

Weak convergence and weak-* convergence: definitions, basic properties and examples

WEAK CONVERGENCE

Definition 1 (Weak convergence and strong convergence in $\mathcal{B}$). *Given a Banach space $\mathcal{B}$ with dual $\mathcal{B}'$, we say that a sequence $u_n \in \mathcal{B}$ converges weakly to $u \in \mathcal{B}$, and we write*

$$u_n \rightharpoonup u,$$

if

$$T(u_n) \rightarrow T(u) \quad \text{for every bounded linear functional } T \in \mathcal{B}'.$$

We will say that the sequence $u_n \in \mathcal{B}$ converges strongly to $u \in \mathcal{B}$, and we will write $u_n \rightarrow u$, if it converges in norm, that is:

$$\lim_{n \rightarrow +\infty} \|u_n - u\|_{\mathcal{B}} = 0.$$

Remark 2 (Uniqueness of the weak limit). *The weak limit is unique. Indeed, suppose that u_n is a sequence in $\mathcal{B}$ converging to both $u \in \mathcal{B}$ and $v \in \mathcal{B}$. Then, we have*

$$T(u) = \lim_{n \rightarrow +\infty} T(u_n) = T(v) \quad \text{for all } T \in \mathcal{B}'.$$

This can be written as

$$T(u - v) = 0 \quad \text{for all } T \in \mathcal{B}'.$$

By the first Hahn-Banach theorem, there is a functional $S \in \mathcal{B}'$ such that

$$S(u - v) = \|u - v\|_{\mathcal{B}}.$$

Thus, $\|u - v\|_{\mathcal{B}} = 0$ and so, $u = v$.

Remark 3 (Strong convergence implies weak convergence). *Suppose that u_n is a sequence in $\mathcal{B}$ that converges strongly to some $u \in \mathcal{B}$. Then, for every bounded linear functional $T \in \mathcal{B}'$ we have*

$$|T(u_n) - T(u)| = |T(u_n - u)| \leq \|T\|_{\mathcal{B}'} \|u_n - u\|_{\mathcal{B}}.$$

Passing to the limit as $n \rightarrow +\infty$, we obtain that

$$T(u) = \lim_{n \rightarrow +\infty} T(u_n) \quad \text{for all } T \in \mathcal{B}'.$$

Thus, by definition, u_n converges to u weakly in $\mathcal{B}$.

WEAK-* CONVERGENCE

Definition 4 (Weak-* convergence and strong convergence in $\mathcal{B}'$). *Let $\mathcal{B}$ be a Banach space with dual $\mathcal{B}'$. We say that a sequence of bounded linear functionals $T_n \in \mathcal{B}'$ converges weakly-* to the bounded linear functional $T \in \mathcal{B}'$, and we write*

$$T_n \rightharpoonup^* T,$$

if

$$T_n(u) \rightarrow T(u) \quad \text{for every } u \in \mathcal{B}.$$

We will say that the sequence T_n converges strongly to T in $\mathcal{B}'$, if it converges in norm, that is:

$$\lim_{n \rightarrow +\infty} \|T_n - T\|_{\mathcal{B}'} = 0.$$

Remark 5 (Uniqueness of the weak-* limit). *Suppose that T_n is a sequence in $\mathcal{B}'$ that converges weakly-* in $\mathcal{B}'$. The, the weak-* limit of T_n is unique. Indeed, suppose that there are two functionals $T \in \mathcal{B}'$ and $S \in \mathcal{B}'$ such that $T_n \rightharpoonup^* T$ and $T_n \rightharpoonup^* S$. Then, by definition*

$$T(u) = \lim_{n \rightarrow +\infty} T_n(u) = S(u) \quad \text{for all } u \in \mathcal{B}.$$

Thus

$$T(u) = S(u) \quad \text{for all } u \in \mathcal{B},$$

which implies that T and S coincide as bounded linear functions on $\mathcal{B}$.

Remark 6 (Strong convergence implies weak-* convergence). Suppose that T_n is a sequence in $\mathcal{B}'$ that converges strongly to some $T \in \mathcal{B}'$. Then, for every $u \in \mathcal{B}$, we have

$$|T_n(u) - T(u)| = |(T_n - T)(u)| \leq \|T_n - T\|_{\mathcal{B}'} \|u\|_{\mathcal{B}}.$$

Passing to the limit as $n \rightarrow +\infty$, we obtain that

$$T(u) = \lim_{n \rightarrow +\infty} T_n(u) \quad \text{for all } u \in \mathcal{B}.$$

Thus, by definition, T_n converges to T weakly-* in $\mathcal{B}'$.

Remark 7 (Weak convergence implies weak-* convergence). Let $\mathcal{B}$ be a Banach space with dual $\mathcal{B}'$ and bidual $\mathcal{B}''$. Suppose that T_n is a sequence in $\mathcal{B}'$ that converges weakly in the Banach space $\mathcal{B}'$ to some $T \in \mathcal{B}'$, that is:

$$\xi(T) = \lim_{n \rightarrow +\infty} \xi(T_n) \quad \text{for every bounded linear functional } \xi \in \mathcal{B}''.$$

Consider now an element $u \in \mathcal{B}$. Then, the functional

$$\delta_u : \mathcal{B}' \rightarrow \mathbb{R}, \quad \delta_u(T) = T(u) \quad \text{for all } T \in \mathcal{B}',$$

is a bounded linear functional on $\mathcal{B}'$. In particular, we have

$$\delta_u(T) = \lim_{n \rightarrow +\infty} \delta_u(T_n),$$

which can also be written as

$$T(u) = \lim_{n \rightarrow +\infty} T_n(u).$$

Since u is an arbitrary element of $\mathcal{B}$, we get that T_n converges to T weakly-* in $\mathcal{B}'$.

Proposition 8. Let $\mathcal{B}$ be a Banach space with dual $\mathcal{B}'$ and bidual $\mathcal{B}''$. If $\mathcal{B}$ is reflexive, then for every sequence $T_n \in \mathcal{B}'$, the following are equivalent:

- (1) T_n converges weakly in $\mathcal{B}'$;
- (2) T_n converges weakly-* in $\mathcal{B}'$.

WEAK CONVERGENCE IN $L^p(\Omega)$ IN THE CASE $p \in (1, +\infty)$

Let $p \in (1, +\infty)$ and $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set. We recall that $L^p(\Omega)$ can be identified with the dual space $(L^q(\Omega))'$, where as usual

$$q := \frac{p}{p-1}.$$

Indeed, we have already shown that every bounded linear functional $T \in (L^q(\Omega))'$ is of the form $T = T_f$ for some $f \in L^p(\Omega)$, where T_f is the functional

$$T_f : L^q(\Omega) \rightarrow \mathbb{R}, \quad T_f(\phi) = \int_{\Omega} f(x)\phi(x) dx.$$

Thanks to the above identification of the functionals in $(L^q(\Omega))'$ with the elements of $L^p(\Omega)$, we have the following proposition.

Proposition 9. Let $p \in (1, +\infty)$ and $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set and let $q := \frac{p}{p-1}$. Given a sequence $f_n \in L^p(\Omega)$ and a function $f \in L^p(\Omega)$, the following are equivalent:

- (1) f_n converges weakly in $L^p(\Omega)$ to f ;
- (2) T_{f_n} converges weakly-* in $(L^q(\Omega))'$ to T_f ;
- (3) for every $g \in L^q(\Omega)$, we have

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f_n(x)g(x)dx = \int_{\Omega} f(x)g(x)dx.$$

Definition 10. Let $p \in (1, +\infty)$, $q := \frac{p}{p-1}$, and $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set. We say that a sequence $f_n \in L^p(\Omega)$ converges weakly a $f \in L^p(\Omega)$ (and we write $f_n \rightharpoonup f$ in $L^p(\Omega)$), if

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f_n(x)g(x)dx = \int_{\Omega} f(x)g(x)dx \quad \text{for all } g \in L^q(\Omega).$$

GENERAL CRITERIA FOR THE WEAK CONVERGENCE OF A SEQUENCE

The next one is a general criteria for the weak convergence of a sequence in a Banach space, which is often used in $L^p(\Omega)$.

Proposition 11. *Let $\mathcal{B}$ be a Banach space with dual $\mathcal{B}'$ and let Y be a dense subset of $\mathcal{B}'$. Suppose that u_n is a sequence in $\mathcal{B}$ and that $u \in \mathcal{B}$ are such that:*

- (a) *the sequence of norms $(\|u_n\|_{\mathcal{B}})_{n \geq 1}$ is bounded;*
- (b) *$T(u_n) \rightarrow T(u)$ for every $T \in Y$.*

Then, u_n converges to u weakly in $\mathcal{B}$.

Proof. Let $S \in \mathcal{B}'$. We will show that

$$S(u) = \lim_{n \rightarrow +\infty} S(u_n).$$

Let

$$M := \sup_n \|u_n\|_{\mathcal{B}}.$$

For every $T \in Y$ we have

$$\begin{aligned} |S(u_n) - S(u)| &\leq |S(u_n) - T(u_n)| + |T(u_n) - T(u)| + |T(u) - S(u)| \\ &\leq \|S - T\|_{\mathcal{B}'} \|u_n\|_{\mathcal{B}} + |T(u_n) - T(u)| + \|S - T\|_{\mathcal{B}'} \|u\|_{\mathcal{B}} \\ &\leq \|S - T\|_{\mathcal{B}'} (M + \|u\|_{\mathcal{B}}) + |T(u_n) - T(u)|. \end{aligned}$$

Since $T \in Y$ we have

$$T(u) = \lim_{n \rightarrow +\infty} T(u_n).$$

Thus, passing to the limit as $n \rightarrow +\infty$, we get

$$\limsup_{n \rightarrow +\infty} |S(u_n) - S(u)| \leq \|S - T\|_{\mathcal{B}'} (M + \|u\|_{\mathcal{B}}).$$

Since $T \in Y$ is arbitrary and since Y is dense in $\mathcal{B}'$, we can choose $\|S - T\|_{\mathcal{B}'}$ arbitrarily small. Thus, we get

$$\limsup_{n \rightarrow +\infty} |S(u_n) - S(u)| = 0,$$

which concludes the proof. □

Similarly, we can write a criteria for the weak-* convergence of a sequence of functionals. The proof is identical.

Proposition 12. *Let $\mathcal{B}$ be a Banach space with dual $\mathcal{B}'$ and let X be a dense subset of $\mathcal{B}$. Suppose that T_n is a sequence in $\mathcal{B}'$ and that $T \in \mathcal{B}'$ are such that:*

- (a) *the sequence of norms $(\|T_n\|_{\mathcal{B}'})_{n \geq 1}$ is bounded;*
- (b) *$T_n(x) \rightarrow T(x)$ for every $x \in X$.*

Then, T_n converges to T weakly- in $\mathcal{B}'$.*

Proof. Let $y \in \mathcal{B}$. We need to show that

$$T(y) = \lim_{n \rightarrow +\infty} T_n(y).$$

We set $M := \sup_n \|T_n\|_{\mathcal{B}'}$, and for every $x \in X$ we compute

$$\begin{aligned} |T_n(y) - T(y)| &\leq |T_n(y) - T_n(x)| + |T_n(x) - T(x)| + |T(x) - S(y)| \\ &\leq \|T_n\|_{\mathcal{B}'} \|y - x\|_{\mathcal{B}} + |T_n(x) - T(x)| + \|T\|_{\mathcal{B}'} \|y - x\|_{\mathcal{B}} \\ &\leq (M + \|T\|_{\mathcal{B}'}) \|y - x\|_{\mathcal{B}} + |T_n(x) - T(x)|. \end{aligned}$$

Since $x \in X$, by hypothesis we have

$$T(x) = \lim_{n \rightarrow +\infty} T_n(x).$$

Thus, passing to the limit as $n \rightarrow +\infty$, we get

$$\limsup_{n \rightarrow +\infty} |T_n(y) - T(y)| \leq (M + \|T\|_{\mathcal{B}'}) \|y - x\|_{\mathcal{B}}.$$

Now, since $x \in X$ is arbitrary and since X is dense in $\mathcal{B}$, we get

$$\limsup_{n \rightarrow +\infty} |T_n(y) - T(y)| = 0,$$

which concludes the proof. $\square$

EXAMPLES

Example 13 (A sequence escaping at infinity). *Let $p \in (1, +\infty)$ and let $f \in L^p(\mathbb{R}^d)$ be fixed. Let x_n be a sequence in $\mathbb{R}^d$ such that $|x_n| \rightarrow +\infty$. For every $n \geq 1$, we define:*

$$f_n(x) := f(x - x_n).$$

Then, f_n converges to 0 weakly in $L^p(\mathbb{R}^d)$.

Proof. We notice that for every $n \geq 1$ we have

$$\|f_n\|_{L^p(\mathbb{R}^d)}^p = \int_{\mathbb{R}^d} |f(x - x_n)|^p dx = \int_{\mathbb{R}^d} |f(y)|^p dy = \|f\|_{L^p(\mathbb{R}^d)}^p.$$

We will apply Proposition 11 with Y being the space $C_c(\mathbb{R}^d)$ of continuous functions with compact support in $\mathbb{R}^d$, which is dense in both $L^p(\mathbb{R}^d)$ and its dual $L^q(\mathbb{R}^d)$. Take $\phi \in C_c(\mathbb{R}^d)$ and $\psi \in C_c(\mathbb{R}^d)$ and consider

$$\begin{aligned} \left| \int_{\mathbb{R}^d} f_n(x) \phi(x) dx \right| &= \left| \int_{\mathbb{R}^d} f(x - x_n) \phi(x) dx \right| \\ &= \left| \int_{\mathbb{R}^d} f(y) \phi(y + x_n) dy \right| \\ &\leq \left| \int_{\mathbb{R}^d} (f(y) - \psi(y)) \phi(y + x_n) dy \right| + \left| \int_{\mathbb{R}^d} \psi(y) \phi(y + x_n) dy \right| \\ &\leq \|f - \psi\|_{L^p(\mathbb{R}^d)} \|\phi\|_{L^q(\mathbb{R}^d)} + \left| \int_{\mathbb{R}^d} \psi(y) \phi(y + x_n) dy \right|. \end{aligned}$$

Now, when x_n is large, the functions $\psi(\cdot)$ and $\phi(\cdot + x_n)$ have disjoint supports so that

$$\int_{\mathbb{R}^d} \psi(y) \phi(y + x_n) dy = 0 \quad \text{for all } n \text{ large enough.}$$

Thus, passing to the limit as $n \rightarrow +\infty$, we get

$$\limsup_{n \rightarrow +\infty} \left| \int_{\mathbb{R}^d} f_n(x) \phi(x) dx \right| \leq \|f - \psi\|_{L^p(\mathbb{R}^d)} \|\phi\|_{L^q(\mathbb{R}^d)}.$$

Since $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, we can choose $\|f - \psi\|_{L^p(\mathbb{R}^d)}$ arbitrarily small. Thus, we get

$$\limsup_{n \rightarrow +\infty} \left| \int_{\mathbb{R}^d} f_n(x) \phi(x) dx \right|,$$

which proves that

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} f_n(x) \phi(x) dx = 0 \quad \text{for all } \phi \in C_c(\mathbb{R}^d).$$

Since $C_d(\mathbb{R}^d)$ is dense in $L^q(\mathbb{R}^d)$, the conclusion follows from Proposition 11. $\square$

Example 14 (A highly oscillating sequence). *In $L^2([0, 1])$, consider the sequence ϕ_n , $n \geq 1$, defined as*

$$\phi_n(x) := \begin{cases} 1 & \text{if } \frac{2k}{2n} \leq x < \frac{2k+1}{2n} \quad \text{for } 0 \leq k \leq n-1, \\ 0 & \text{if } \frac{2k+1}{2n} \leq x < \frac{2k+2}{2n} \quad \text{for } 0 \leq k \leq n-1. \end{cases}$$

Then, ϕ_n converges weakly in $L^2([0, 1])$ to the constant $\frac{1}{2}$.

Proof. We first notice that

$$\|\phi_n\|_{L^2([0,1])}^2 = \int_0^1 \phi_n^2(x) dx = \frac{1}{2} \quad \text{for all } n \geq 1.$$

We next consider a continuous function $g \in C([0,1])$ and using the absolute continuity of g , we compute

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_0^1 \phi_n(x) g(x) dx &= \lim_{n \rightarrow +\infty} \sum_{k=0}^{n-1} \int_{\frac{2k}{2n}}^{\frac{2k+1}{2n}} g(x) dx \\ &= \lim_{n \rightarrow +\infty} \sum_{k=0}^{n-1} \frac{1}{2n} g\left(\frac{2k+1}{2n}\right) \\ &= \frac{1}{2} \lim_{n \rightarrow +\infty} \sum_{k=0}^{n-1} \frac{1}{n} g\left(\frac{2k+1}{2n}\right) \\ &= \frac{1}{2} \lim_{n \rightarrow +\infty} \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} g(x) dx = \frac{1}{2} \int_0^1 g(x) dx. \end{aligned}$$

Thus, we can apply Proposition 11 with $Y = C([0,1])$, which is a dense subspace of $L^2([0,1])$. This concludes the proof. $\square$

Remark 15. We notice that the sequence from the previous example does not converge strongly in L^2 . Indeed, suppose that ϕ_n converges strongly in $L^2([0,1])$ to some ϕ . Then, it also converges weakly to ϕ . By the uniqueness of the weak limit, we have that ϕ is the constant $1/2$. But the L^2 norm of the constant $1/2$ on $[0,1]$ is $1/2$, while the L^2 norm of ϕ_n is $1/\sqrt{2}$.

Example 16 (Oscillating sequences and Hilbert basis). In $L^2([0,\pi])$, consider the sequence $u_n(x) := \sin(nx)$. Then, u_n converges weakly in $L^2([0,\pi])$ to 0. More in general, if $\mathcal{H}$ is a Hilbert space (not necessarily separable) and if $(\phi_n)_{n \geq 1}$ is a sequence of orthonormal vectors (not necessarily complete), then $\phi_n \rightharpoonup 0$ in $\mathcal{H}$.

Proof. Let $x \in \mathcal{H}$. For every $n \geq 1$, we have

$$\begin{aligned} 0 &\leq \left\| x - \sum_{i=1}^n \langle x, e_i \rangle e_i \right\|^2 = \|x\|^2 - 2 \langle x, \sum_{i=1}^n \langle x, e_i \rangle e_i \rangle + \left\| \sum_{i=1}^n \langle x, e_i \rangle e_i \right\|^2 \\ &= \|x\|^2 - 2 \sum_{i=1}^n \langle x, e_i \rangle^2 + \sum_{i=1}^n \langle x, e_i \rangle^2 \\ &= \|x\|^2 - \sum_{i=1}^n \langle x, e_i \rangle^2. \end{aligned}$$

Since n is arbitrary, we get that

$$\|x\|^2 \geq \sum_{i=1}^{+\infty} \langle x, e_i \rangle^2,$$

which implies

$$\lim_{i \rightarrow +\infty} \langle x, e_i \rangle = 0.$$

$\square$

Example 17. Let $p \in (1, +\infty)$, $f \in L^p(\mathbb{R}^d)$ and let $R_n \rightarrow +\infty$. For every $n \geq 1$, let

$$f_n(x) := \frac{1}{R_n^{d/p}} f(x/R_n)$$

Then, $f_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$.

Proof. Thanks to the change of coordinates $y = x/R_n$, we have

$$\|f_n\|_{L^p(\mathbb{R}^d)}^p = \int_{\mathbb{R}^d} \frac{1}{R_n^d} |f|^p(x/R_n) dx = \int_{\mathbb{R}^d} |f(y)|^p dy.$$

Let now $\phi \in C_c(\mathbb{R}^d)$. Suppose that ϕ is supported in B_ρ and that $|\phi| \leq M$. Then,

$$\begin{aligned}
\left| \int_{\mathbb{R}^d} \phi(x) f_n(x) dx \right| &= \left| \int_{\mathbb{R}^d} \phi(x) \frac{1}{R_n^{d/p}} f(x/R_n) dx \right| \\
&= \left| \int_{\mathbb{R}^d} \phi(R_n y) \frac{1}{R_n^{d/p}} f(y) R_n^d dy \right| \\
&\leq M \int_{B_{\rho/R_n}} \frac{1}{R_n^{d(1/p-1)}} f(y) dy \\
&\leq M \frac{1}{R_n^{d(1/p-1)}} |B_{\rho/R_n}|^{1/q} \left(\int_{B_{\rho/R_n}} |f(y)|^p dy \right)^{1/p} \\
&= M \frac{1}{R_n^{d(1/p-1)}} \frac{|B_\rho|^{1/q}}{R_n^{d/q}} \left(\int_{B_{\rho/R_n}} |f(y)|^p dy \right)^{1/p} \\
&= M |B_\rho|^{1/q} \left(\int_{B_{\rho/R_n}} |f(y)|^p dy \right)^{1/p}.
\end{aligned}$$

Passing to the limit as $n \rightarrow +\infty$, we get that

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} \phi(x) f_n(x) dx = 0.$$

Thus, by Proposition 11, we get that f_n converges to zero weakly in $L^p(\mathbb{R}^d)$. □