal{B}$ is the closure of $V_{\mathbb{Q}}$. □

Remark 2. *We notice that the converse implication is not true. Indeed, for every Lebesgue measurable space $\Omega \subset \mathbb{R}^d$ of non-zero Lebesgue measure, we have that $L^1(\Omega)$ is separable and $L^\infty(\Omega)$ is the dual of $L^1(\Omega)$, but $L^\infty(\Omega)$ is NOT separable.*

Corollary 3. *Suppose that $\mathcal{B}$ is a reflexive Banach space. Then, $\mathcal{B}$ is separable if and only if $\mathcal{B}'$ is separable.*

Remark 4. *The above corollary provides another proof of the fact that, given a Lebesgue measurable set $\Omega \subset \mathbb{R}^d$ of non-zero Lebesgue measure, the space $L^1(\Omega)$ is not reflexive.*