Second Clarkson inequality

Clarkson's inequality with exponent $p \in (1,2]$

Proposition 1. *Let $p \in (1,2)$ be a real number and let $q := \frac{p}{p-1}$. Then, the following inequality holds:*

$$\left|\frac{a+b}{2}\right|^{\frac{p}{p-1}} + \left|\frac{a-b}{2}\right|^{\frac{p}{p-1}} \leq \left(\frac{|a|^p}{2} + \frac{|b|^p}{2}\right)^{\frac{1}{p-1}},$$

for every $a, b \in \mathbb{R}$.

Proof. We proceed in two steps following the proof of Friedrichs.

Step 1. For every $t \in [0,1]$ it holds

$$(1+t)^{\frac{p}{p-1}} + (1-t)^{\frac{p}{p-1}} \leq 2(1+t^p)^{\frac{1}{p-1}}.$$

For a fixed $t \in (0,1)$, consider the auxiliary function

$$F : [t^{p-1}, 1] \to \mathbb{R}\,, \qquad F(s) = \big(1 + s^{-\frac{1}{p-1}}t\big)(1+st)^{\frac{1}{p-1}} + \big(1 - s^{-\frac{1}{p-1}}t\big)(1-st)^{\frac{1}{p-1}}.$$

Since

$$F(t^{p-1}) = 2(1+t^p)^{\frac{1}{p-1}},$$

and

$$F(1) = (1+t)(1+t)^{\frac{1}{p-1}} + (1-t)(1-t)^{\frac{1}{p-1}} = (1+t)^{\frac{p}{p-1}} + (1-t)^{\frac{p}{p-1}}.$$

Thus, we only need to prove that F is non-decreasing on $[t^{p-1}, 1]$. Computing the derivative in s, we get

$$\begin{aligned}
F'(s) &= -\frac{1}{p-1}s^{-\frac{1}{p-1}-1}t(1+st)^{\frac{1}{p-1}} + \big(1+s^{-\frac{1}{p-1}}t\big)t\frac{1}{p-1}(1+st)^{\frac{1}{p-1}-1} \\
&\quad + \frac{1}{p-1}s^{-\frac{1}{p-1}-1}t(1-st)^{\frac{1}{p-1}} - \big(1-s^{-\frac{1}{p-1}}t\big)t\frac{1}{p-1}(1-st)^{\frac{1}{p-1}-1} \\
&= \frac{t}{p-1}\Big(- s^{-\frac{1}{p-1}-1}(1+st)^{\frac{1}{p-1}} + \big(1+s^{-\frac{1}{p-1}}t\big)(1+st)^{\frac{1}{p-1}-1} \\
&\qquad\qquad + s^{-\frac{1}{p-1}-1}(1-st)^{\frac{1}{p-1}} + \big(s^{-\frac{1}{p-1}}t - 1\big)(1-st)^{\frac{1}{p-1}-1}\Big) \\
&= \frac{t}{p-1}\Big(\Big(- s^{-\frac{1}{p-1}-1}(1+\cancel{st}) + \big(1+\cancel{s^{-\frac{1}{p-1}}t}\big)\Big)(1+st)^{\frac{1}{p-1}-1} \\
&\qquad\qquad + \Big(s^{-\frac{1}{p-1}-1}(1-\cancel{st}) + \big(\cancel{s^{-\frac{1}{p-1}}t} - 1\big)\Big)(1-st)^{\frac{1}{p-1}-1}\Big) \\
&= \frac{t}{p-1}\Big(\Big(1 - s^{-\frac{1}{p-1}-1}\Big)(1+st)^{\frac{1}{p-1}-1} \\
&\qquad\qquad - \Big(1 - s^{-\frac{1}{p-1}-1}\Big)(1-st)^{\frac{1}{p-1}-1}\Big) \\
&= \frac{t}{p-1}\Big(1 - s^{-\frac{1}{p-1}-1}\Big)\Big((1+st)^{\frac{1}{p-1}-1} - (1-st)^{\frac{1}{p-1}-1}\Big),
\end{aligned}$$

which implies that

$$F'(s) \leq 0 \quad \text{for all} \quad s \in [t^{p-1}, 1].$$

Step 2. Let now $a, b \in \mathbb{R}$ be such that $a \neq 0$ and $b \neq 0$. Suppose that $|a| < |b|$ and take $t = |a|/|b|$. Since $t \leq 1$, Step 1 implies

$$\left|\frac{|a|+|b|}{2}\right|^{\frac{p}{p-1}} + \left|\frac{|a|-|b|}{2}\right|^{\frac{p}{p-1}} \leq \left(\frac{|a|^p}{2} + \frac{|b|^p}{2}\right)^{\frac{1}{p-1}}.$$

Now, since

$$\left|\frac{|a|+|b|}{2}\right|^{\frac{p}{p-1}} + \left|\frac{|a|-|b|}{2}\right|^{\frac{p}{p-1}} = \left|\frac{a+b}{2}\right|^{\frac{p}{p-1}} + \left|\frac{a-b}{2}\right|^{\frac{p}{p-1}},$$

we get the claim. □