Besicovitch's covering theorem

This note is dedicated to the following covering theorem.

Teorema 1 (Besicovitch). *There are integers $P(d)$ and $Q(d)$ depending only on the dimension d with the following properties. Let A be a bounded subset of $\mathbb{R}^d$ and let $\mathcal{F}$ be a collection of closed balls with centers in A such that each point in A is a center of some ball:*

$$\mathcal{F} = \big\{\overline{B}_{r_i}(x_i)\ :\ x_i \in A,\ r_i > 0\big\}.$$

- *There is a finite or countable collection of balls $B_i \in \mathcal{F}$ such that:*
  - *the collection of balls B_i covers A;*
  - *every point in $\mathbb{R}^d$ belongs to at most $P(d)$ balls B_i.*
- *There are families of balls $\mathcal{F}_j$, $j = 1, \dots, Q(d)$, such that:*
  - *the balls in each collection $\mathcal{F}_j$ are disjoint;*
  - *A is covered by the balls in the collection $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_{Q(d)}$.*

A counting lemma

Osservazione 2. *Consider a triangle with vertices O, A and B such that*

$$|AB| \ge |OA| \qquad \text{and} \qquad |AB| \ge |OB|.$$

Then the angle $\angle AOB$ is at least $\pi/3$. We can re-write this statement as follows:

Suppose that $a, b \in \mathbb{R}^d$ are two non-zero vectors such that

$$|a| \le |a - b| \qquad \text{and} \qquad |b| \le |a - b|.$$

Then,

$$\left|\frac{a}{|a|} - \frac{b}{|b|}\right| \ge 1.$$

Lemma 3. *Suppose that*

$$\mathcal{F} = \big\{\overline{B}_{r_i}(x_i)\big\}_{i=1,\dots,k}$$

is a finite collection of closed balls in $\mathbb{R}^d$ such that:

- *the balls have a non-empty intersection:*

$$0 \in \overline{B}_{r_i}(x_i) \quad \text{for every} \quad i = 1, \dots k;$$

- *no ball contains a center of another ball:*

$$x_i \notin \overline{B}_{r_j}(x_j) \quad \text{when} \quad i \neq j.$$

Then

$$k \le C_d,$$

where C_d is a dimensional constant.

Dimostrazione. Notice that when $i \neq j$ we have that

$$x_i \notin \overline{B}_{r_j}(x_j) \qquad \text{and} \qquad x_j \notin \overline{B}_{r_i}(x_i).$$

This means that:

$$r_i \le |x_i - x_j| \qquad \text{and} \qquad r_j \le |x_i - x_j|.$$

Now, the first point implies that

$$|x_i| \le r_i \le |x_i - x_j| \qquad \text{and} \qquad |x_j| \le r_j \le |x_i - x_j|.$$

By the previous remark, we get that

$$\left| \frac{x_i}{|x_i|} - \frac{x_j}{|x_j|} \right| \geq 1.$$

Set $v_i = x_i/|x_i|$. Then, the balls

$$\left\{ \overline{B}_{1/2}(v_i) \right\}_{i=1,\dots,k}$$

are mutually disjoint. This concludes the proof. □

Proof of the Besicovitch's theorem for finite sets

In this section we will suppose that the set of centers A is a finite set.

In particular, the family of balls $\mathcal{F}$ can be written as

$$B_j := \overline{B}_{r_j}(x_j),\ j = 1, \dots, M,$$

for some finite $M \in \mathbb{N}$. Without loss of generality, we can suppose that the radii r_j are ordered:

$$r_1 \geq r_2 \geq r_3 \geq \dots \geq r_M > 0.$$

Part 1. We construct the subcollection $\mathcal{F}'$ of balls

$$\overline{B}_{r_{j_k}}(x_{j_k}),\ k = 1, \dots, m,$$

as follows:

First element. As first element of $\mathcal{F}'$ we pick the ball $\overline{B}_{r_1}(x_1)$. That is $j_1 := 1$.

Iterative construction. Suppose that we have selected balls $\overline{B}_{r_{j_1}}(x_{j_1}), \dots, \overline{B}_{r_{j_n}}(x_{j_n}) \in \mathcal{F}$, for some $n \in [1, M]$. We choose $B_{r_{j_{n+1}}}(x_{j_{n+1}})$ as follows: j_{n+1} is the smallest index $k > j_n$ for which the center x_k of the ball $\overline{B}_{r_k}(x_k) \in \mathcal{F}$ does not belong to any of the previously selected balls $\overline{B}_{r_{j_1}}(x_{j_1}), \dots, B_{r_{j_n}}(x_{j_n})$. If there is not such an index, we have that $\overline{B}_{r_{j_1}}(x_{j_1}), \dots, \overline{B}_{r_{j_n}}(x_{j_n})$ cover A, so we set $M := N$ and

$$\mathcal{F}' := \left\{ \overline{B}_{r_{j_1}}(x_{j_1}), \dots, \overline{B}_{r_{j_M}}(x_{j_M}) \right\}.$$

Sparse centers property. We claim that the collection $\mathcal{F}'$ has the following property:

$$x_{j_m} \notin \overline{B}_{r_{j_n}}(x_{j_n}) \qquad \text{and} \qquad x_{j_n} \notin \overline{B}_{r_{j_m}}(x_{j_m}),$$

for all $1 \leq m < n \leq M$.

- Since $m < n$, then in the construction of $\mathcal{F}'$ the ball $\overline{B}_{r_{j_n}}(x_{j_n})$ has been selected after the ball $\overline{B}_{r_{j_m}}(x_{j_m})$, but this means that the center x_{j_n} does not belong to any of $B_{r_1}(x_1), \dots, B_{r_{n-1}}(x_{n-1})$, in particular,

$$x_{j_n} \notin \overline{B}_{r_{j_m}}(x_{j_m}).$$

- Thanks to the previous point we have that If $k < n$, then x_{j_n} does not belong to $B_{r_{j_k}}(x_{j_k})$. This means that

$$|x_{j_n} - x_{j_m}| > r_{j_m}.$$

  Since $m < n$, we have that

$$r_{j_n} \leq r_{j_m},$$

  so we get

$$|x_{j_n} - x_{j_m}| > r_{j_n},$$

  which means that

$$x_{j_m} \notin \overline{B}_{r_{j_n}}(x_{j_n}).$$

Bounded overlap property. Consider now a point $x \in \mathbb{R}^d$. Suppose that x belongs to a finite number N of balls in $\mathcal{F}'$. Since, these balls have the sparse centers property, the counting lemma implies that N is less that a dimensional constant C_d.

Part 2. We next divide the collection $\mathcal{F}'$ into subcollections $\mathcal{F}'_1, \dots, \mathcal{F}'_N$ of disjoint balls. The algorithm for constructing $\mathcal{F}'_1, \dots, \mathcal{F}'_N$ is the following.

Initialization. We create the family $\mathcal{F}'_1$ and we set $B_{r_{j_1}}(x_{j_1})$ to be the first element of $\mathcal{F}'_1$.

Iteration. Suppose that we have divided the balls

$$\overline{B}_{r_{j_1}}(x_{j_1}),\ \overline{B}_{r_{j_2}}(x_{j_2}), \dots,\ \overline{B}_{r_{j_{k-1}}}(x_{j_{k-1}}),$$

into families $\mathcal{F}'_1, \dots, \mathcal{F}'_n$ of disjoint balls. Consider the ball $\overline{B}_{r_{j_k}}(x_{j_k})$. We add $\overline{B}_{r_{j_k}}(x_{j_k})$ to the first collection $\mathcal{F}'_i$, $i = 1, \dots, n$, for which $B_{r_{j_k}}(x_{j_k})$ is disjoint from all balls in $\mathcal{F}'_i$. If there is not such a collection we create a new collection $\mathcal{F}'_n$. We proceed this way until all the balls in $\mathcal{F}'$

$$\overline{B}_{r_{j_1}}(x_{j_1}),\ \overline{B}_{r_{j_2}}(x_{j_2}), \dots,\ \overline{B}_{r_{j_M}}(x_{j_M}),$$

are divided into families $\mathcal{F}'_1, \dots, \mathcal{F}'_N$ of disjoint balls.

Bound on the number of collections $\mathcal{F}'_1, \dots, \mathcal{F}'_N$. We will prove that

$$N \le Q(d).$$

Let the ball $\overline{B}_{r_{j_{k_N}}}(x_{j_{k_N}})$ be the first element of the collection $\mathcal{F}'_N$. Then, by construction, there are balls

$$\overline{B}_{r_{j_{k_1}}}(x_{j_{k_1}}) \in \mathcal{F}'_1, \dots, \overline{B}_{r_{j_{k_{N-1}}}}(x_{j_{k_{N-1}}}) \in \mathcal{F}'_{N-1},$$

such that:

(a) the balls $\overline{B}_{r_{j_{k_i}}}(x_{j_{k_i}})$, $i = 1, \dots, N-1$, have been selected before $\overline{B}_{r_{j_{k_N}}}(x_{j_{k_N}})$, so in particular we have

$$r_{j_{k_i}} \ge r_{j_{k_N}} \quad \text{for every} \quad i = 1, \dots, N-1;$$

(b) $\overline{B}_{r_{j_{k_i}}}(x_{j_{k_i}}) \cap \overline{B}_{r_{j_{k_N}}}(x_{j_{k_N}}) \neq \emptyset \quad \text{for every} \quad i = 1, \dots, N-1.$

Consider the ball $\overline{B}_{2r_{j_{k_N}}}(x_{j_{k_N}})$. Thanks to the points (a) and (b) above, for each $i = 1, \dots, N-1$, we can find a closed ball B_i of radius $r_{j_{k_N}}$ such that;

$$B_i \subset \overline{B}_{r_{j_{k_i}}}(x_{j_{k_i}}) \cap \overline{B}_{2r_{j_{k_N}}}(x_{j_{k_N}}).$$

We notice that

$$\begin{aligned}
(N-1)\omega_d 2^{-d}(r_{j_{k_N}})^d &= \sum_{i=1}^{N-1} |B_i| \\
&= \int_{B_{2r_{j_{k_N}}}(x_{j_{k_N}})} \sum_{i=1}^{N-1} \chi_{B_i} \\
&\le \int_{B_{2r_{j_{k_N}}}(x_{j_{k_N}})} \sum_{i=1}^{N-1} \chi_{B_{r_{j_{k_i}}}(x_{j_{k_i}})} \\
&\le P(d)|B_{2r_{j_{k_N}}}(x_{j_{k_N}})| = P(d)\omega_d 2^d (r_{j_{k_N}})^d.
\end{aligned}$$

Thus, simplifying the term $\omega_d(r_{j_{k_N}})^d$, we get

$$N - 1 \le 4^d P(d),$$

which concludes the proof of Theorem 1 for finite sets A. $\square$

Proof of the Besicovitch's theorem

In this section we will prove the Besicovitch's theorem for a general bounded set A.

We can suppose that to every $x \in A$ there is exactly one ball $\overline{B}_{r(x)}(x) \in \mathcal{F}$. Indeed, we can simply start by selecting a subcollection of $\mathcal{F}$ by choosing one ball for every center $x \in A$.

Suppose that the collection $\mathcal{F}$ contains a ball $\overline{B}_{r(x)}(x)$ such that $r(x) \geq \operatorname{diam}(A)$. Then, since $x \in A$, the ball $\overline{B}_{r(x)}(x)$ contains A, so the existence of the family $\mathcal{F}'$ trivially follows.

In what follows we will suppose that

$$r(x) \leq \operatorname{diam}(A) \quad \text{for all} \quad x \in A.$$

Proof of Part 1 of Theorem 1. We start by constructing the family $\mathcal{F}'$. We first set

$$M_1 := \sup\{r(x) \ : \ \overline{B}_{r(x)}(x) \in \mathcal{F}\},$$

and we consider the family $\mathcal{F}_1$ of all balls $\overline{B}_{r(x)}(x) \in \mathcal{F}$ such that

$$\frac{M_1}{2} < r(x) \leq M_1.$$

We denote by A_1 the family of centers $x \in A$ such that $\overline{B}_{r(x)}(x) \in \mathcal{F}_1$.

Construction of $\mathcal{F}_1'$. We now create a subcollection $\mathcal{F}_1'$ of $\mathcal{F}_1$ as follows:

- **Initialization.** We pick any ball $\overline{B}_{r(x_{1,1})}(x_{1,1}) \in \mathcal{F}_1$ and we add it to $\mathcal{F}_1'$.
- **Selection algorithm.** Suppose that we have picked balls

  $$\overline{B}_{r(x_{1,1})}(x_{1,1}), \overline{B}_{r(x_{1,2})}(x_{1,2}), \dots, \overline{B}_{r(x_{1,k-1})}(x_{1,k-1}) \in \mathcal{F}_1'.$$

  If there is a point $y \in A_1$ not belonging to any of these balls,

  $$y \in A_1 \setminus \Big(\overline{B}_{r(x_{1,1})}(x_{1,1}) \cup \overline{B}_{r(x_{1,2})}(x_{1,2}) \cup \dots \cup \overline{B}_{r(x_{1,k-1})}(x_{1,k-1})\Big),$$

  then we set

  $$x_{1,k} := y,$$

  and we add the ball $\overline{B}_{r(x_{1,k})}(x_{1,k})$ to $\mathcal{F}_1'$.

  Remark: The sequence of radii $r(x_{1,1}),\ r(x_{1,2}),\ \dots,\ r(x_{1,k})$ obtained this way might not be decreasing.

- **The algorithm terminates.** We will show that the algorithm terminates after a finite number of steps. Indeed, suppose that we have selected balls

  $$\overline{B}_{r(x_{1,1})}(x_{1,1}),\ \overline{B}_{r(x_{1,2})}(x_{1,2}),\ \dots,\ \overline{B}_{r(x_{1,k})}(x_{1,k}) \in \mathcal{F}_1'.$$

  Since

  $$x_{1,k} \notin \overline{B}_{r(x_{1,i})}(x_{1,i}) \quad \text{for} \quad i = 1, \dots, k-1\,,$$

  we have that

  $$|x_{1,k} - x_{1,i}| > r(x_{1,i}) > \frac{M_1}{2}.$$

  By the same argument, we have that

  $$|x_{1,i} - x_{1,j}| > \frac{M_1}{2} \quad \text{for all} \quad 1 \leq i < j \leq k,$$

  which means that the balls in the family

  $$\Big\{\overline{B}_{M_1/2}(x_{1,j}) \ : \ j = 1, \dots, k\Big\},$$

  are mutually disjoint. Since all these points are contained in some fixed ball of radius $2\operatorname{diam}(A)$, we get that the algorithm will terminate after a finite number of steps.

- **Definition of $\mathcal{F}_1'$.** When the process terminates we have the subcollection $\mathcal{F}_1' \subset \mathcal{F}$ of balls

  $$\mathcal{F}_1' = \Big\{\overline{B}_{r(x_{1,1})}(x_{1,1}),\ \overline{B}_{r(x_{1,2})}(x_{1,2}),\ \dots,\ \overline{B}_{r(x_{1,N_1})}(x_{1,N_1})\Big\}.$$

  Moreover, by construction $\mathcal{F}_1'$ has the following properties:

(1) the balls $\overline{B}_{r(x_{1,j})}(x_{1,j})$, $j = 1, \ldots, N_1$, cover A_1;
(2) $M_1/2 < r(x_{1,j}) \le M_1$ for all $j = 1, \ldots, N_1$;
(3) the balls $\overline{B}_{M_1/2}(x_{1,j})$, $j = 1, \ldots, N_1$ are mutually disjoint.

Construction of $\mathcal{F}_2'$. We define the family of centers A_2 as follows. We first set

$$\widetilde{A}_2 := A \setminus \Big(\overline{B}_{r(x_{1,1})}(x_{1,1}) \cup \overline{B}_{r(x_{1,2})}(x_{1,2}) \cup \cdots \cup \ \overline{B}_{r(x_{1,N_1})}(x_{1,N_1})\Big).$$

We notice that, since $\mathcal{F}_1'$ is a cover of A_1, we have by construction

$$\widetilde{A}_2 \cap A_1 = \emptyset,$$

that is

$$r(x) \le \frac{M_1}{2} \quad \text{for all} \quad x \in \widetilde{A}_2.$$

We next define M_2 as

$$M_2 := \sup\Big\{r(x) \ : \ x \in \widetilde{A}_2\Big\},$$

and we notice that

$$M_2 \le \frac{M_1}{2}.$$

We define A_2 as the family of all centers $x \in \widetilde{A}_2$ such that

$$\frac{M_2}{2} \le r(x) \le M_2.$$

We say that $\overline{B}_{r(x)}(x) \in \mathcal{F}_2$ if $x \in A_2$.

We construct the subcollection $\mathcal{F}_2'$ of $\mathcal{F}_2$ as follows.

- **Initialization.** We pick any ball $\overline{B}_{r(x_{2,1})}(x_{2,1}) \in \mathcal{F}_2$ and we add it to $\mathcal{F}_2'$.
- **Selection algorithm.** Suppose that we have picked balls

  $$\overline{B}_{r(x_{2,1})}(x_{2,1}), \ldots, \overline{B}_{r(x_{2,k-1})}(x_{2,k-1}) \in \mathcal{F}_2'.$$

  If there is a point $y \in A_2$ that does not belong to any of these balls,

  $$y \in A_2 \setminus \Big(\overline{B}_{r(x_{2,1})}(x_{2,1}) \cup \cdots \cup \overline{B}_{r(x_{2,k-1})}(x_{2,k-1})\Big),$$

  then we set

  $$x_{2,k} := y,$$

  and we add the ball $\overline{B}_{r(x_{1,k})}(x_{1,k})$ to $\mathcal{F}_1'$. The algorithm terminates by the argument we have used in the construction of $\mathcal{F}_1'$.
- **Definition of $\mathcal{F}_2'$.** When the process terminates we have the subcollection $\mathcal{F}_2' \subset \mathcal{F}_2$ of balls

  $$\mathcal{F}_2' = \Big\{\overline{B}_{r(x_{2,1})}(x_{2,1}), \ \ldots, \ \overline{B}_{r(x_{2,N_2})}(x_{2,N_2})\Big\},$$

  with the following properties:

  (1) the balls $\overline{B}_{r(x_{2,j})}(x_{2,j})$, $j = 1, \ldots, N_2$, cover A_2;

  (2) $M_2/2 < r(x_{2,j}) \le M_2 \le M_1/2$ for all $j = 1, \ldots, N_2$;

  (3) the balls $\overline{B}_{M_2/2}(x_{1,j})$, $j = 1, \ldots, N_2$ are mutually disjoint.

Construction of $\mathcal{F}_k'$. Suppose that we have constructed the families

$$\mathcal{F}_j' \quad \text{for} \quad j = 1, \ldots, k-1,$$

where:

each $\mathcal{F}'_j$ consists of N_j balls

$$\mathcal{F}'_j = \left\{ \overline{B}_{r(x_{j,1})}(x_{j,1}),\ \ldots,\ \overline{B}_{r(x_{j,N_j})}(x_{j,N_j}) \right\},$$

- with centers
$$x_{j,1},\ x_{j,2},\ \ldots,\ x_{j,N_j}\ ,$$
all lying in the set
$$\widetilde{A}_j := A \setminus \left(\bigcup_{i=1}^{j-1} \bigcup_{\overline{B} \in \mathcal{F}'_i} \overline{B} \right),$$
- and radii
$$\frac{M_j}{2} < r(x_{j,1}) \le M_j\ , \quad \ldots\ , \quad \frac{M_j}{2} <\ r(x_{j,N_j}) \le M_j\ ,$$
where M_j is given by
$$M_j := \sup \left\{ r(x)\ :\ x \in \widetilde{A}_j \right\}.$$

Furthermore,

- the balls in $\mathcal{F}'_j$ cover the set
$$A_j := \left\{ x \in \widetilde{A}_j\ :\ \frac{M_j}{2} < r(x) \le M_j \right\}.$$

Then, we construct $\mathcal{F}'_k$ as follows:

- We first define the set
$$\widetilde{A}_k := A \setminus \left(\bigcup_{i=1}^{k-1} \bigcup_{\overline{B} \in \mathcal{F}'_i} \overline{B} \right),$$
in which we have eliminated all the points in A lying in one of the previously selected balls.
- If $\widetilde{A}_k \neq \emptyset$, then we compute
$$M_k := \sup \left\{ r(x)\ :\ x \in \widetilde{A}_k \right\},$$
- and we consider the set of centers (which is non-empty thanks to the definition of M_k)
$$A_k := \left\{ x \in \widetilde{A}_k\ :\ \frac{M_k}{2} < r(x) \le M_k \right\},$$
and we define the corresponding collection of balls $\mathcal{F}_k$ as
$$\mathcal{F}_k := \left\{ \overline{B}_{r(x)}(x) \in \mathcal{F}\ :\ x \in A_k \right\}.$$
- Finally we construct the subcollection $\mathcal{F}'_k$ of $\mathcal{F}_k$ as follows:
  - **Initialization.** We pick any ball $\overline{B}_{r(x_{k,1})}(x_{k,1}) \in \mathcal{F}_k$ and we add it to $\mathcal{F}'_k$.
  - **Selection algorithm.** Suppose that we have picked balls
  $$\overline{B}_{r(x_{k,1})}(x_{k,1}), \ldots, \overline{B}_{r(x_{k,\ell-1})}(x_{k,\ell-1}) \in \mathcal{F}'_k.$$
  If there is a point $y \in A_k$ that does not belong to any of these balls,
  $$y \in A_k \setminus \left(\overline{B}_{r(x_{k,1})}(x_{k,1}) \cup \cdots \cup \overline{B}_{r(x_{k,\ell-1})}(x_{k,\ell-1}) \right),$$
  then we set
  $$x_{k,\ell} := y,$$
  and we add the ball $\overline{B}_{r(x_{k,\ell})}(x_{k,\ell}) := \overline{B}_{r(y)}(y)$ to $\mathcal{F}'_k$.
  - **The algorithm terminates.** We notice that the algorithm terminates after a finite number of steps thanks to the same argument we have used in the construction of $\mathcal{F}'_1$.

– **Definition of $\mathcal{F}'_k$.** When the process terminates we have a finite subcollection $\mathcal{F}'_k$ of $\mathcal{F}_2$ of balls

$$\mathcal{F}'_k = \Big\{ \overline{B}_{r(x_{k,1})}(x_{k,1}),\ \ldots,\ \overline{B}_{r(x_{k,N_k})}(x_{k,N_k}) \Big\},$$

with the following properties:

– **Main properties of $\mathcal{F}'_k$:**

(1) the balls $\overline{B}_{r(x_{k,j})}(x_{2,j})$, $j = 1, \ldots, N_k$, cover A_k;

(2) $M_k/2 < r(x_{k,j}) \leq M_k$ for all $j = 1, \ldots, N_k$;

(3) the balls $\overline{B}_{M_k/2}(x_{k,j})$, $j = 1, \ldots, N_k$ are mutually disjoint.

In the next three lemmas we do not require that the procedure has terminated, but only that we have already constructed n families following the procedure described above.

Lemma 4 (Any collection of balls, each belonging to a different family, is sparse)**.** *Suppose that we have constructed the families*

$$\mathcal{F}'_1, \ldots, \mathcal{F}'_n$$

following the procedure described above. Consider two indices

$$1 \leq j < k \leq n,$$

and two balls

$$\overline{B}_{r(x_j)}(x_j) \in \mathcal{F}'_j \qquad \text{and} \qquad \overline{B}_{r(x_k)}(x_k) \in \mathcal{F}'_k.$$

Then:

(1) $r(x_k) < r(x_j)$*;*
(2) $x_k \notin \overline{B}_{r(x_j)}(x_j)$*;*
(3) $x_j \notin \overline{B}_{r(x_k)}(x_k)$*.*

Dimostrazione. The first two claims follow directly from the construction of $\mathcal{F}'_k$, while the last claim follows from (1) and (2). □

Lemma 5 (Counting balls in different families)**.** *Suppose that we have constructed the families*

$$\mathcal{F}'_1, \ldots, \mathcal{F}'_n$$

following the procedure described above. If there are $k \leq n$ distinct indices

$$1 \leq j_1 < j_2 < \cdots < j_k \leq n,$$

corresponding to k balls

$$\overline{B}_{r(x_{j_1})}(x_{j_1}) \in \mathcal{F}'_{j_1}\ ,\ldots,\ \overline{B}_{r(x_{j_k})}(x_{j_k}) \in \mathcal{F}'_{j_k}$$

all containing the same point $x \in \mathbb{R}^d$,

$$x \in \overline{B}_{r(x_{j_1})}(x_{j_1}) \cap \cdots \cap \overline{B}_{r(x_{j_k})}(x_{j_k}),$$

then $k \leq C_d$, where C_d is the dimensional constant from Lemma **??***.*

Dimostrazione. This follows immediately from to the previous Lemma 4 and Lemma **??**. □

Lemma 6 (Counting balls in the same family)**.** *Suppose that we have constructed the families*

$$\mathcal{F}'_1, \ldots, \mathcal{F}'_n$$

following the procedure described above. Let N_n be the number of balls in the collection $\mathcal{F}'_n$. If there are $k \leq N_n$ distinct indices

$$1 \leq j_1 < j_2 < \cdots < j_k \leq N_n,$$

corresponding to n balls

$$\overline{B}_{r(x_{j_1})}(x_{j_1})\ ,\ldots,\ \overline{B}_{r(x_{j_k})}(x_{j_k}) \in \mathcal{F}'_n$$

all containing the same point $x \in \mathbb{R}^d$,

$$x \in \overline{B}_{r(x_{j_1})}(x_{j_1}) \cap \cdots \cap \overline{B}_{r(x_{j_k})}(x_{j_k}),$$

then $n \leq C_d$, where C_d is a dimensional constant.

Dimostrazione. We recall that all the radii $r(x_{j_1}),\dots,r(x_{j_k})$ satisfy the inequalities
$$\frac{M_n}{2}<r(x_{j_1})\le M_n\ ,\ \dots\ ,\ \frac{M_n}{2}<r(x_{j_k})\le M_n\,.$$
In particular, since we have a point $x\in\mathbb{R}^d$ contained in all balls, we get that
$$\overline{B}_{M_n/2}(x_{j_1}),\dots,\ \overline{B}_{M_n/2}(x_{j_k})$$
are all contained in the ball $\overline{B}_{2M_n}(x)$. On the other hand, we know that these balls are disjoint, so we get
$$k\omega_d(M_n/2)^d=|B_{M_n/2}(x_{j_1})|+\dots+|B_{M_n/2}(x_{j_k})|\le|B_{2M_n}(x)|=\omega_d(2M_n)^d,$$
which implies that $k\le 4^d$. □

We now continue with the proof of Part 1 of the Besicovitch's theorem.

If for some $N\ge 1$ we have that
$$A\setminus\left(\bigcup_{i=1}^N\bigcup_{\overline{B}\in\mathcal{F}_i'}\overline{B}\right)=\emptyset,$$
then the procedure stops and we consider the collection of balls
$$\mathcal{F}':=\mathcal{F}_1'\cup\mathcal{F}_2'\cup\dots\cup\mathcal{F}_N'.$$
In this case, we have that $\mathcal{F}'$ covers A. Suppose that the procedure described above does not terminate. Then, we have a sequence of collections
$$\mathcal{F}_n',\quad n\in\mathbb{N},$$
and we set
$$\mathcal{F}':=\bigcup_{n=1}^{+\infty}\mathcal{F}_n'.$$
Suppose that
$$x\notin\mathcal{F}_1'\cup\dots\cup\mathcal{F}_N'.$$
Then,
$$r(x)\notin\bigcup_{j=1}^N\left(\frac{M_j}{2},M_j\right],$$
which implies that
$$r(x)\le\frac{M_N}{2}.$$
Since M_N is arbitrary, we get that
$$r(x)\le\lim_{N\to+\infty}M_N=0,$$
which is a contradiction. This implies that $\mathcal{F}'$ covers A.

It remains to observe that there is a dimensional constant $P(d)$ such that each point $x\in\mathbb{R}^d$ lies in at most $P(d)$ balls of $\mathcal{F}'$. Indeed, this is an immediate consequence of Lemma 5 and Lemma 6. This concludes the proof of Part 1.

Proof of Part 2 of Theorem 1. Given the collection of balls $\mathcal{F}'$ constructed in Part 1, we notice that:

- $\mathcal{F}'$ contains a finite or a countable number of balls;
- we can order the balls in $\mathcal{F}'$ in a sequence
$$\mathcal{F}':=\Big\{\overline{B}_{r_j}(x_j)\ :\ j\in\mathbb{N}\Big\},$$
in such a way that
$$r_j\le r_k\quad\text{whenever}\quad j\ge k.$$

We next divide $\mathcal{F}'$ into subcollections $\mathcal{F}_1'',\dots,\mathcal{F}_N''$ of disjoint balls through the following algorithm.

Initialization. We create the family $\mathcal{F}_1''$ and we set $B_{r_1}(x_1)$ to be the first element of $\mathcal{F}_1''$.

Iteration. Suppose that we have divided the balls
$$\overline{B}_{r_1}(x_1),\ \overline{B}_{r_2}(x_2),\dots,\ \overline{B}_{r_{k-1}}(x_{k-1}),$$
into families $\mathcal{F}_1'',\dots,\mathcal{F}_n''$ of disjoint balls. Consider the ball $\overline{B}_{r_k}(x_k)$. We add $\overline{B}_{r_k}(x_k)$ to

the first collection $\mathcal{F}''_i$, $i = 1, \ldots, n$, for which $\overline{B}_{r_k}(x_k)$ is disjoint from all balls in $\mathcal{F}''_i$.

If there is not such a collection we create a new collection $\mathcal{F}''_n$. We proceed this way until all the balls in $\mathcal{F}'$

$$\overline{B}_{r_1}(x_1),\ \overline{B}_{r_2}(x_2), \ldots,\ \overline{B}_{r_k}(x_k), \ldots$$

are divided into (a priori countably many) families of disjoint balls.

Bound on the number of subcollections. Suppose that we have created N distinct collections $\mathcal{F}''_1, \ldots, \mathcal{F}''_N$ by following the above procedure. We will show that

$$N \leq Q(d).$$

Let the ball $\overline{B}_{r_{k_N}}(x_{k_N})$ be the first element of the collection $\mathcal{F}''_N$. Then, by construction, there are balls

$$\overline{B}_{r_{k_1}}(x_{k_1}) \in \mathcal{F}''_1, \ldots, \overline{B}_{r_{k_{N-1}}}(x_{k_{N-1}}) \in \mathcal{F}''_{N-1},$$

such that:

(a) the balls $\overline{B}_{r_{k_i}}(x_{k_i})$, $i = 1, \ldots, N-1$, have been selected before the ball $\overline{B}_{r_{k_N}}(x_{k_N})$, so in particular we have

$$r_{k_i} \geq r_{k_N} \quad \text{for every} \quad i = 1, \ldots, N-1;$$

(b) $\overline{B}_{r_{k_i}}(x_{k_i}) \cap \overline{B}_{r_{k_N}}(x_{k_N}) \neq \emptyset \quad$ for every $\quad i = 1, \ldots, N-1$.

Consider the ball $\overline{B}_{2r_{k_N}}(x_{k_N})$. Thanks to the points (a) and (b) above, for each $i = 1, \ldots, N-1$, we can find a closed ball B_i of radius r_{k_N} such that;

$$B_i \subset \overline{B}_{r_{k_i}}(x_{k_i}) \cap \overline{B}_{2r_{k_N}}(x_{k_N}).$$

We notice that

$$\begin{aligned}
(N-1)\omega_d 2^{-d}(r_{k_N})^d &= \sum_{i=1}^{N-1} |B_i| \\
&= \int_{B_{2r_{k_N}}(x_{k_N})} \sum_{i=1}^{N-1} \chi_{B_i} \\
&\leq \int_{B_{2r_{k_N}}(x_{k_N})} \sum_{i=1}^{N-1} \chi_{B_{r_{k_i}}(x_{k_i})} \\
&\leq P(d)|B_{2r_{k_N}}(x_{k_N})| = P(d)\omega_d 2^d (r_{k_N})^d.
\end{aligned}$$

Thus, simplifying the term $\omega_d(r_{k_N})^d$, we get

$$N - 1 \leq 4^d P(d),$$

which concludes the proof of part 2 of Theorem 1 in the general case. $\square$