

Weak convergence, weak-* convergence and norm

In this section we explore the relation between the weak convergence and the norms of a sequence in a Banach space, as well as the relation between the weak-* convergence of a sequences of bounded linear functionals in the dual $\mathcal{B}'$ and the sequence of their norms as operators on $\mathcal{B}$. In particular, we will prove:

- a weak-* compactness theorem for the bounded sequences (Theorems 1, 2, 3);
- the semincontinuity with respect to the weakly (and weakly-*) convergence (Theorems 4, 5, 6);
- the boundedness of the weak and the weak-* converging sequences (Theorems 7, 8, 9);
- the Radon-Riesz theorem, which relates the strong convergence, the weak convergence and the convergence of the norms of a sequence in $L^p(\Omega)$ (Theorem 10);
- the coupling of a weakly converging sequence with a strongly converging sequence gives a converging sequence (Theorems 11, 13, and 14).

WEAK-* COMPACTNESS OF THE BOUNDED SEQUENCES

Theorem 1. *Let $\mathcal{B}$ be a separable Banach space. Let T_n be a bounded sequence in $\mathcal{B}'$. Then, there exists a subsequence T_{n_k} and a bounded linear functional $T \in \mathcal{B}'$ such that*

$$T_n(x) \rightarrow T(x) \quad \text{for all } x \in \mathcal{B}.$$

Proof. Let $L > 0$ be a constant such that

$$\|T_n\|_{\mathcal{B}'} \leq L \quad \text{for all } n \geq 1.$$

Let

$$\mathcal{C} := \{x_j : j \in \mathbb{N}\},$$

be a countable dense subset of $\mathcal{B}$. Since $\mathcal{C}$ is countable, we can suppose that the sequence $(T_n(x_j))_{n \geq 1}$ converges (as $n \rightarrow +\infty$) for every $j \in \mathbb{N}$. We can then define

$$T(x_j) := \lim_{n \rightarrow +\infty} T_n(x_j)$$

We will next show that, for any $x \in \mathcal{B}$, the sequence $(T_n(x))_{n \geq 1}$ is a Cauchy sequence. Indeed, fixed $x_j \in \mathcal{C}$, we have

$$\begin{aligned} |T_n(x) - T_m(x)| &\leq |T_n(x) - T_n(x_j)| + |T_n(x_j) - T_m(x_j)| + |T_m(x) - T_m(x_j)| \\ &\leq 2L|x - x_j| + |T_n(x_j) - T_m(x_j)| \end{aligned}$$

Then, since $\mathcal{C}$ is dense, we get that the sequence $T_n(x)$ is a Cauchy sequence in $\mathbb{R}$. Then, we can define $T(x)$ as the limit

$$T(x) := \lim_{n \rightarrow +\infty} T_n(x).$$

We notice that the map $T : \mathcal{B} \rightarrow \mathbb{R}$, is linear: this is due to the fact that T is the pointwise limit of the linear maps T_n . Moreover, since

$$|T(x)| = \lim_{n \rightarrow +\infty} |T_n(x)| \leq L|x|,$$

we have that T is also bounded. □

Two immediate consequences of Theorem 1 we obtain the following corollaries.

Theorem 2. *In a separable Hilbert space $\mathcal{H}$, every bounded sequence admits a subsequence that converges weakly in $\mathcal{H}$.*

Theorem 3. *Let $p \in (1, +\infty)$ and let Ω be a Lebesgue measurable set in $\mathbb{R}^d$. Then, every bounded sequence $f_n \in L^p(\Omega)$ admits a subsequence that converges weakly in $L^p(\Omega)$.*

SEMICONINUITY OF THE NORM WITH RESPECT TO THE WEAK CONVERGENCE

Theorem 4. *Let $p \in (1, +\infty)$ and let $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set. Let $f_n \in L^p(\Omega)$ be a sequence that converges weakly to a function $f \in L^p(\Omega)$. Then*

$$\|f\|_{L^p(\Omega)} \leq \liminf_{n \rightarrow +\infty} \|f_n\|_{L^p(\Omega)}$$

Proof. We know that there is a function $g \in L^q(\Omega)$, where as usual $q := \frac{p}{p-1}$, such that

$$\|f\|_{L^p(\Omega)} = \int_{\Omega} f(x)g(x)dx \quad \text{and} \quad \|g\|_{L^q(\Omega)} = 1.$$

We recall that g can be written explicitly as

$$g := \frac{1}{\|f\|_{L^p(\mathbb{R}^d)}^{p-1}} f|f|^{p-2}.$$

By the choice of g and the Hölder inequality, we have

$$\|f\|_{L^p(\Omega)} = \int_{\Omega} f(x)g(x)dx = \lim_{n \rightarrow \infty} \int_{\Omega} f_n(x)g(x)dx \leq \liminf_{n \rightarrow +\infty} \|f_n\|_{L^p(\mathbb{R}^d)}.$$

□

The above theorem can be easily generalized to weakly converging sequences in a Banach space $\mathcal{B}$ and to weakly- $\star$ converging sequences in $\mathcal{B}'$.

Theorem 5. *Let $\mathcal{B}$ be a Banach space. For any sequence $x_n \in \mathcal{B}$ converging weakly in $\mathcal{B}$ to some $x \in \mathcal{B}$, we have*

$$\|x\|_{\mathcal{B}} \leq \liminf_{n \rightarrow +\infty} \|x_n\|_{\mathcal{B}}.$$

Proof. By the first theorem of Hahn-Banach, we have that there is a functional $T \in \mathcal{B}'$ such that

$$T(x) = \|x\|_{\mathcal{B}} \quad \text{and} \quad \|T\|_{\mathcal{B}'} = 1.$$

Since $x_n \rightharpoonup x$ in $\mathcal{B}$, we have that $T(x_n) \rightarrow T(x)$. Thus,

$$\|x\|_{\mathcal{B}} = T(x) = \lim_{n \rightarrow +\infty} T(x_n) \leq \liminf_{n \rightarrow +\infty} \|T\|_{\mathcal{B}'} \|x_n\|_{\mathcal{B}} = \liminf_{n \rightarrow +\infty} \|x_n\|_{\mathcal{B}},$$

which concludes the proof. □

Theorem 6. *Let $\mathcal{B}$ be a Banach space with dual $\mathcal{B}'$. For any sequence $T_n \in \mathcal{B}'$ converging weakly- $\star$ in $\mathcal{B}'$ to some $T \in \mathcal{B}'$, we have*

$$\|T\|_{\mathcal{B}'} \leq \liminf_{n \rightarrow +\infty} \|T_n\|_{\mathcal{B}'}.$$

Proof. Fix $\varepsilon > 0$. Since

$$\|T\|_{\mathcal{B}'} = \sup \left\{ T(y) : y \in \mathcal{B}, \|y\|_{\mathcal{B}} = 1 \right\},$$

we can find some $x_{\varepsilon} \in \mathcal{B}$ such that

$$\|x_{\varepsilon}\|_{\mathcal{B}} = 1 \quad \text{and} \quad \|T\|_{\mathcal{B}'} - \varepsilon \leq T(x_{\varepsilon}) \leq \|T\|_{\mathcal{B}'}.$$

Since $T_n \rightharpoonup^* T$ in $\mathcal{B}'$, we have that $T_n(x_{\varepsilon}) \rightarrow T(x_{\varepsilon})$. Thus,

$$\|T\|_{\mathcal{B}'} \leq \varepsilon + T(x_{\varepsilon}) = \varepsilon + \lim_{n \rightarrow +\infty} T_n(x_{\varepsilon}) \leq \varepsilon \liminf_{n \rightarrow +\infty} \|T_n\|_{\mathcal{B}'} \|x_{\varepsilon}\|_{\mathcal{B}} = \varepsilon + \liminf_{n \rightarrow +\infty} \|T_n\|_{\mathcal{B}'}.$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\|T\|_{\mathcal{B}'} \leq \liminf_{n \rightarrow +\infty} \|T_n\|_{\mathcal{B}'},$$

which concludes the proof. □

THE SEQUENCES CONVERGING WEAKLY IN $\mathcal{B}$ ARE BOUNDED

Theorem 7. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{B}'$ be its dual. Let $x_n \in \mathcal{B}$ be a weakly converging sequence in $\mathcal{B}$. Then, the sequence of norms $\|x_n\|_{\mathcal{B}}$ is bounded.*

As an immediate consequence, we obtain the following

Theorem 8. *Let $p \in (1, +\infty)$ and let $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set. Then, every sequence $f_n \in L^p(\Omega)$ that converges weakly in $L^p(\Omega)$ is bounded in $L^p(\Omega)$.*

Proof of Theorem 7. Suppose that there is a sequence y_n which converges weakly in $\mathcal{B}$ to some $y \in \mathcal{B}$, but is not bounded. Then, the sequence $x_n := y_n - y \in \mathcal{B}$ converges weakly to 0 in $\mathcal{B}$, but for the sequence of norms $\|x_n\|$ we have

$$\|x_n\|_{\mathcal{B}} = \|y_n - y\|_{\mathcal{B}} \geq \|y_n\|_{\mathcal{B}} - \|y\|_{\mathcal{B}},$$

so $\|x_n\|_{\mathcal{B}}$ remains unbounded, so we have:

$$x_n \rightharpoonup 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|x_n\|_{\mathcal{B}} = +\infty.$$

Thanks to first theorem of Hahn-Banach, for every $n \in \mathbb{N}$, we can find a bounded linear functional $T_n : \mathcal{B} \rightarrow \mathbb{R}$ such that

$$T_n(x_n) = \|x_n\|_{\mathcal{B}} \quad \text{and} \quad \|T_n\|_{\mathcal{B}'} = 1.$$

We extract a subsequence $(x_{n_k})_{k \geq 1}$ with the following properties:

- $\|x_{n_1}\|_{\mathcal{B}} \geq 4$;
- $\|x_{n_{k+1}}\|_{\mathcal{B}} \geq \|x_{n_k}\|_{\mathcal{B}}^2$ for all $k \geq 1$;
- $|T_{n_k}(x_{n_\ell})| \leq 1$ for all $k \geq 1$ and all $\ell \geq k + 1$.

For simplicity, we relabel the subsequence, as follows:

$$x_k := x_{n_k}, \quad T_k := T_{n_k},$$

and we set

$$A_k^2 := \|u_k\|_{\mathcal{B}} = T_k(x_k).$$

We notice that by construction we have that

$$A_1 \geq 2 \quad \text{and} \quad A_{k+1} \geq A_k^2 \quad \text{for all } k \geq 1,$$

and

$$A_{k+j} \geq 2^j A_k \quad \text{for all } k \geq 1 \quad \text{and all } j \geq 1.$$

In particular, the sequence of bounded linear functionals

$$\sum_{j=1}^n \frac{1}{A_j} T_j \quad n \geq 1,$$

is a Cauchy sequence in $\mathcal{B}'$ and so we can define the bounded linear functional $T \in \mathcal{B}'$ as the limit

$$T := \sum_{j=1}^{+\infty} \frac{1}{A_j} T_j.$$

Since by assumption $x_k \rightharpoonup 0$, we have that

$$\lim_{k \rightarrow +\infty} T(x_k) = 0.$$

On the other hand, by the choice of the subsequence, we have:

$$\begin{aligned}
T(x_k) &= \sum_{j=1}^{k-1} \frac{1}{A_j} T_j(x_k) + \frac{1}{A_k} T_k(x_k) + \sum_{j=k+1}^{+\infty} \frac{1}{A_j} T_j(x_k) \\
&\geq -\sum_{j=1}^{k-1} \frac{1}{A_j} |T_j(x_k)| + A_k - \sum_{j=k+1}^{+\infty} \frac{1}{A_j} |T_j(x_k)| \\
&\geq -\sum_{j=1}^{k-1} \frac{1}{A_j} + A_k - \sum_{j=k+1}^{+\infty} \frac{1}{A_j} |T_j| \|x_k\|_{\mathcal{B}} \\
&\geq -\sum_{j=1}^{k-1} \frac{1}{A_j} + A_k - \|x_k\|_{\mathcal{B}} \sum_{j=k+1}^{+\infty} \frac{1}{A_j} \\
&= -1 + A_k - A_k^2 \sum_{i=1}^{+\infty} \frac{1}{A_{k+i}}.
\end{aligned}$$

In order to estimate the right-hand side from below we need to estimate the series

$$A_k^2 \sum_{i=1}^{+\infty} \frac{1}{A_{k+i}},$$

for which we use the fast growth of the sequence A_k . Precisely, we notice that, for all $k \geq 1$ and all $i \geq 1$, we have

$$A_{k+i} \geq A_{k+i-1}^2 \geq \dots \geq A_{k+1}^{2(i-1)} \geq A_k^{2i}.$$

Thus,

$$A_k^2 \sum_{i=1}^{+\infty} \frac{1}{A_{k+i}} \leq A_k^2 \sum_{i=1}^{+\infty} \frac{1}{A_k^{2i}} = \sum_{j=0}^{+\infty} \frac{1}{A_k^{2j}} = \frac{1}{1 - A_k^{-2}} \leq \frac{1}{1 - \frac{1}{2}} = 2.$$

which gives that

$$T(x_k) \geq A_k - 3 \rightarrow +\infty,$$

which is a contradiction. $\square$

THE SEQUENCES CONVERGING WEAKLY-* IN $\mathcal{B}'$ ARE BOUNDED

The proof of the next theorem is identical to the one of Theorem 7. We give it here for the sake of completeness.

Theorem 9. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{B}'$ be its dual. Let $T_n \in \mathcal{B}'$ be a weakly converging sequence in $\mathcal{B}'$. Then, the sequence of norms $\|T_n\|_{\mathcal{B}'}$ is bounded.*

Proof. Suppose that there is a sequence $S_n \in \mathcal{B}'$ which converges weakly-* in $\mathcal{B}'$ to some $S \in \mathcal{B}'$, but is not bounded. Then, the sequence $T_n := S_n - S \in \mathcal{B}'$ converges weakly to 0 in $\mathcal{B}'$, but we still have that

$$\lim_{n \rightarrow +\infty} \|T_n\|_{\mathcal{B}'} = \lim_{n \rightarrow +\infty} \|S_n - S\|_{\mathcal{B}'} \geq \lim_{n \rightarrow +\infty} \|S_n\|_{\mathcal{B}'} - \|S\|_{\mathcal{B}'} = +\infty.$$

Since

$$T_n = \sup \left\{ T_n(x) : x \in \mathcal{B}, \|x\|_{\mathcal{B}} = 1 \right\},$$

we can find $x_n \in \mathcal{B}$ such that

$$\frac{1}{2} \|T_n\|_{\mathcal{B}'} \leq T_n(x_n) \leq \|T_n\|_{\mathcal{B}'} \quad \text{and} \quad \|x_n\|_{\mathcal{B}} = 1.$$

We next extract a subsequence $(T_{n_k})_{k \geq 1}$ with the following properties:

- $\|T_{n_1}\|_{\mathcal{B}'} \geq 4$;
- $\|T_{n_{k+1}}\|_{\mathcal{B}'} \geq \|T_{n_k}\|_{\mathcal{B}'}^2$ for all $k \geq 1$;
- $|T_{n_\ell}(x_{n_k})| \leq 1$ for all $\ell \geq 2$ and all $1 \leq k \leq \ell - 1$.

We relabel the subsequence, as follows:

$$T_k := T_{n_k}, \quad x_k := x_{n_k},$$

and we set

$$A_k := \sqrt{\|T_k\|_{\mathcal{B}}}.$$

By construction we have

$$A_1 \geq 2 \quad \text{and} \quad A_{k+1} \geq A_k^2 \quad \text{for all} \quad k \geq 1,$$

and

$$A_{k+j} \geq 2^j A_k \quad \text{for all} \quad k \geq 1 \quad \text{and all} \quad j \geq 1.$$

In particular, the sequence

$$\sum_{j=1}^n \frac{1}{A_j} x_j \quad n \geq 1,$$

is a Cauchy sequence in $\mathcal{B}$ and so it converges to some

$$x := \sum_{j=1}^{+\infty} \frac{1}{A_j} x_j.$$

Since by assumption $T_k \rightarrow 0$, we have that

$$\lim_{k \rightarrow +\infty} T_k(x) = 0.$$

On the other hand, by the choice of the subsequence, we have:

$$\begin{aligned} T_k(x) &= \sum_{j=1}^{k-1} \frac{1}{A_j} T_k(x_j) + \frac{1}{A_k} T_k(x_k) + \sum_{j=k+1}^{+\infty} \frac{1}{A_j} T_k(x_j) \\ &\geq - \sum_{j=1}^{k-1} \frac{1}{A_j} |T_k(x_j)| + \frac{1}{2} \frac{\|T_k\|_{\mathcal{B}'}}{A_k} - \sum_{j=k+1}^{+\infty} \frac{1}{A_j} |T_k(x_j)| \\ &\geq - \sum_{j=1}^{k-1} \frac{1}{A_j} + \frac{1}{2} A_k - \sum_{j=k+1}^{+\infty} \frac{1}{A_j} \|T_k\|_{\mathcal{B}'} \|x_j\|_{\mathcal{B}} \\ &\geq - \sum_{j=1}^{k-1} \frac{1}{A_j} + \frac{1}{2} A_k - \|T_k\|_{\mathcal{B}} \sum_{j=k+1}^{+\infty} \frac{1}{A_j} \\ &= -1 + \frac{1}{2} A_k - A_k^2 \sum_{i=1}^{+\infty} \frac{1}{A_{k+i}}. \end{aligned}$$

Next, we notice that by the choice of the subsequence, for all $k \geq 1$ and all $i \geq 1$, we have

$$A_{k+i} \geq A_{k+i-1}^2 \geq \dots \geq A_{k+1}^{2(i-1)} \geq A_k^{2i}.$$

Thus,

$$A_k^2 \sum_{i=1}^{+\infty} \frac{1}{A_{k+i}} \leq A_k^2 \sum_{i=1}^{+\infty} \frac{1}{A_k^{2i}} = \sum_{j=0}^{+\infty} \frac{1}{A_k^{2j}} = \frac{1}{1 - A_k^{-2}} \leq \frac{1}{1 - \frac{1}{2}} = 2.$$

which gives that

$$T_k(x) \geq \frac{1}{2} A_k - 3 \rightarrow +\infty,$$

which is a contradiction. □

TEOREMA DI RADON-RIESZ IN $L^p(\Omega)$

Theorem 10 (Radon-Riesz). *Let $p \in (1, +\infty)$ and $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set. Let $f_n \in L^p(\Omega)$ be a given sequence and let $f \in L^p(\Omega)$ a given function. Then, the following are equivalent:*

- (1) $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$ and $\|f_n\|_{L^p(\Omega)} \rightarrow \|f\|_{L^p(\Omega)}$;
- (2) $f_n \rightarrow f$ strongly in $L^p(\Omega)$.

Proof. We notice that we always have that (2) implies (1). Thus, it is sufficient to prove that (1) implies (2). We consider several different cases.

Proof in the case $p = 2$. A direct computation gives

$$\|f_n - f\|_{L^2(\Omega)}^2 = \int_{\Omega} f_n^2 dx - \int_{\Omega} f^2 dx + 2 \int_{\Omega} (f - f_n) f \rightarrow 0.$$

Proof in the case $p \geq 2$. By the Clarkson's inequality

$$\left\| \frac{f_n - f}{2} \right\|_{L^p(\Omega)}^p \leq \frac{1}{2} \|f_n\|_{L^p(\Omega)}^p + \frac{1}{2} \|f\|_{L^p(\Omega)}^p - \left\| \frac{f_n + f}{2} \right\|_{L^p(\Omega)}^p.$$

Since $\frac{1}{2}(f_n + f) \rightharpoonup f$ in $L^p(\mathbb{R}^d)$, we have that

$$\|f\|_{L^p(\Omega)}^p \leq \liminf_{n \rightarrow +\infty} \left\| \frac{f_n + f}{2} \right\|_{L^p(\Omega)}^p,$$

and so

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \left\| \frac{f_n - f}{2} \right\|_{L^p(\Omega)}^p &\leq \limsup_{n \rightarrow +\infty} \left(\frac{1}{2} \|f_n\|_{L^p(\Omega)}^p + \frac{1}{2} \|f\|_{L^p(\Omega)}^p - \left\| \frac{f_n + f}{2} \right\|_{L^p(\Omega)}^p \right) \\ &\leq \|f\|_{L^p(\Omega)}^p - \liminf_{n \rightarrow +\infty} \left\| \frac{f_n + f}{2} \right\|_{L^p(\Omega)}^p \\ &\leq \|f\|_{L^p(\Omega)}^p - \|f\|_{L^p(\Omega)}^p = 0, \end{aligned}$$

which concludes the proof in the case $p \geq 2$.

Proof in the case $1 < p < 2$. In this case we will use the second Clarkson inequality

$$\left| \frac{a+b}{2} \right|^{\frac{p}{p-1}} + \left| \frac{a-b}{2} \right|^{\frac{p}{p-1}} \leq \left(\frac{|a|^p}{2} + \frac{|b|^p}{2} \right)^{\frac{1}{p-1}},$$

which in terms of the L^p norms reads as

$$\int_{\Omega} \left(\left| \frac{f_n - f}{2} \right|^{\frac{p}{p-1}} + \left| \frac{f_n + f}{2} \right|^{\frac{p}{p-1}} \right)^{p-1} \leq \frac{1}{2} \int_{\Omega} |f_n|^p dx + \frac{1}{2} \int_{\Omega} |f|^p dx.$$

Thanks to the reverse Minkowski inequality

$$\left(\int_{\Omega} |g|^{p-1} \right)^{\frac{1}{p-1}} + \left(\int_{\Omega} |h|^{p-1} \right)^{\frac{1}{p-1}} \leq \left(\int_{\Omega} |g+h|^{p-1} \right)^{\frac{1}{p-1}},$$

applied to

$$g = \left| \frac{f_n - f}{2} \right|^{\frac{p}{p-1}} \quad \text{and} \quad h = \left| \frac{f_n + f}{2} \right|^{\frac{p}{p-1}},$$

we get

$$\int_{\Omega} \left(\left| \frac{f_n - f}{2} \right|^{\frac{p}{p-1}} + \left| \frac{f_n + f}{2} \right|^{\frac{p}{p-1}} \right)^{p-1} \geq \left(\left(\int_{\Omega} \left| \frac{f_n - f}{2} \right|^p \right)^{\frac{1}{p-1}} + \left(\int_{\Omega} \left| \frac{f_n + f}{2} \right|^p \right)^{\frac{1}{p-1}} \right)^{p-1},$$

which together with the Clarkson's inequality gives

$$\left(\left(\int_{\Omega} \left| \frac{f_n - f}{2} \right|^p \right)^{\frac{1}{p-1}} + \left(\int_{\Omega} \left| \frac{f_n + f}{2} \right|^p \right)^{\frac{1}{p-1}} \right)^{p-1} \leq \frac{1}{2} \int_{\Omega} |f_n|^p dx + \frac{1}{2} \int_{\Omega} |f|^p dx.$$

This can also be written as

$$\left(\int_{\Omega} \left|\frac{f_n - f}{2}\right|^p\right)^{\frac{1}{p-1}} + \left(\int_{\Omega} \left|\frac{f_n + f}{2}\right|^p\right)^{\frac{1}{p-1}} \leq \left(\frac{1}{2} \int_{\Omega} |f_n|^p dx + \frac{1}{2} \int_{\Omega} |f|^p dx\right)^{\frac{1}{p-1}}.$$

Thanks to the semincontinuity of the L^p norm, we have

$$\int_{\Omega} |f|^p \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \left|\frac{f_n + f}{2}\right|^p.$$

Thus, we get

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \left(\int_{\Omega} \left|\frac{f_n - f}{2}\right|^p\right)^{\frac{1}{p-1}} &\leq \lim_{n \rightarrow +\infty} \left(\frac{1}{2} \int_{\Omega} |f_n|^p dx + \frac{1}{2} \int_{\Omega} |f|^p dx\right)^{\frac{1}{p-1}} - \liminf_{n \rightarrow +\infty} \left(\int_{\Omega} \left|\frac{f_n + f}{2}\right|^p\right)^{\frac{1}{p-1}} \\ &= \left(\int_{\Omega} |f|^p dx\right)^{\frac{1}{p-1}} - \liminf_{n \rightarrow +\infty} \left(\int_{\Omega} \left|\frac{f_n + f}{2}\right|^p\right)^{\frac{1}{p-1}} \leq 0, \end{aligned}$$

which concludes the proof. $\square$

WEAK CONVERGENCE TIMES STRONG CONVERGENCE

Theorem 11. Let Ω be a Lebesgue measurable set in $\mathbb{R}^d$. Let $p \in (1, +\infty)$ and $q := \frac{p}{p-1}$.

Let f_n be a sequence in $L^p(\Omega)$ that converges strongly in $L^p(\Omega)$ to some $f \in L^p(\Omega)$.

Let g_n be a succession in $L^q(\Omega)$ that converges weakly in $L^q(\Omega)$ to some $g \in L^q(\Omega)$.

Then

$$\int_{\Omega} f(x)g(x)dx = \lim_{n \rightarrow +\infty} \int_{\Omega} f_n(x)g_n(x)dx.$$

Proof. We notice that

$$\int_{\Omega} f_n(x)g_n(x)dx - \int_{\Omega} f(x)g(x)dx = \int_{\Omega} (f_n(x) - f(x))g_n(x)dx + \int_{\Omega} (g_n(x) - g(x))f(x)dx.$$

We first notice that since f is fixed and $g_n \rightharpoonup g$ in $L^q(\Omega)$, we have

$$\lim_{n \rightarrow +\infty} \int_{\Omega} (g_n(x) - g(x))f(x)dx = 0.$$

On the other hand, by the Hölder inequality we have

$$\left| \int_{\Omega} (f_n(x) - f(x))g_n(x)dx \right| \leq \|f_n - f\|_{L^p(\Omega)} \|g_n\|_{L^q(\Omega)}.$$

Now, since $f_n \rightarrow f$ strongly in $L^p(\Omega)$, we have

$$\lim_{n \rightarrow +\infty} \|f_n - f\|_{L^p(\Omega)} = 0.$$

Moreover, since $g_n \rightharpoonup g$ in $L^q(\Omega)$, we know that the sequence of norms $\|g_n\|_{L^q(\Omega)}$ is bounded. Thus,

$$\limsup_{n \rightarrow +\infty} \left| \int_{\Omega} (f_n(x) - f(x))g_n(x)dx \right| \leq \limsup_{n \rightarrow \infty} \|f_n - f\|_{L^p(\Omega)} \|g_n\|_{L^q(\Omega)} = 0,$$

so we get

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f_n(x)g_n(x)dx - \int_{\Omega} f(x)g(x)dx = 0,$$

which concludes the proof. $\square$

Remark 12. We notice that there are sequences in $L^p(\Omega)$ and $L^q(\Omega)$ such that: f_n converges weakly in $L^p(\Omega)$ to some $f \in L^p(\Omega)$, and g_n converges weakly in $L^q(\Omega)$ to some $g \in L^q(\Omega)$, but

$$\int_{\Omega} f(x)g(x)dx \neq \lim_{n \rightarrow +\infty} \int_{\Omega} f_n(x)g_n(x)dx.$$

For instance, if $\Omega = [0, \pi]$ and $p = 2$, we can take the sequences $f_n(x) = g_n(x) = \sin(nx)$. We know that $\phi_n \rightharpoonup 0$ weakly in $L^2([0, \pi])$ to 0, so $f = g = 0$. On the other hand, for every $n \geq 1$, we have

$$\int_0^\pi f_n(x)g_n(x)dx = \int_0^\pi \sin^2(nx)dx = \frac{\pi}{2}.$$

Thus,

$$\lim_{n \rightarrow +\infty} \int_0^\pi f_n(x)g_n(x) dx = \frac{\pi}{2} \quad \text{while} \quad \int_0^\pi f(x)g(x) dx = 0.$$

We can easily generalize Theorem 11 to sequences in a Banach space and its dual. Precisely, we have the following. We notice that the proofs of Theorem 11, Theorem 13, and Theorem 14 are obtained from the same estimate. We also notice that an essential ingredient in both proofs is the fact that the weakly converging sequences in $\mathcal{B}$ and the weakly-* converging sequences in $\mathcal{B}'$ are bounded.

Theorem 13. *Let $\mathcal{B}$ be a Banach space with dual $\mathcal{B}'$. Let $x_n \in \mathcal{B}$ be a sequence converging strongly in $\mathcal{B}$ to some $x \in \mathcal{B}$. Let $T_n \in \mathcal{B}'$ be a sequence converging weakly-* in $\mathcal{B}'$ to some $T \in \mathcal{B}'$. Then,*

$$T(x) = \lim_{n \rightarrow +\infty} T_n(x_n).$$

Proof. For every $n \geq 1$, we estimate

$$\begin{aligned} |T_n(x_n) - T(x)| &\leq |T_n(x_n) - T_n(x)| + |T_n(x) - T(x)| \\ &\leq \|T_n\|_{\mathcal{B}'} \|x_n - x\|_{\mathcal{B}} + |T_n(x) - T(x)|. \end{aligned}$$

Since $T_n \rightharpoonup^* T$, we know that the sequence of norms $\|T_n\|_{\mathcal{B}'}$ is bounded, while the strong convergence of x_n implies that $\|x_n - x\|_{\mathcal{B}} \rightarrow 0$. The weak-* convergence of T_n , on the other hand, gives that $T_n(x) \rightarrow T(x)$. Thus,

$$\lim_{n \rightarrow +\infty} |T_n(x_n) - T(x)| \leq \limsup_{n \rightarrow +\infty} \|T_n\|_{\mathcal{B}'} \lim_{n \rightarrow +\infty} \|x_n - x\|_{\mathcal{B}} + \lim_{n \rightarrow +\infty} |T_n(x) - T(x)| = 0,$$

which concludes the proof. $\square$

Theorem 14. *Let $\mathcal{B}$ be a Banach space with dual $\mathcal{B}'$. Let $x_n \in \mathcal{B}$ be a sequence converging weakly in $\mathcal{B}$ to some $x \in \mathcal{B}$. Let $T_n \in \mathcal{B}'$ be a sequence converging strongly in $\mathcal{B}'$ to some $T \in \mathcal{B}'$. Then,*

$$T(x) = \lim_{n \rightarrow +\infty} T_n(x_n).$$

Proof. For every $n \geq 1$, we estimate

$$\begin{aligned} |T_n(x_n) - T(x)| &\leq |T_n(x_n) - T(x_n)| + |T(x_n) - T(x)| \\ &\leq \|T_n - T\|_{\mathcal{B}'} \|x_n\|_{\mathcal{B}} + |T(x_n) - T(x)|. \end{aligned}$$

Since $x_n \rightharpoonup x$, we know that the sequence of norms $\|x_n\|_{\mathcal{B}}$ is bounded, while the strong convergence of T_n implies that $\|T_n - T\|_{\mathcal{B}'} \rightarrow 0$. Finally, the weak convergence of x_n to x , implies that $T(x_n) \rightarrow T(x)$. Thus,

$$\lim_{n \rightarrow +\infty} |T_n(x_n) - T(x)| \leq \lim_{n \rightarrow +\infty} \|T_n - T\|_{\mathcal{B}'} \limsup_{n \rightarrow +\infty} \|x_n\|_{\mathcal{B}} + \lim_{n \rightarrow +\infty} |T(x_n) - T(x)| = 0,$$

which concludes the proof. $\square$